Constructing BPS Monopoles with Spherical Symmetry

Christopher Lang

University of Waterloo, Department of Pure Mathematics

Joint work with Benoit Charbonneau, Anuk Dayaprema, Ákos Nagy & Haoyang Yu

Email: cjlang@uwaterloo.ca

Setup

The **Bogomolny equation** $F_\nabla = \star D_\nabla \Phi$ relates

- the connection ∇, with curvature F_∇, of a vector bundle E over $\mathbb{R}^3$;
- an endomorphism Φ of E called the Higgs field

A **BPS monopole** is a solution to the Bogomolny equation with finite energy. This equation is difficult to solve, so efforts to produce explicit BPS monopoles have focused on symmetric monopoles [4, 2].

BPS monopoles are equivalent to certain solutions of Nahm's equations. Introduced by Nahm in [3], Nahm's equations are a system of ordinary differential equations of three $\mathfrak{su}(n)$-valued functions on an interval of $\mathbb{R}$. Such a triple $\mathcal{T} = (T_1, T_2, T_3)$ satisfy **Nahm's equations** if

$$\dot{T}_1 = [T_2, T_3]; \quad \dot{T}_2 = [T_3, T_1]; \quad \dot{T}_3 = [T_1, T_2]. \tag{1}$$

Though much simpler than the Bogomolny equation, Nahm's equations are still difficult to solve; however, we can simplify them further if we search for symmetric solutions.

The main result of our paper is a structure theorem for generating spherically symmetric solutions to Nahm's equations [1]. We use this theorem to produce Ansätze for families of spherically symmetric solutions to Nahm's equations. We use these Ansätze to produce two new explicit BPS monopoles with spherical symmetry.

Symmetric Solutions to Nahm's Equations

In order to understand symmetric BPS monopoles, we must first study symmetric solutions to Nahm's equations. We define the following commuting actions.

Definition 1

Let $\mathcal{T} = (T_1, T_2, T_3)$. For $A \in \mathrm{SO}(3)$ and $U \in \mathrm{SU}(n)$ define

$$^{A}T_i := \sum_{j=1}^{3} A_{ij} T_j; \tag{2}$$
$$_{U}T_i := U T_i U^{-1}. \tag{3}$$

These actions preserve Nahm's equations. The former corresponds to a pull-back on the monopole via the same rotation and the latter produces the same monopole, up to gauge.

Definition 2

Let $A \in \mathrm{SO}(3)$. If $\exists U_A \in \mathrm{SU}(n)$ such that $^{A}\mathcal{T} =_{U_A^{-1}} \mathcal{T}$, then $\mathcal{T}$ is A-equivariant.

We focus on the spherically symmetric case, where $\mathcal{T}$ is $\mathrm{SO}(3)$-equivariant. Note that we have not made any comment on the map $A \mapsto U_A$.

Theorem 1

If $\mathcal{T}$ is spherically symmetric and $U\colon \mathrm{SO}(3) \to \mathrm{SU}(n)$ is twice continuously differentiable at the identity, then $\exists Y_1, Y_2, Y_3 \in \mathfrak{su}(n)$ such that

$$[Y_i, T_j] = \sum_{k=1}^{3} \epsilon_{ijk} T_k, \tag{4}$$
$$[[Y_i, Y_j], T_l] = \sum_{k=1}^{3} \epsilon_{ijk} [Y_k, T_l]. \tag{5}$$

Conversely, if these equations are satisfied, then $\mathcal{T}$ is spherically symmetric.

We consider triples of Y_1, Y_2, Y_3 that satisfy the $\mathfrak{so}(3)$ commutator relations:

$$[Y_i, Y_j] = \sum_{k=1}^{3} \epsilon_{ijk} Y_k. \tag{6}$$

Consider the standard basis $\{X_1, X_2, X_3\}$ of $\mathfrak{so}(3)$. The linear map $\rho\colon \mathfrak{so}(3) \to \mathfrak{su}(n)$ that sends $X_i \mapsto Y_i$ gives a representation $(\mathbb{C}^n, \rho)$ of $\mathfrak{so}(3)$. Let (V_k, ρ_k) be the irreducible, k-dimensional complex representation of $\mathfrak{so}(3)$ and let

$$(\hat{V}, \hat{\rho}) := (\mathbb{C}^n, \rho) \otimes ((\mathbb{C}^n)^*, \rho^*) \otimes (V_3, \rho_3). \tag{7}$$

The values of $\mathcal{T}$ can be viewed as an element of $\hat{V}$ and the action of $\mathfrak{so}(3)$ on $\mathcal{T}$ is $\hat{\rho}(X)(\mathcal{T}) := {}^{X}\mathcal{T} + [\rho(X), \mathcal{T}]$. If $\mathcal{T}$ is spherically symmetric, then $\hat{\rho}(X)(\mathcal{T}) = 0$, for all $X \in \mathfrak{so}(3)$. That is, $\mathcal{T}$ lies in the trivial component of $(\hat{V}, \hat{\rho})$. Thus, $\mathcal{T}$ is spherically symmetric exactly if $\mathcal{T}$ takes values in the direct sum of the one-dimensional irreducible components of $(\hat{V}, \hat{\rho})$.

Theorem 2: Structure Theorem

Consider $(\mathbb{C}^n, \rho)$ whose decomposition into irreducible summands is

$$(\mathbb{C}^n, \rho) \simeq \bigoplus_{a=1}^{k} (V_{n_a}, \rho_{n_a}). \tag{8}$$

Let $Y_{i,a} := \rho_{n_a}(X_i)$. For t_0 in the domain of $\mathcal{T}$ and $i \in \{1, 2, 3\}$, $T_i(t_0)$ can be written in block form according to the above decomposition:

$$T_i(t_0) = \begin{bmatrix} (T_i)_{11} & \cdots & (T_i)_{1k} \\ \vdots & \ddots & \vdots \\ (T_i)_{k1} & \cdots & (T_i)_{kk} \end{bmatrix}, \tag{9}$$

with $(T_i)_{ab} = -(T_i)_{ba}^* \in V_{n_a} \otimes V_{n_b}^*$. Then, up to a ρ-invariant gauge,

1. there exists $c_a \in \mathbb{R}$ such that $(T_i)_{aa} = c_a Y_{i,a}$;
2. if $a \neq b$ and $|n_a - n_b| \neq 2$, then $(T_i)_{ab} = 0$;
3. if $n_a = n_b + 2$, then $\exists c_{ab} \in \mathbb{C}$ such that $(T_i)_{ab} = c_{ab} B_i^{n_a}$.

Spherically Symmetric BPS Monopoles

We will use this theorem to construct spherically symmetric solutions to Nahm's equations. Let $n \in \mathbb{N}$ and suppose that $(\mathbb{C}^{2n+2}, \rho) \simeq (V_{n+2}, \rho_{n+2}) \oplus (V_n, \rho_n)$. Then there exists real, analytic functions f_0, f_1, g such that

$$T_i = \begin{bmatrix} f_1 Y_i^{n+2} & g B_i^n \\ -g (B_i^n)^* & f_0 Y_i^n \end{bmatrix}. \tag{10}$$

Nahm's equations tell us that

$$\dot{f}_0 = f_0^2 - \frac{2(n+2)}{n(n+1)} g^2; \tag{11}$$
$$\dot{f}_1 = f_1^2 + \frac{2}{n+1} g^2; \tag{12}$$
$$\dot{g} = \left(\frac{n+3}{2} f_1 - \frac{n-1}{2} f_0 \right) g. \tag{13}$$

Solving these equations and using Nahm transform, we have computed two new spherically symmetric BPS monopoles in the cases $n = 1$ and $n = 2$ above.

In the case $n = 1$, we get a $\mathrm{SU}(4)$ monopole with asymptotic Φ given by

$$\Phi(r) = i \cdot \mathrm{diag}(1, 1, -1, -1) - \frac{i}{2r} \cdot \mathrm{diag}(2, 2, -2, -2) + O(r^{-2}). \tag{14}$$

In the case $n = 2$, we get a $\mathrm{SU}(5)$ monopole with asymptotic Φ given by

$$\Phi(r) = \frac{i}{5} \cdot \mathrm{diag}(4, 4, 4, -6, -6) - \frac{i}{2r} \cdot \mathrm{diag}(2, 2, 2, -3, -3) + O(r^{-2}). \tag{15}$$

Note that these are neither maximal nor minimal symmetry breaking. Figure 1 shows the pointwise norm squared of Φ and the energy density of the monopole.

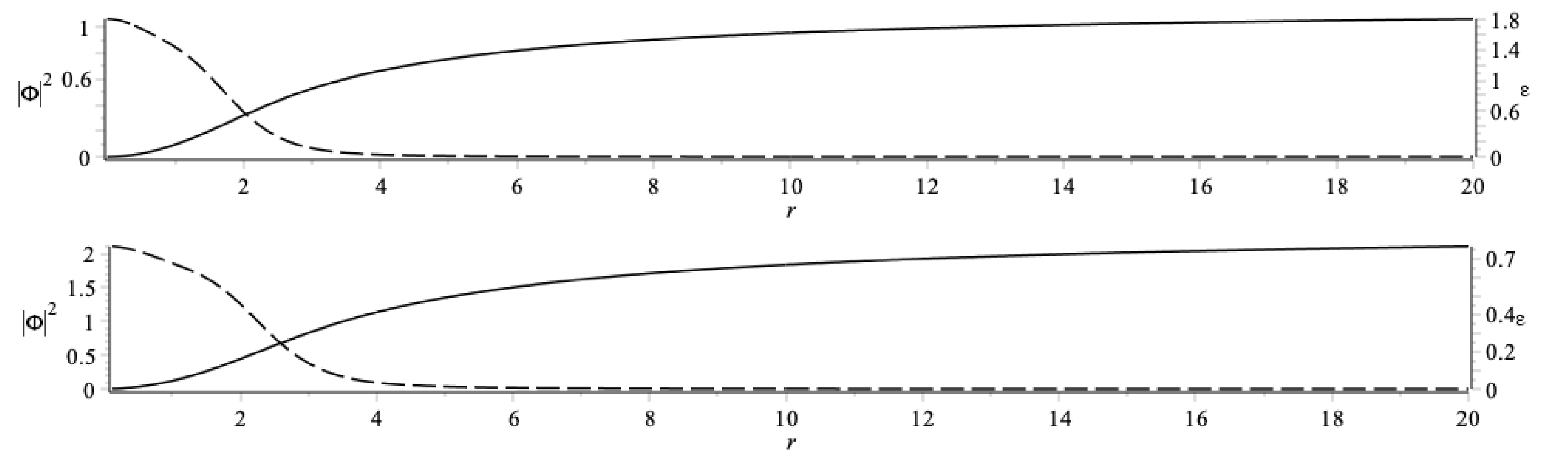

Figure 1: Norm squared of the Higgs field (solid) and energy density (dashed) of the spherically symmetric monopole in the cases $n = 1, 2$ (top and bottom, respectively).

References

[1] B. Charbonneau, A. Dayaprema, C. Lang, Á. Nagy, and H. Yu. Construction of Nahm data and BPS monopoles with continuous symmetries, 2021.

[2] G.'t Hooft. Magnetic monopoles in unified gauge theories. *Nuclear Physics B*, 79(2):276–284, 1974.

[3] W Nahm. The construction of all self-dual multimonopoles by the ADHM method. 1982.

[4] M. Prasad and C. Sommerfield. Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon. *Phys. Rev. Lett.*, 35:760–762, Sep 1975.

Acknowledgements

A portion of the paper was done while I received NSERC USRA support and the rest of the paper and this poster were done with support from NSERC CGS-D.