Topology and Groups

Week 10, Thursday

1 Borel space construction

It is a theorem of Milnor that, given a group G, there exists a contractible CW complex EG admitting a properly discontinuous G-action.

1. What can we say about $BG := EG/G$?

2. Let X be a space with a G-action. The space $X_G := (X \times EG)/G$ is called the *Borel space* associated to the G-action. (Here the G-action on $X \times EG$ is $(x, e) \overset{g}{\mapsto} (gx, ge)$.) If X is contractible, what is the fundamental group of $(X \times EG)/G$?

3. The Borel space comes equipped with a map $F\colon X_G \to X/G$ (defined by $F([x, e]) = [x]$). Show that the preimage $F^{-1}(x)$ is a $K(G_x, 1)$-space, where G_x denotes the stabiliser of $x \in X$ under the G-action.

4. Now suppose that X is a tree (contractible 1-dimensional CW complex) and that X/G is a tree with two vertices connected by one edge. Suppose moreover that the action is *rigid*, in other words each $g \in G$ acts by cellular homeomorphisms (taking edges to edges) such that if g takes an edge E to itself then it fixes E pointwise). Let G_E be the stabiliser of an edge E and let G_x, G_y be the stabilisers of the endpoints of E. Show that
$$G \cong G_x \star_{G_E} G_y,$$
being careful to describe the homomorphisms $G_E \to G_x$ and $G_E \to G_y$ over which you are amalgamating. (Hint: This looks like Van Kampen's theorem. To which space are you supposed to apply Van Kampen's theorem to get this result? How should you decompose that space?)

5. The group $PSL(2, \mathbf{Z})$ acts by Möbius transformations with integer coefficients on the upper-half plane. Take the unit semicircle C and form the tree $X := \bigcup_{g \in PSL(2,\mathbf{Z})} gC$. Sketch X (use Sage if you like).

The quotient $X/PSL(2,\mathbf{Z})$ is a single edge (the circle segment between $e^{2\pi i/3}$ and i). What more would you need to do to show that $PSL(2,\mathbf{Z}) \cong (\mathbf{Z}/2) \star \mathbf{Z}/3$?

6. Show that the infinite-dimensional sphere S^∞ is contractible and hence give an example of a $K(\mathbf{Z}/2,1)$-space.