Moduli Spaces of Euclidean Monopoles

Jaime Mendizabal
University College London
London School of Geometry and Number Theory

24th June 2021

Monopoles

- G: simply connected, compact, semisimple, real Lie group
  - $\mathfrak{g}$: its Lie algebra
- P: principal G-bundle over $\mathbb{R}^3$
- A: principal connection on P
- Φ: section of the adjoint bundle $\mathrm{ad}(P)$ (*Higgs field*)

The pair (A, Φ) is a *(G-)monopole* if it is a solution to the Bogomolny equations

$$\star F_A - \mathrm{d}_A\Phi = 0,$$

and it has finite energy

$$\|F_A\|_{L^2} = \|\mathrm{d}_A\Phi\|_{L^2} < \infty.$$

Configuration Space

The space $\mathscr{C}$ of pairs (A, Φ) is called the *configuration space*.

It is an infinite-dimensional affine space.

Gauge Transformations

The group

$$\mathscr{G} = \Gamma(\mathrm{Ad}(P))$$

of gauge transformations of the bundle P acts on pairs $(A, \Phi) \in \mathscr{C}$.

Gauge transformations preserve the Bogomolny equations and the energy functional.

Mass and Charge

We expect a certain asymptotic behaviour,

$$\Phi = \mu - \frac{1}{2r}\kappa + \cdots,$$

$$\star F_A = \mathrm{d}_A\Phi = \frac{1}{2r^2}\kappa \mathrm{d}r + \cdots,$$

for elements $\mu, \kappa \in \mathfrak{g}$. These are the *mass* and *charge*. We must have $[\mu, \kappa] = 0$.

The charge κ determines the behaviour of the connection A on very large spheres around the origin. In particular, we must have the integrality condition $\exp(2\pi\kappa) = 1_G$.

Degree / Magnetic Charges

The Higgs field Φ must take values in the adjoint orbit of μ near infinity, so we have a map

$$\Phi_\infty \colon S^2_\infty \to O_\mu.$$

The homotopy class of this map is the *degree* of the monopole. It determines the *magnetic charges*.

The group $\pi_2(O_\mu)$ is always of the form $\mathbb{Z}^s$, so this can be expressed as a finite amount of integers.

Example

If $G = \mathrm{SU}(n)$ and we associate a complex vector bundle $E = P \times_G \mathbb{C}^n$, then the mass determines the eigenvalues of Φ at infinity and the magnetic charges determine the degree of the corresponding eigenbundles.

Holomorphic Charges

For the same magnetic charges, the specific κ further determines the behaviour of the connection A.

This is expressed through the *holomorphic charges*. There is a finite amount of choices for these given the magnetic charges.

Example

In the previous example, the holomorphic charges determine the holomorphic type of the eigenbundles.

For example, a degree 2 bundle could be $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or $\mathcal{O}(0) \oplus \mathcal{O}(2)$ (when restricted to a large sphere).

Deformations

Given the discrete nature of the degree, a continuous defomation can't change it.

However, it is possible for a continuous deformation to change the holomorphic charges (cf. jumping lines).

Nonetheless, these continuous deformations can't have finite L^2-length, since they change the Higgs field up to order r^{-1}.

The Moduli Space

We want to construct the moduli space of monopoles modulo gauge transformations.

We can't continuously deform between different degrees, so we can try to build the moduli space $\mathcal{M}_{\mu,m}$ for a fixed mass μ and magnetic charges m.

We can also try to impose a metric on the moduli space given by the L^2 norms of deformations. In this case, we have to consider the moduli space $\mathcal{M}_{\mu,\kappa}$, for a fixed mass μ and charge κ.

The Framed Moduli Space

Instead of the usual moduli space, we can consider *framed monopoles*, essentially monopoles together with a specific asymptotic isomorphism to the model for a given mass μ and charge κ.

The resulting moduli space $\widetilde{\mathcal{M}}_{\mu,\kappa}$ is a fibration over the usual moduli space $\mathcal{M}_{\mu,\kappa}$. The fibres correspond to the group of automorphisms of the asymptotic model.

A Hyper-Kähler Structure

The configuration space $\mathscr{C}$ can be given a quaternionic structure. If $(a, \varphi) \in \Omega^1(\mathrm{ad}(P)) \oplus \Omega^0(\mathrm{ad}(P))$, consider

$$(a, \varphi) \longleftrightarrow \varphi + a_1 i + a_2 j + a_3 k \qquad (\text{where } a = a_1 \mathrm{d}x_1 + a_2 \mathrm{d}x_2 + a_3 \mathrm{d}x_3).$$

Take $\mathscr{C}_{\mu,\kappa}$ to be the configuration space of pairs asymptotical to this model, and $\mathscr{G}_{\mu,\kappa}$ to be the group of gauge transformations asymptotic to the identity (in suitable ways).

Then, $\mathscr{C}_{\mu,\kappa}$ has a hyper-Kähler structure using the L^2 norm, and $\mathscr{G}_{\mu,\kappa}$ acts in a hyper-Kähler manner. The corresponding moment map gives precisely the Bogomolny equations. Hence, formally, we can write $\widetilde{\mathcal{M}}_{\mu,\kappa} = \mathscr{C}_{\mu,\kappa} /\!\!/\!\!/ \mathscr{G}_{\mu,\kappa}$.

Separating Monopoles

Some infinite paths along the moduli space correspond to monopoles of a certain charge "separating" into several monopoles of lower charges.

For $G = \mathrm{SU}(2)$, one can compactify the moduli spaces of a given charge into a manifold with corners using moduli spaces of lower charge.

A Stratified Space

The space $\widetilde{\mathcal{M}}_{\mu,m}$ is stratified, with strata corresponding to the different possible holomorphic charges.

The L^2 metric will provide a metric on each stratum, and the paths between strata must have infinite length.

Further Work

- Proving that $\widetilde{\mathcal{M}}_{\mu,\kappa}$ is always a smooth hyper-Kähler manifold for all μ and κ
  - Compute the index of the corresponding linearised operator using micro-local analysis
  - Apply the implicit function theorem and obtain a slice
- Studying the completeness of $\widetilde{\mathcal{M}}_{\mu,\kappa}$
  - Apply a version of Uhlenbeck compactness to finite-length paths
- Studying the relationships for different μ and κ
  - Separating monopoles
  - Stratified moduli spaces
  - Varying μ

Selected References

Atiyah, M. F. and Hitchin, N. J. *The Geometry and Dynamics of Magnetic Monopoles.* M.B. Porter Lectures. Princeton University Press, 1988.

Fritzsch, K., Kottke, C. and Singer, M. A. *Monopoles and the Sen Conjecture: Part I.* 2018. URL: https://arxiv.org/abs/1811.00601.

Jaffe, A. M. and Taubes, C. H. *Vortices and Monopoles. Structure of Static Gauge Theories.* Progress in Physics 2. Birkhäuser, 1980.

Jarvis, S. 'Euclidean Monopoles and Rational Maps'. In: *Proceedings of the London Mathematical Society* 77.1 (1998), pp. 170–192.

Kottke, C. 'A Callias-Type Index Theorem with Degenerate Potentials'. In: *Communications in Partial Differential Equations* 40.2 (2015), pp. 219–264.

Murray, M. K. 'Stratifying Monopoles and Rational Maps'. In: *Communications in Mathematical Physics* 125.4 (1989), pp. 661–674.

Murray, M. K. and Singer, M. A. 'A Note on Monopole Moduli Spaces'. In: *Journal of Mathematical Physics* 44.8 (2003), pp. 3517–3531.

Uhlenbeck, K. K. 'Connections with L^p Bounds on Curvature'. In: *Communications in Mathematical Physics* 83.1 (1982), pp. 31–42.