Topology and Groups

Week 1, Monday

1 How this module is going to work

I have divided the content from this module up into small, bitesize chunks. Each chunk has a webpage, containing:

- a video lecture (usually 15-20 minutes),
- associated lecture notes,
- one or two pre-class questions.

Each week we have a two-hour and a one-hour face-to-face session in class. These sessions will be primarily for group-work and class-discussions, in which we will:

- discuss the pre-class questions and consolidate our understanding of the videos/notes,
- work through more complicated and interesting examples and the details of some of the proofs omitted from videos,
- try to get an intuitive feel for topology (going beyond the formal definitions, working with pictures, etc),
- try to develop your abilities at writing and reading proofs (when is a proof rigorous? how should I structure my own proofs? can I convince others that my theorem is true?)

For each in-class session, you will need to watch some videos/read the associated notes and do the accompanying pre-class questions. The stuff you would ordinarily think of as "homework" we will do in class.

As term progresses, you will develop a portfolio of classwork, which I will take a look at periodically to give you formative feedback. There will also be two assessed projects (which will contribute 10% of the final grade) and, of course, an exam at the end of the year (90% of the final grade) which will examine everything we have seen in videos and in class.

2 Structure

To find out which videos you need to watch in preparation for each session, check the course webpage:

`http://www.homepages.ucl.ac.uk/~ucahjde/tg/html/index.html`

2.1 Index of videos

This page has an index of all videos, numbered like 1.01, 1.02, ..., 8.06. This is **not** the *pedagogical* order in which we will watch them for class: it is the best *logical* order for you to *revise from*.

For example, at some point I introduce CW complexes (4.01) then state Van Kampen's theorem (5.(01-03)) which allows you to compute the fundamental group of any CW complex. This quickly gives you powerful tools and allows you to do computations. Important but technical points about CW complexes (the homotopy extension property, 4.(02-03)) are delayed until after Van Kampen's theorem. When you come to revise the subject, however, you will probably be glad to clump all the material about CW complexes together in 4.(01-03).

2.2 Week-by-week plan

Therefore there is also a week-by-week plan. You will see something like (preparation for next lecture):

Week 1

- Thursday *(32.01)*
  - 1.02, 1.03 (Paths, loops, concatenation, fundamental group)

with links to the relevant videos (in this case 1.02 and 1.03). The *(32.01)* means that the total length of all these videos is 32 minutes and 1 second.

If you click on the link, you will get a page with an embedded video of the lecture. There are also notes below, to help you follow. These notes are annotated with timings that correspond to positions in the video.

Below the notes, there will be one or two pre-class questions (PCQs) which you should do before you come to class.

3 The fundamental theorem of algebra

Theorem 3.1. *A nonconstant complex polynomial has a complex root.*

Proof. Let $p\colon \mathbf{C} \to \mathbf{C}$, $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, be a complex polynomial. Assume that p has no complex root, in other words that there is no point $x \in \mathbf{C}$ for which $p(x) = 0$. We will show that $n = 0$, in other words that p has only a constant term.

Consider the circle of radius R in the complex plane. The points in this circle are precisely those of the form $Re^{i\theta}$. Let γ_R be the image of this circle under the map $p\colon \mathbf{C} \to \mathbf{C}$. We can think of γ_R as a loop in $\mathbf{C}$:

$$\gamma_R(\theta) = p(Re^{i\theta}).$$

Crucially, because $p(z) \neq 0$ for all $z \in \mathbf{C}$, the loop γ_R is a loop in $\mathbf{C} \setminus \{0\}$.

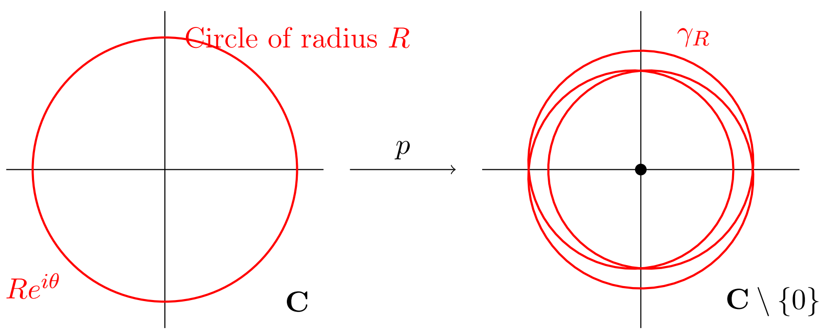


Figure 1. *In this figure, we see the circle of radius R in the domain of p and its image γ_R in the image of p ($\mathbf{C} \setminus \{0\}$). In this example, $p(z) = z^3 - z/2$ and $R = 2$ and we see that the winding number of γ_R is 3.*

When $R = 0$, $\gamma_0(\theta) = p(0)$ is independent of θ. In other words, γ_0 is the *constant loop* at the point $p(0) \in \mathbf{C} \setminus \{0\}$.

When R is very large, the term z^n dominates in p, so $\gamma_R(\theta) \approx \delta(\theta)$, where $\delta(\theta) = R^n e^{in\theta}$.

Claim: There is a *homotopy invariant* notion of *winding number around the origin* for paths in $\mathbf{C} \setminus \{0\}$ which gives zero for the constant loop and n for the loop $\delta(\theta) = R^n e^{in\theta}$.

Homotopy invariant means, roughly, invariant under continuous deformations; in our situation, that means that the winding number of γ_R around the origin should be independent of R. Since γ_0 has winding number zero and γ_R has winding number n for large R, this implies $n = 0$. $\square$

The rest of this module will be about defining this notion of winding number, the notion of homotopy and homotopy invariance, and generalising it to other spaces. In a more general setting, the spaces we're interested in (in this example $\mathbf{C} \setminus \{0\}$) will have an associated group (in this example the integers $\mathbf{Z}$) called the *fundamental group* and loops will have winding "numbers" which are elements of this group.

This will have many applications, including:

- the Brouwer fixed point theorem (any continuous map from the 2-dimensional disc to itself has a fixed point).
- the fact that a trefoil knot cannot be unknotted.
- the fact that the three Borromean rings cannot be unlinked from one another (despite the fact that they can be unlinked in pairs ignoring the third).

4 Discussion

1. Consider the rough sketch proof and highlight all the steps which seem to you not to be fully justified. Make a list of things that need to be done to make this into a rigorous proof. (Later in the module, we will revisit this proof and fill in all the gaps, hopefully to your satisfaction).

5 Classwork

In your learning groups, assuming at least one person per group has a phone or other computer, try the following programming exercise.

1. Using your favourite mathematical computing engine, e.g.

    - Sage `https://sagecell.sagemath.org`
    - WolframAlpha `http://m.wolframalpha.com`

    plot the loops $\gamma_R(t) = p(Re^{i2\pi t})$, $t \in [0,1]$, for $p(z) = z^3 - 10z + 5$ and $R = 0, 0.5, 1, 2, 3.2, 4$.

    - What are the winding numbers around the origin?
    - Why do they change as R varies? (Why does this not contradict the proof of the fundamental theorem of algebra that we just saw?)

    We will need volunteers to sketch their curves on the board.

(Hint: In Sage, if I wanted to plot the parametric curve $(x(t), y(t))$ in the plane, I would do something like the following:

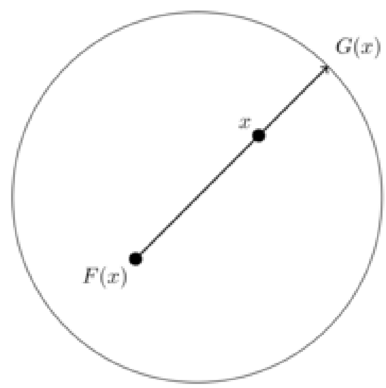
```
t = var('t')
parametric_plot((x(t),y(t)),(t,0,1))
```

where $x(t)$ and $y(t)$ are functions of t.)

6 Brouwer's fixed point theorem

Theorem 6.1. *Let D^2 be the closed unit disc in $\mathbf{R}^2$. Any continuous map $F\colon D^2 \to D^2$ has a fixed point, i.e. a point $x \in D^2$ such that $F(x) = x$.*

Proof. Assume, for a contradiction, that $F(x) \neq x$ for all $x \in D^2$. Then we can define a map $G\colon D^2 \to \partial D^2$ (where ∂D^2 denotes the boundary circle) as follows: consider the ray starting at $F(x)$, passing through x and let $G(x)$ to be the unique point of intersection of this ray with ∂D^2.



If there were a fixed point, we would not know which ray to draw (as $F(x) = x$), so this map is only well-defined because we have assumed there are no fixed points.

Now the map G gives us a contradiction as follows. We can see that $G(y) = y$ for all $y \in \partial D^2$. Consider the 1-parameter family of loops $\gamma_s(t) = se^{2\pi i t}$ contracting γ_1 (the boundary loop) to γ_0 (a point). Then $\delta_s(t) := G(\gamma_s(t))$ is a 1-parameter family of loops in ∂D^2 contracting $\delta_1 = \gamma_1$ to a point $G(0)$. You can't contract γ_1 whilst staying amongst continuous loops on the circle. $\square$

7 Discussion

1. Consider the rough sketch proof and highlight all the steps which seem to you not to be fully justified. Make a list of things that need to be done to make this into a rigorous proof. (Later in the module, we will revisit this proof and fill in all the gaps, hopefully to your satisfaction).

8 Questionnaire

1. What do you hope to learn from this module?

2. What do you hope to do with this new knowledge?

3. What do you expect the face-to-face sessions to do for you?

4. What do you expect the videos to do for you?

5. How many hours per week do you think it will take to learn all you need to know from this course? Include everything: lectures, homework, etc.