Generalized Seiberg-Witten Equations on Gibbons-Hawking spaces

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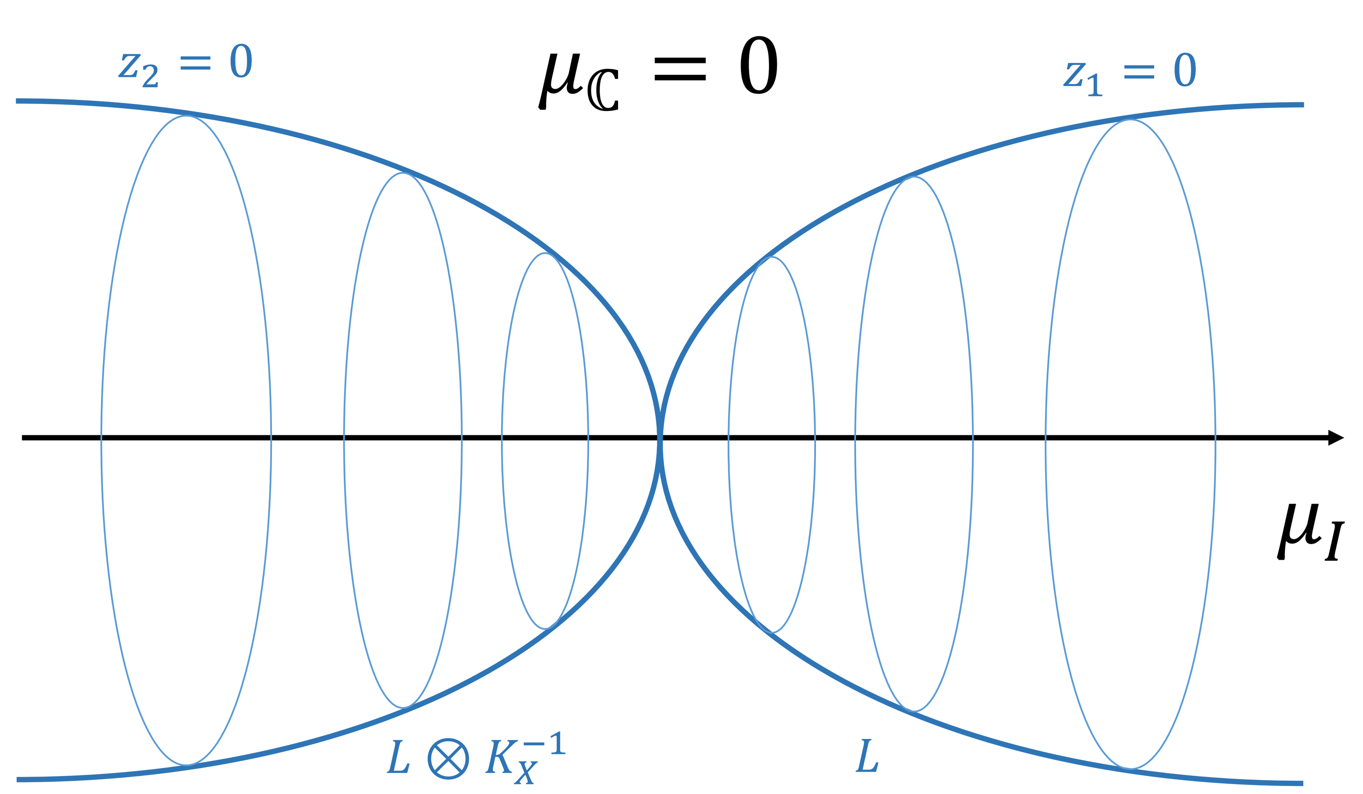
Seiberg-Witten equations

Let P be a $Spin^{\mathbb{C}}(4) = (Sp(1)_+ \times Sp(1)_- \times U(1))_{\pm}$ prinicpal bundle over a simply connected Riemannian manifold X with determinant line bundle L^2, obtained as a double cover $P \xrightarrow{2:1} P_{SO(4)} \times_X P_{L^2}$, where $P_{SO(4)}$ is the frame bundle of X.

- The positive Spinor bundle $S^+ = P \times_{\rho_+} \mathbb{H}$, $\rho_+(q_+, q_-, \lambda).v = q_+ \cdot v \cdot \lambda$ $Sp(1)_+$ acts permuting, $Sp(1)-$ trivially and $U(1)$ hyperKähler.
- The $U(1)$ action admits a hyperKähler moment map
$$\mu = i \cdot \mu_I + j \cdot \mu_{\mathbb{C}} : \mathbb{H} \to \mathrm{Im}(\mathbb{H})$$
- A connection a on L^2 lifts together with the Levi-Cevita connection a_{LC} on $P_{SO(4)}$ to a connection A on P.
- Spinors are sections $u \in \Gamma(X, S^+) = Map(P, \mathbb{H})^{Spin^{\mathbb{C}}(4)}$
- The **Seiberg-Witten equations:**
$$\not{D}^A(u) = 0 \ , \quad F_a^+ = \mu \circ u$$
- If X is **Kähler**, the bundle P can be reduced to an $(U(1) \times Sp(1)_- \times U(1))_{\pm}$ bundle Q.
- Identifying $\mathbb{H} \simeq \mathbb{C}^2$ we have $Q \times_{\rho_+} \mathbb{C}^2 = L \oplus (K_X^{-1} \otimes L)$, K_X the canonical bundle on X.
- Assuming L admits a holomorphic structure, the SW-Equations can be simplified, writing $u = (\alpha, \beta)$:
$$\overline{\partial}_a \alpha + \overline{\partial}_a^{\,*} \beta = 0 \ , (F_a^+)^{1,1} = \mu_I \circ u = \frac{1}{4}(|\alpha|^2 - |\beta|^2) \ , F_a^{2,0} = \mu_{\mathbb{C}} \circ u = \alpha \cdot \overline{\beta}$$
- Depending on L, either $\alpha \equiv 0$ or $\beta \equiv 0$, a determines a holomorphic structure and $\overline{\beta}$ (or α) is a holomorphic section of $K_X \otimes L^{-1}$ (or L).

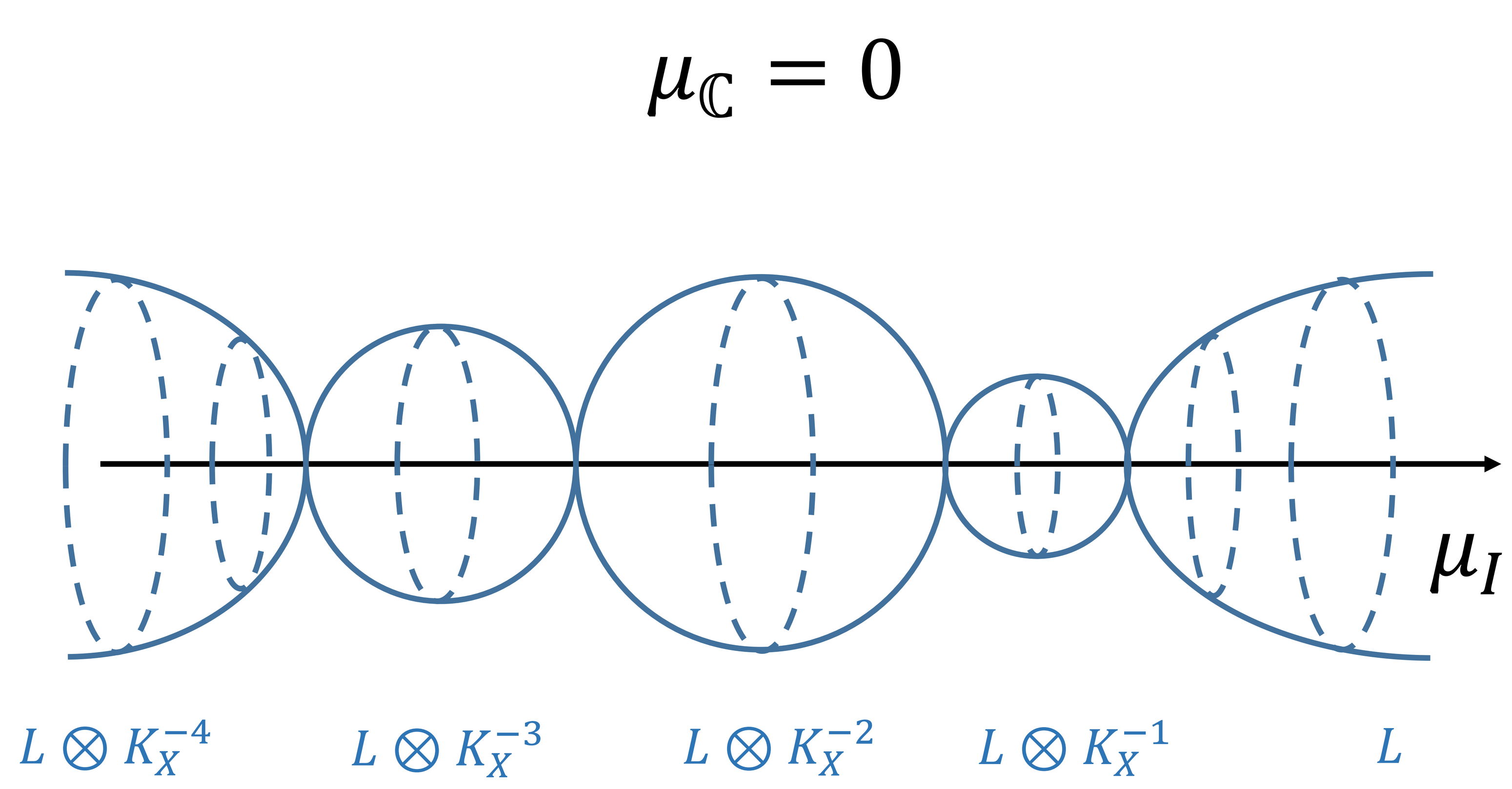


- By modding out the action of the (complexified) gauge group, one gets rid of the $(F_a^+)^{1,1} = \dots$ equation.
- The moduli space is $\mathbb{P}(H^0(X, L))$ (or $K_X \otimes L^{-1}$ respectively), the space of holomorphic sections of L up to a scalar.

Generalized Seiberg-Witten equations

Let Q be a $G = (U(1)_+ \times Sp(1)_- \times U(1))_{\pm}$ principal bundle over a simply connected **Kähler** manifold X with determinant line bundle L^2, obtained as a double cover $Q \xrightarrow{2:1} P_{U(2)} \times_X P_{L^2}$ and M a Gibbons-Hawking space as described on the right.

- The positive **non-linear** Spinor bundle $W^+ = Q \times_{\rho_+} M$, where $U(1)_+$ permutes the complex structures J and K, while preserving I, $Sp(1)_-$ acts trivially and $U(1)$ hyperKähler.
- The $U(1)$ action admits a hyperKähler moment map $\mu = i \cdot \mu_I + j \cdot \mu_{\mathbb{C}} : M \to \mathrm{Im}(\mathbb{H})$.
- A connection a on L^2 lifts together with the Levi-Cevita connection a_{LC} on $P_{U(2)}$ to a connection A on Q.
- Spinors are sections $u \in \Gamma(X, W^+) = Map(Q, M)^G$
- The **generalized Seiberg-Witten equations:**
$$\not{D}^A(u) = 0 \ , \quad F_a^+ = \mu \circ u$$
Here $\not{D}^A$ is a **non-linear** Dirac operator.
- Assuming L admits a holomorphic structure, the gen. SW-Equations can be simplified:
$$F_a^{2,0} = \mu_{\mathbb{C}} \circ u = 0$$
and thus the connection a determines a holomorphic structure on Q, and u is a holomorphic map w.r.t. this structure.



- The image of u is fully contained in one of the spheres in the niveau set $\mu_{\mathbb{C}} = 0$, the particular sphere is determined by L.
- By modding out the action of the (complexified) gauge group, one gets rid of the $(F_a^+)^{1,1} = \dots$ equation.
- The moduli space is the set of all meromorphic sections of the line bundle $L \otimes K_X^{-k}$ with empty indeterminancy set, where k depends on the sphere.

Gibbons-Hawking spaces

Theorem 1. *Let M be a simply-connected, complete four dimensional hyperKähler manifold. If there exists a hyperKähler S^1 action on M admitting a hyperKähler moment map, then M is obtained by the Gibbons-Hawking Ansatz [3].*

We are interested in Multi-Taub-NUT spaces, which are obtained by taking an S^1 principal bundle over $\mathbb{R}^3 \backslash \{p_1, \dots p_n\}$, which locally around each point p_k is the extended Hopf bundle $R^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$. They admit a hyperKähler metric which can be extended "over the singularities", i.e. by adding n single points one obtains a complete hyperKähler manifold.
By requiring the points $p_1, \dots, p_n$ to lie on a straight line, one obtains a permuting S^1 action rotating the complex structures J and K, while fixing I. A hyperKähler S^1 action is given by the S^1-principle bundle action.

Further Results

For transversality reasons, one pertubs the second equation $F_a^+ = \mu \circ u + \phi$ for $\phi \in \mathcal{H}_{\overline{\partial}}^{2,0}(X)$.
For a generic pertubation ϕ, the equations are transversal and the moduli space consists of finetly many single points, more explicitly:

Theorem 2. *If L admits a holomorphic structure, the moduli space of the pertubed gen. SW equations with pertubation ϕ and target space the Multi-Taub-NUT space with n spheres consists of holomorphic sections of L dividing ϕ^n, up to scalars.*

Open Problems/Future work

- Can one explicitly describe the moduli space if L does not admit a holomorphic structure?
- One needs to show that we indeed obtain invariants of complex structures admitting Kähler metrics (one would like to take a path of complex structures and than do a Cobordism argument, but what if L stops being holomorphic for some of these?).
- Generalize to symplectic manifolds and almost complex structures.

References

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