Associative submanifolds of the 7-sphere

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Outline

- **The G_2 geometry of the 7-sphere**
- **Simple examples and basic theory**
- **Rigidity results and group orbits**
- **Ruled associative submanifolds and Chen's equality**

G_2 and Spin(7)

G_2

- $\exists$ closed 3-form φ_0 on $\mathbb{R}^7$ such that

$$G_2 = \mathrm{Stab}(\varphi_0) \subseteq SO(7).$$

- $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O} \Rightarrow \varphi_0(x, y, z) = g_0(x \times y, z)$.
- On M^7, admissible 3-form $\varphi \leftrightarrow G_2$ structure.

Spin(7)

- $\exists$ closed self-dual 4-form Φ_0 on $\mathbb{R}^8$ such that

$$\mathrm{Spin}(7) = \mathrm{Stab}(\Phi_0) \subseteq SO(8).$$

- $\mathbb{R}^8 \cong \mathbb{O} \Rightarrow \Phi_0(x, y, z, w) = g_0(x \times y \times z, w)$.
- On M^8, admissible 4-form $\Phi \leftrightarrow$ Spin(7) structure.

The G_2 structure on $\mathcal{S}^7$

Consider $\mathbb{R}^8 \setminus \{0\} \cong \mathbb{R}^+ \times \mathcal{S}^7$.

- $\Phi_0 = *\Phi_0 \Rightarrow \Phi_0 = r^3 dr \wedge \varphi + r^4 *\varphi$.
- $d\Phi_0 = 0 \Rightarrow d\varphi = 4*\varphi$ and $d*\varphi = 0$.
- φ is a nearly parallel G_2 structure on $\mathcal{S}^7 \cong \mathrm{Spin}(7)/G_2$.
- $(\mathcal{S}^7, \varphi)$ is a nearly G_2 manifold.

M^7 has a nearly parallel G_2 structure $\Leftrightarrow$ the cone $\mathbb{R}^+ \times M^7$ has a torsion-free Spin(7) structure Φ , i.e. $\nabla\Phi = 0$.

Theorem

- $\nabla\varphi = 0 \Leftrightarrow d\varphi = d^*\varphi = 0 \Leftrightarrow \mathrm{Hol}(g_\varphi) \subseteq G_2$.
- $\nabla\Phi = 0 \Leftrightarrow d\Phi = 0 \Leftrightarrow \mathrm{Hol}(g_\Phi) \subseteq \mathrm{Spin}(7)$.

Submanifolds of $\mathcal{S}^7$

$(\mathcal{S}^7, \varphi) \rightsquigarrow$

- $\varphi|_U \leq \mathrm{vol}_U$ for all oriented tangent 3-planes U;
- $*\varphi|_V \leq \mathrm{vol}_V$ for all oriented tangent 4-planes V;
- $*\varphi|_V = \mathrm{vol}_V \Leftrightarrow \varphi|_V \equiv 0$.

Definition

- $A^3 \subseteq \mathcal{S}^7$ *is associative* $\Leftrightarrow \varphi|_A = \mathrm{vol}_A$.
- $C^4 \subseteq \mathcal{S}^7$ *is coassociative* $\Leftrightarrow *\varphi|_C = \mathrm{vol}_C \Leftrightarrow \varphi|_C \equiv 0$.

Proposition

There are no coassociative submanifolds of $\mathcal{S}^7$.

Proof: C coassociative $\Rightarrow \varphi|_C \equiv 0 \Rightarrow d\varphi|_C \equiv 0$. $d\varphi = 4*\varphi \Rightarrow$ $*\varphi|_C = \mathrm{vol}_C \equiv 0 \Rightarrow$ Contradiction.

Complex geometry

Identify $\mathbb{R}^8 \cong \mathbb{C}^4$.

- ω_0 Kähler form, Ω_0 holomorphic volume form $\Rightarrow$ $\Phi_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \operatorname{Re}\Omega_0$.
- $S^4 \subseteq \mathbb{C}^4$ complex surface $\Leftrightarrow \frac{1}{2}\omega_0 \wedge \omega_0|_S = \mathrm{vol}_S$ and $\Omega_0|_S = 0$.
- $L^4 \subseteq \mathbb{C}^4$ special Lagrangian $\Leftrightarrow \omega_0|_L = 0$ and $\operatorname{Re}\Omega_0|_L = \mathrm{vol}_L$.

$A \subseteq \mathcal{S}^7$ associative $\Leftrightarrow$ the cone $N = \mathbb{R}^+ \times A$ satisfies $\Phi_0|_N = \mathrm{vol}_N$.

Proposition

- $\Sigma^2 \subseteq \mathbb{CP}^3$ *holomorphic curve* $\Rightarrow$ *the Hopf lift of* Σ *to* $\mathcal{S}^7$ *is associative.*
- $A^3 \subseteq \mathcal{S}^7$ *minimal Legendrian* $\Rightarrow$ A *is associative.*

$\mathcal{S}^6$ geometry

$\mathcal{S}^6 \hookrightarrow \operatorname{Im}\mathbb{O} \rightsquigarrow$

- almost complex structure J given by $J_x u = x \times u$.
- almost symplectic form ω given by $\omega(u, v) = g(Ju, v)$.

Definition

- $\Sigma^2 \subseteq \mathcal{S}^6$ *is a pseudoholomorphic curve* $\Leftrightarrow \omega|_\Sigma = \mathrm{vol}_\Sigma$.
- $L^3 \subseteq \mathcal{S}^6$ *is Lagrangian* $\Leftrightarrow \omega|_L = 0$.

Identify $\mathbb{R}^8 \cong \mathbb{R} \oplus \operatorname{Im}\mathbb{O}$.

Proposition

- $\Sigma^2 \subseteq \mathcal{S}^6$ *pseudoholomorphic curve* $\Leftrightarrow$ $\{(\cos t, \sigma \sin t) \, : \, \sigma \in \Sigma, \; t \in (0, \pi)\} \subseteq \mathcal{S}^7$ *is associative.*
- $L^3 \subseteq \mathcal{S}^6$ *Lagrangian* $\Leftrightarrow \{0\} \times L \subseteq \mathcal{S}^7$ *is associative.*

Properties of associative 3-folds

Theorem (Harvey & Lawson 1982)

Associative submanifolds of $\mathcal{S}^7$ are minimal.

A associative $\Leftrightarrow T_xA \subseteq \mathbb{R}^7 \cong \operatorname{Im}\mathbb{O} \rightsquigarrow$ associative subalgebra of $\mathbb{O}$.

Theorem (Harvey & Lawson 1982)

Given $P^2 \subseteq \mathcal{S}^7$ real analytic there locally exists associative A containing P. Moreover, A is locally unique.

Associative 3-folds in $\mathcal{S}^7$ locally depend on 4 functions of 2 variables.

Constant curvature

Question (Chern 1971)

Does an isometric minimal immersion $\mathcal{S}^3(\kappa) \to \mathcal{S}^7$ have to be totally geodesic?

Theorem (L-)

Let $A(\kappa) \subseteq \mathcal{S}^7$ be associative with constant curvature κ. Then $\kappa = 1, \frac{1}{16}$ or 0 and, in each case, $A(\kappa)$ is unique up to rigid motion.

- $A(1)$: totally geodesic orbit of $\mathrm{SU}(2) \curvearrowright \mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathbb{R}^8$.
- $A(\frac{1}{16})$ (Ejiri 1981): Lagrangian orbit in $\mathcal{S}^6$ of $\mathrm{SO}(3) \curvearrowright \mathcal{H}_3(\mathbb{R}^3) \cong \mathbb{R}^7$.
- $A(0)$ (Harvey & Lawson 1982): minimal Legendrian orbit of $\mathrm{U}(1)^3 \curvearrowright \mathbb{C}^4 \cong \mathbb{R}^8$.

Group orbits

(Mashimo 1986) Lagrangian group orbits in $\mathcal{S}^6$.

(Marshall 1999) Minimal Legendrian group orbits in $\mathcal{S}^{2n-1}$.

Theorem (L-)

Let $\mathrm{G} \subseteq \mathrm{Spin}(7)$ *be a 3-dimensional Lie subgroup and let* $A \subseteq \mathcal{S}^7$ *be an associative* G*-orbit. Then either*

- $A \subseteq \mathcal{S}^6$ *is Lagrangian; or*
- $\mathrm{G} = \mathrm{U}(1)^3$ *and* $A = A(0) \cong T^3$*; or*
- $\mathrm{G} = \mathrm{SU}(2) \curvearrowright S^3\mathbb{C}^2 \cong \mathbb{R}^8$ *and either*
    - $A = A' \cong \mathrm{SU}(2)$ *or*
    - $A = A'' \cong \mathrm{SU}(2)/\mathbb{Z}_3$.

A': first known associative 3-fold not arising from other geometries.

Scalar and sectional curvature

Question (Chern 1970)

For a minimal submanifold A of a sphere, is the set of possible constant values of the scalar curvature S_A discrete?

Proposition (Li & Li 1992)

$A^3 \subseteq \mathcal{S}^7$ *associative* $\Rightarrow S_A$ *does not take values in* $[4, 6)$ *, i.e. if* $S_A \geq 4$*, A is totally geodesic.*

Proposition (Dillen et al 1987, Leung 1995)

Let $A \subseteq \mathcal{S}^7$ be associative and K_A be the sectional curvature of A.

- $\inf K_A > \frac{5}{12} \Rightarrow A$ *totally geodesic.*
- $A \subseteq \mathcal{S}^6$, $\inf K_A > \frac{1}{16} \Rightarrow A$ *totally geodesic.*

Ruled associative 3-folds and $\mathcal{C}$

Definition

$A^3 \subseteq \mathcal{S}^7$ is ruled if it is fibered by oriented geodesic circles.

- Hopf lifts of holomorphic curves in $\mathbb{CP}^3$ and products with pseudoholomorphic curves in $\mathcal{S}^6$ are ruled.
- The group orbits $A(0)$ and A' are not ruled, but A'' is ruled.

Ruled $A^3 \subseteq \mathcal{S}^7 \longleftrightarrow \Sigma^2 \subseteq \mathcal{C}^{12} = \{$oriented geodesic circles in $\mathcal{S}^7\}$.

$\mathcal{C} = \mathrm{Gr}_+(2, 8) \cong \mathrm{Spin}(7)/\,\mathrm{U}(3) \rightsquigarrow$

Spin(7)-invariant almost complex structure on $\mathcal{C}$.

Proposition (Fox 2008)

Ruled associative $A \subseteq \mathcal{S}^7 \longleftrightarrow$ pseudoholomorphic curve Σ in $\mathcal{C}$.

Pseudoholomorphic curves in $\mathcal{C}$

Given $\mathrm{U}(3) \subseteq \mathrm{Spin}(7)$ $\exists$ unique $\mathrm{SU}(4) \subseteq \mathrm{Spin}(7)$ containing $\mathrm{U}(3)$.

$\mathrm{Spin}(6) \cong \mathrm{SU}(4) \Rightarrow \mathcal{S}^6 \cong \mathrm{Spin}(7)/\,\mathrm{SU}(4) \rightsquigarrow \mathbb{CP}^3$ fibration $\mathcal{C} \xrightarrow{\pi} \mathcal{S}^6$.

Theorem (Salamon 1985)

Let $\Sigma \subseteq \mathcal{C}$ be a pseudoholomorphic curve.

- *$\pi(\Sigma)$ is a point $\Leftrightarrow$ $\Sigma \subseteq \mathbb{CP}^3$ is a holomorphic curve.*
- *$\pi(\Sigma)$ is not a point $\Leftrightarrow$ $\pi(\Sigma) \subseteq \mathcal{S}^6$ is a minimal surface.*

Theorem (Fox 2008)

Let $\iota : \Sigma^2 \to \mathcal{S}^6$ be a minimal immersion of a Riemann surface. There is a holomorphic $\mathbb{CP}^1$ subbundle $\mathcal{X}(\Sigma)$ of $\iota^*(\mathcal{C})$ such that

- *$\Gamma^2 \subseteq \mathcal{X}(\Sigma)$ defines a pseudoholomorphic lift of Σ to $\mathcal{C}$ $\Leftrightarrow$ Γ is a holomorphic curve.*

Chen's equality

Theorem (Chen 1993)

$A^3 \subseteq \mathcal{S}^7$ associative $\Rightarrow \delta_A := \frac{1}{2}S_A - \inf K_A \leq 2$.
Moreover, $\delta_A = 2$ (Chen's equality) $\Rightarrow$ A is ruled.

Let $\Sigma \subseteq \mathcal{C}$ be a pseudoholomorphic curve.

- $\mathbb{CP}^3$ fibration $\pi : \mathcal{C} \to \mathcal{S}^6 \rightsquigarrow$ splitting $T^{(1,0)}\mathcal{C} = \mathcal{H} \oplus \mathcal{V}$.
- There exist $\alpha^{\mathcal{H}}$ and $\alpha^{\mathcal{V}}$ triples of $(1,0)$-forms such that $\alpha^{\mathcal{H}}|_\Sigma = 0$ or $\alpha^{\mathcal{V}}|_\Sigma = 0 \Leftrightarrow \Sigma$ horizontal or vertical.
- Let $\beta = \alpha^{\mathcal{H}} \times \alpha^{\mathcal{V}}$ (i.e. $\beta_1 = \alpha_2^{\mathcal{H}} \circ \alpha_3^{\mathcal{V}} - \alpha_3^{\mathcal{H}} \circ \alpha_2^{\mathcal{V}}$ etc).

Definition

Pseudoholomorphic curve $\Sigma \subseteq \mathcal{C}$ is linear $\Leftrightarrow \beta|_\Sigma = 0$.

Chen's equality and minimal 2-spheres

Theorem (L-)

- *Associative 3-folds in $\mathcal{S}^7$ satisfying Chen's equality $\leftrightarrow$ linear pseudoholomorphic curves in $\mathcal{C}$.*
- *$\Sigma \subseteq \mathcal{C}$ linear $\Rightarrow \Gamma = \pi(\Sigma) \subseteq \mathcal{S}^6$ is an isotropic minimal surface , i.e. $\{h_\Gamma(v, v) : v \in T_x\Gamma, |v| = 1\}$ is a circle $\forall x$.*

(Calabi 1967) A minimal $\mathcal{S}^2$ in $\mathcal{S}^6$ is isotropic.

$\rightsquigarrow$ horizontal (hence linear) pseudoholomorphic curve in $\mathcal{C}$.

$\rightsquigarrow$ associative 3-fold in $\mathcal{S}^7$ satisfying Chen's equality.

Theorem (L-)

Non-totally geodesic minimal $\mathcal{S}^2 \subseteq \mathcal{S}^6 \rightsquigarrow$ 1-parameter family of isometric associative immersions in $\mathcal{S}^7$ satisfying Chen's equality.

Summary

- Many examples using submanifolds of $\mathbb{C}^4$ and $\mathcal{S}^6$.
- Constant curvature and homogeneous examples, including example not arising from known geometries.
- Ruled examples defined by minimal surfaces in $\mathcal{S}^6$ and holomorphic data.
- Classification of examples satisfying Chen's equality using linear pseudoholomorphic curves in $\mathcal{C}$.
- 1-parameter families of isometric associative immersions in $\mathcal{S}^7$ using minimal 2-spheres in $\mathcal{S}^6$.