The construction of complete, (non)-compact hyperkähler manifolds

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Gibbons-Hawking Ansatz

The Gibbons-Hawking ansatz is an explicit construction of an 4-dimensional hyperkähler metric on a circle bundle. This construction goes as follows: Let U be an open subset of $\mathbf{R}^3$ and let P be a circle bundle over U. Equip P with a connection θ and a strictly positive function h on U such that $*\mathrm{d}h = \mathrm{d}\theta$. Now consider the following metric:

$$g^{gh} = h\pi^* g_{\mathbf{R}^3} + h^{-1}\theta^2$$

Because $h > 0$, g^{gh} is non-degenerate bilinear form and hence g^{gh} is a Riemannian metric. Next we define almost complex structures I_1, I_2, I_3 by the following relations:

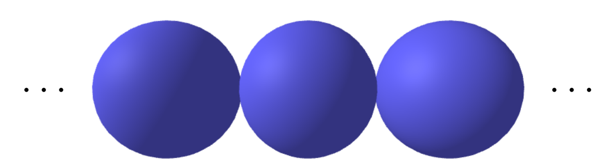
$$\theta \circ I_i = h\mathrm{d}x_i \qquad \mathrm{d}x_i \circ I_i = -h^{-1}\theta \qquad \mathrm{d}x_j \circ I_i = -\mathrm{d}x_k \qquad \mathrm{d}x_k \circ I_i = \mathrm{d}x_j.$$

One can check that $I_i^2 = -1$ and $I_i \circ I_j = I_k$ for all $\epsilon^{ijk} = -1$. This concludes that I_i defines an almost complex structure on P which satisfy the quaternion relations. To show that g^{gh} is Hyper-Kähler it is sufficient to show that the corresponding Kähler forms ω_i are closed . Explicitly ω_i are given by $\omega_i(\cdot,\cdot) = \mathrm{d}x_i \wedge \theta + h\mathrm{d}x_j \wedge \mathrm{d}x_k$. and by the relation $*\mathrm{d}h = \mathrm{d}\theta$ they are closed.

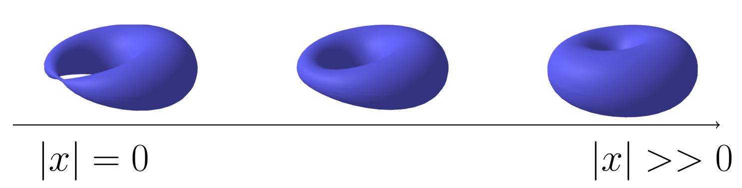
Examples

Using the Gibbons-Hawking ansatz one can describe some famous Hyper-Kähler metrics. In this section we list some of them and their properties

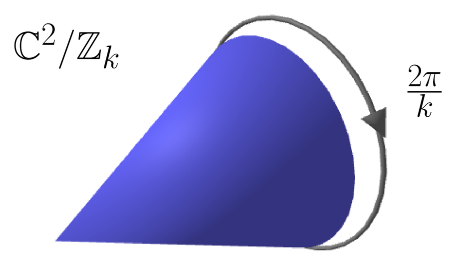
- $h = \mathrm{constant}$
  The first example one can construct is the **flat metric on** $\mathrm{R}^3 \times S^1$. The constant determines the size of the circle radius.
- $h = \frac{1}{r}$
  For this harmonic function, the metric gives the **Eguchi–Hanson space**. This is a non-compact, asymptotically locally Euclidean (ALE) metric on the cotangent bundle of the 2-sphere T^*S^2. Although h is ill-defined at $x = 0$, the metric around the origin approximates the flat metric on $\mathbb{C}^2$ and hence can be extended to a complete metric.
- $h = \frac{1}{r} + \mathrm{const}$
  This metric is called the **Taub-NUT** metric. Even more, for any constant $C \geq 0$ and any set of distinct points $\{p_i\}$ the harmonic function $h(x) = C + \sum_{i=1}^{n} \frac{1}{2|x-p_i|}$ generates an hyperkähler manifold which are called **(Multi)-Taub-NUT**. These manifolds retracts to a wedge sum of $n-1$ spheres and are complete.



- $h = \mathrm{const} + \sum_{k\in\mathbb{Z}} \frac{1}{|x+k|}$
  We can also apply the Gibbons-Hawking ansatz on quotients of $\mathbb{R}^3$. This will give metrics like the **Ooguri-Vafa metric**. This metric has an torus-fibration, with at the origin a singular fiber which is a pinched torus.



- $h = \frac{k}{r} + \mathrm{const}$ with $k \in \mathbb{Z}$ and $|k| > 1$. (non-example)
  The constant in the numerator determines the Chern class of the S^1-bundle. If this constant is larger than one metric cannot be extended to the origin and we will get an **orbifold singularity**. Locally the space looks like $\mathbb{C}^2/\mathbb{Z}_k$.



- $h \simeq 1 - \frac{2}{r}$
  Related to the Gibbons-Hawking Ansatz is the **Atiyah-Hitchin manifold**. It is the moduli space of centred charge 2 monopoles on $\mathbf{R}^3$. It's metric cannot be given by the Gibbons-Hawking Ansatz, but the induced metric on the double cover is asymptotically equal to g^{gh} when $h(x) = C + \frac{-4}{2|x|}$. The Atiyah-Hitchin manifold is diffeomorphic to the complement of $\mathrm{R}P^2$ in S^4 and it retracts to $\mathbf{R}P^2$.

New construction

We consider a quotient of $\mathbf{R}^3$ by a lattice. That is, we pick $B \in \{\mathbf{R}^3, \mathbf{R}^2 \times S^1, \mathbf{R} \times \mathbb{T}^2, \mathbb{T}^3\}$ and we endow B with a flat metric. On B there is the antipodal map $\tau\colon B \to B$ and let $\{q_1, \ldots, q_m\}$ be the set of fixed points. Next pick $2n$ distinct points $p_1, \ldots, p_{2n}$ in B that are not fixed by the $\mathbb{Z}_2$ action. Also assume that the set $\{p_i\}$ is invariant under τ. That is, for each p_i there is a p_j such that $\tau(p_i) = p_j$. It turns out we need to impose some restrictions on the number of poles. We wrote them down in the following table:

B	$\mathbf{R}^3$	$\mathbf{R}^2 \times S^1$	$\mathbf{R} \times \mathbb{T}^2$	$\mathbb{T}^3$
Restriction on n	*none*	$n \leq 4$	$n \leq 8$	$n = 16.$

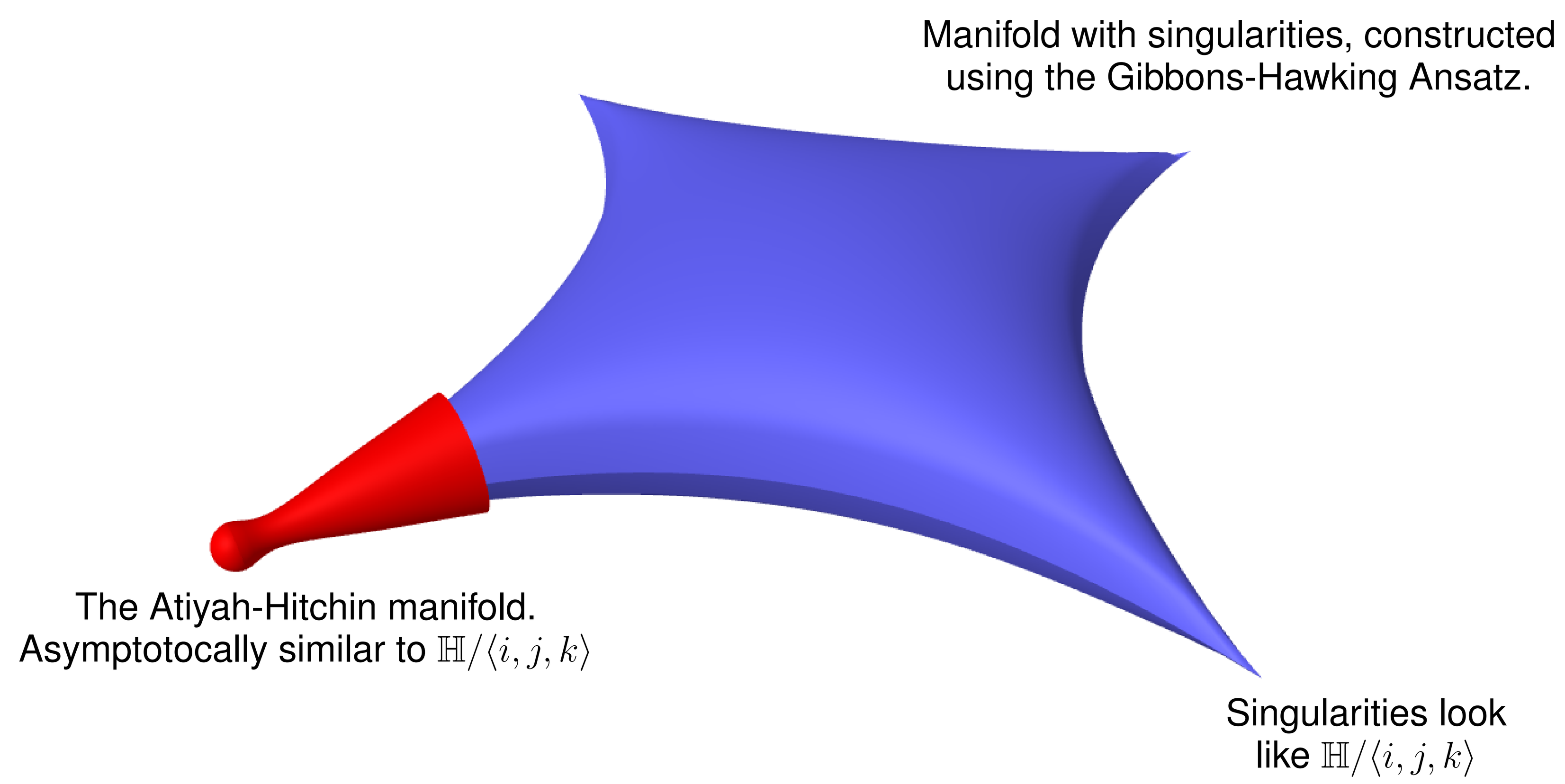
Now let $\delta > 0$ such that the set of balls $\{B_\delta(p_i)\} \cup \{B_\delta(q_i)\}$ are pairwise disjoint. Denote B' for the complement of the union of these balls in B. On B' we construct a smooth function h with the following properties:

1. $\Delta h = 0$,
2. $h(x) \simeq \frac{1}{2|x-p_i|}$ around p_i,
3. $h(x) \simeq \frac{-4}{2|x-p_i|}$ around q_i,
4. $h(\tau(x)) = h(x)$ and
5. $h > 0$.

Using h, we find a circle bundle P over B with a connection θ such that $*\mathrm{d}h = \mathrm{d}\theta$. Next we lift the involution τ to $\tilde{\tau}$ on P such that

$$\tilde{\tau}(p \cdot e^{it}) = \tilde{\tau}(p) \cdot e^{-it}$$

The $\mathbb{Z}_2$-action of $\tilde{\tau}$ is free and $P/\mathbb{Z}_2$ is a smooth manifold. Using the asymptotic behaviour of h we can calculate the local topology around the poles in $P/\mathbb{Z}_2$. Around p_i this space is diffeomorphic to $\mathbf{C}^2 \backslash \{0\}$ and around q_i, $P/\mathbb{Z}_2$ is locally diffeomorphic to $(\mathbb{H} - \{0\})/\langle i, j, k\rangle$. This corresponds to the asymptotic geometry of a Taub-NUT resp. Atiyah-Hitchin manifold. Hence we construct a new manifold that is the union of $P/\mathbb{Z}_2$ with n copies of Taub-NUT and m copies of Atiyah-Hitchin where we identify the boundaries. This can be done in a smooth manner and hence we get a smooth manifold.



Using partition of unity we combine both metrics into a new approximate Hyper-Kähler metric. We can estimate the error of being Hyper-Kähler and so using the inverse function theorem we will perturb the metric such that it becomes Hyper-Kähler globally.