Z-critical connections and Bridgeland stability

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D-branes and mirror symmetry

Open strings in type II string theory must end on special substructures of spacetime, *D-branes*.

- In type IIA string theory, D-branes are Lagrangians inside (X, ω) a Calabi–Yau 6-fold, equipped with a flat unitary connection on a line bundle, called **A-branes**.
- In type IIB string theory, D-branes are complex submanifolds of (X, ω) equipped with a holomorphic vector bundle (or sheaf, or complex of sheaves), called **B-branes**.
- **Mirror symmetry** predicts that type IIB string theory on X is equivalent to type IIA string theory on $\hat{X}$, the mirror Calabi–Yau.

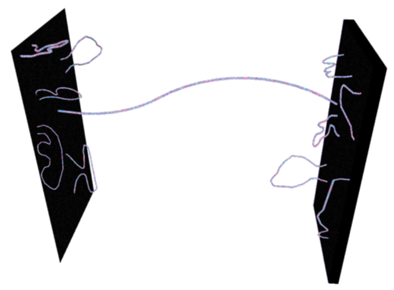

Figure: Opens strings between D-branes

- Physical D-branes must satisfy a supersymmetry condition, called BPS D-branes. In type IIA string theory BPS A-branes are **special Lagrangian** submanifolds. In type IIB string theory they are **stable vector bundles** (or sheaves, or complexes).
- A "Hitchin–Kobayashi-type correspondence" should exist which relates special Lagrangians to a stability condition for Lagrangian submanifolds (Thomas–Yau conjecture) and relating stable sheaves to some kind of special connections on vector bundles (or sheaves, or complexes).
- In the **large volume limit** where (X, ω) is replaced by $(X, k\omega)$ and $k \to \infty$, one expects this to converge to quantum field theory, where vector bundles admit Yang–Mills connections.

Bridgeland stability and asymptotic Z-stability

- Bridgeland stability is an algebro-geometric notion of stability on complexes of *coherent sheaves*. Defined by Bridgeland as a formalisation of Π-stability, introduced by Douglas in 2000 to characterise BPS B-branes.
- We consider the simplified notion of **asymptotic Z-stability**, where $Z : K(X) \to \mathbb{C}$ is a **central charge**, a homomorphism from the ring of vector bundles (or sheaves, or complexes) on X to $\mathbb{C}$. A central charge defines a Z-**slope** or **phase** by $\varphi(E) = \arg Z(E)$.

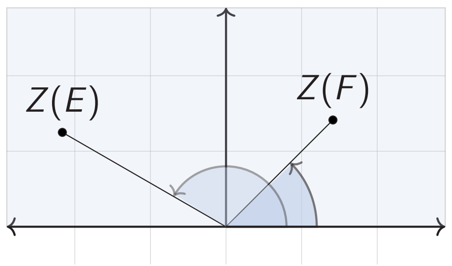

Figure: $\varphi(F) < \varphi(E)$

- We consider *polynomial central charges* of the form

$$Z_k(E) = \int_X \sum_{d=0}^{n} \rho_d k^d [\omega]^d . \mathrm{Ch}(E).U$$

for a class $U = 1 + N$ with $N \in H^{>0}(X, \mathbb{R})$. This has slope function φ_k.
- A holomorphic vector bundle $E \to (X, \omega)$ is **asymptotically Z-stable** if $\varphi_k(F) < \varphi_k(E)$ for all proper non-zero coherent subsheaves $F \subset E$, for all $k \gg 0$.
- In the large volume limit as $k \to \infty$, stability of Z_k approaches *slope stability*:

$$\frac{\deg F}{\mathrm{rk}\, F} < \frac{\deg E}{\mathrm{rk}\, E}.$$

There are implications: slope stability $\implies$ asymptotic Z-stability $\implies$ slope semistability.

Z-critical connections

Let $E \to (X, \omega)$ be a holomorphic vector bundle on a compact Kähler manifold. Each choice of Hermitian metric h on E induces a Chern connection $A(h)$, which gives Chern–Weil representatives $\tilde{\mathrm{Ch}}_i(h)$ of the Chern characters of E as powers of the curvature $F(h)$. Given a central charge Z_k and a representative differential form $\tilde{U}$ for U, define an $\mathrm{End}(E)$-valued (n, n)-form by

$$\tilde{Z}_k(h) = \left[\sum_{d=0}^{n} \rho_d k^d \omega^d \wedge \tilde{\mathrm{Ch}}(h) \wedge \tilde{U} \right]^{(n.n)}.$$

Say A is a Z_k-**critical connection** if

$$\mathrm{Im}(e^{-i\varphi(E)} \tilde{Z}_k(h)) = 0. \qquad (\dagger)$$

- This equation is fully non-linear and in general is not elliptic, but is elliptic for all $k \gg 0$.
- When E is a line bundle, this says that the central charge form $\tilde{Z}_k(A)$ has **constant phase**. In higher rank the Im operator takes the *skew-Hermitian* part of the endomorphism.
- In the large volume limit as $k \to \infty$, The Z_k-critical equation approaches the **Hermitian Yang–Mills equation**.

On the space of all integrable unitary connections $\mathcal{A}(h)$ on E, there exists a Kähler form

$$\Omega_A(a, b) = \int_X \mathrm{tr}(a \wedge b \wedge \mathrm{Re}(e^{-i\varphi(E)} \tilde{Z}'_k(A)))$$

where Z'_k is the formal derivative with respect to the curvature F_A. On a locus $\mathcal{A}(h)^{sub}$ of *subsolutions* the Z_k-critical equation is elliptic, Ω is non-degenerate, and the Z_k-equation is the moment map equation for the action of the gauge group on $\mathcal{A}(h)$ with respect to Ω.

Correspondence in the large volume limit

Theorem: *(Dervan–M.–Sektnan) A sufficiently smooth holomorphic vector bundle $E \to (X, \omega)$ admits a Z_k-critical Chern connection for all $k \gg 0$ if and only if it is asymptotically Z_k-stable.*

- In the limit as $k \to \infty$ there is a correspondence between slope polystable vector bundles and Hermitian Yang–Mills connections. By simultaneously perturbing from $k = \infty$ to $k \gg 0$ and from the polystable graded object $\mathrm{Gr}(E)$ of E to E, one can match the error terms to get a Z_k-critical connection precisely when E is asymptotically Z_k-stable.
- Sufficient smoothness means that the associated graded object is a locally-free sheaf, i.e. a holomorphic vector bundle $\mathrm{Gr}(E)$ smoothly isomorphic to E.

Outlook

- Can the notion of Z-critical connections be extended to complexes of vector bundles, or away from the large volume limit? Does there existence a correspondence to Bridgeland stable objects?
- Can one use the Kähler geometry of $\mathcal{A}(h)$ to define Bridgeland stability conditions on $\mathcal{D}^b \operatorname{Coh} X$? This is the "categorical Kähler geometry" program of Haiden–Katzarkov–Kontsevich–Pandit.