Lecture 14: Narasimhan-Seshadri theorem II

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Today we continue the proof of...

Theorem (Narasimhan-Seshadri, Donaldson)

An indecomposable Hermitian holomorphic vector bundle $\mathcal{E}$ on a Riemann surface (M, g) is stable if and only if there is a compatible unitary connection on $\mathcal{E}$ with constant central curvature

$$\star F_\nabla = -2\pi i \mu(\mathcal{E}).$$

...or equivalently...

Theorem

Every stable $\mathcal{G}_\mathbb{C}$-orbit on $\mathcal{A}$ contains a unique $\mathcal{G}$-orbit of solutions to $\mathcal{YM}^{-1}(0)$ where

$$\mathcal{YM}(\nabla) = N\left(\frac{\star F_\nabla}{2\pi i} + \mu\right)$$

and N is this funny norm we defined last time.

Now we start filling in the details from the sketch proof. We will write $\mathfrak{Orb}(\mathcal{E})$ for the orbit $\mathcal{G}_{\mathbb{C}} \cdot \nabla$ where ∇ is a connection compatible with $\mathcal{E}$.

Lemma

Let $\mathcal{E}$ be a holomorphic bundle over M. Then either $\inf_{\mathfrak{Orb}(\mathcal{E})} \mathcal{YM}$ is attained in the orbit $\mathfrak{Orb}(\mathcal{E})$ or there is a holomorphic bundle $\mathcal{F} \not\cong \mathcal{E}$ such that

- $\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{E})$, $\deg(\mathcal{F}) = \deg(\mathcal{E})$,
- $\inf_{\mathfrak{Orb}(\mathcal{F})} \mathcal{YM} \leq \inf_{\mathfrak{Orb}(\mathcal{E})} \mathcal{YM}$,
- $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$.

Here's some preliminary stuff. First recall the concept of *weak convergence* in a Hilbert space means: $v_i \to v$ weakly if $\langle v_i, w\rangle \to \langle v, w\rangle$ for all w.

Exercise

Give an example of a sequence of functions in $L^2([0,1])$ *which weakly converge to* 0 *but don't actually converge. If a sequence of* L^2_1*-functions on a Riemann surface weakly converge in* L^2_1 *why must they converge in* L^4*? (Hint: Uniform boundedness principle). Note that* $\mathcal{YM}$ *is not obviously continuous with respect to the weak topology on* L^2_1*-connections. However you should show that if* $\nabla_i \to \nabla$ *weakly in* L^2_1 *then* $\mathcal{YM}(\nabla) = \liminf \mathcal{YM}(\nabla_i)$*.*

Theorem (Uhlenbeck compactness)

Let $\nabla_i \in \mathcal{A}$ *be a sequence of* L^2_1*-connections with bounded curvature* $||F_i||_{L^2}$*. Then there is a subsequence and a collection of* L^2_2*-gauge transformations* u_i *such that* $u_i\nabla_i$ *converges weakly in* L^2_1*.*

We will prove this theorem in a few lectures' time. This is the main analytical input.

Proof.

Let ∇_i be an infimising sequence for $\mathcal{YM}|_{\mathfrak{Orb}(\mathcal{E})}$. The curvature of the ∇_i are L^2-bounded since N is equivalent to L^2 and certainly they are bounded in N. Uhlenbeck compactness gives us a subsequence with gauge transformations u_i such that $u_i\nabla_i \to \nabla$ weakly in L^2_1. Moreover,

$$\mathcal{YM}(\nabla) \leq \liminf \mathcal{YM}(\nabla_i) = \inf_{\mathfrak{Orb}(E)} \mathcal{YM}$$

Now ∇ defines a holomorphic structure $\mathcal{E}_\nabla$ with $\mathfrak{Orb}(\mathcal{E}_\nabla) = \mathcal{G}_{\mathbb{C}}\nabla$ (we need to be slightly careful here because ∇ is only L^2_1 - we'll sort this out later). We need to show that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_\nabla) \neq 0$: the dichotomy of the theorem is then just the question of whether $\mathcal{E} \cong \mathcal{E}_\nabla$ or not. □

Proof (continued):

To study holomorphic homomorphisms $\mathcal{E} \to \mathcal{E}_\nabla$ we use the connection ∇_{Hom} induced by ∇_0 and ∇ on the tensor product $E^* \otimes E$. This is compatible with the holomorphic structure coming from $\mathcal{E}^* \otimes \mathcal{E}_\nabla$ and gives a $\bar{\partial}$-operator

$$\bar{\partial}_{\mathrm{Hom}} : \Omega^0(\mathrm{Hom}(E,E)) \to \Omega^{0,1}(\mathrm{Hom}(E,E))$$

whose holomorphic sections $\bar{\partial}_{\mathrm{Hom}}\sigma$ are precisely the holomorphic homomorphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_\nabla)$. If there are none of these then $\bar{\partial}_{\mathrm{Hom}}$ has no kernel and since it is an elliptic operator, there is a C such that for all σ

$$C||\sigma||_{L^2_1} \leq ||\bar{\partial}_{\mathrm{Hom}}\sigma||_{L^2}$$

(this is the enhanced elliptic inequality for operators D with vanishing kernel - it would hold more generally for $\sigma \in \ker(D)^\perp$). The Sobolev inequality implies $||\sigma||_{L^2_1} \geq C'||\sigma||_{L^4}$ so $||\bar{\partial}_{\mathrm{Hom}}\sigma||_{L^2} \geq C''||\sigma||_{L^4}$. □

Proof (concluded):

The idea will be to derive a contradiction by showing that for large i, $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathrm{Hom}(\mathcal{E}, \mathcal{E}_{\nabla_i}) = 0$. We have

$$\left(\bar{\partial}_{\mathrm{Hom},i} - \bar{\partial}_{\mathrm{Hom}}\right)\sigma = (\nabla_i - \nabla)^{0,1}\sigma$$

(here $\bar{\partial}_{\mathrm{Hom},i}$ is the connection on $E^* \otimes E$ coming from ∇_0 and ∇_i) so the Hölder inequality gives

$$||\left(\bar{\partial}_{\mathrm{Hom}} - \bar{\partial}_{\mathrm{Hom},i}\right)\sigma||_{L^2} \leq C'''||\nabla_i - \nabla||_{L^4}||\sigma||_{L^4}$$

and

$$||\bar{\partial}_{\mathrm{Hom},i}\sigma||_{L^2} \geq \left(C'' - C'''||\nabla_i - \nabla||_{L^4}\right)||\sigma||_{L^4}$$

However uniform boundedness, a weakly convergent sequence is bounded in L^2_1 and by Rellich compactness $L^2_1 \hookrightarrow L^4$ is compact so $\nabla_i \to \nabla$ (strongly) in L^4. This means that even for large i, $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_{\nabla_i}) = 0$, contradicting the fact that ∇_i is compatible with $\mathcal{E}$ and hence there should be an isomorphism $\mathcal{E} \to \mathcal{E}_{\nabla_i}$! □

We will finish today by showing that for any L_1^2-connection ∇' there is an L_2^2 complexified gauge transformation taking it to a smooth connection, which proves that the L_2^2-complexified gauge orbits on the L_1^2-completion of $\mathcal{A}$ are in bijection with the isomorphism classes of holomorphic vector bundles.

Lemma

Fix an L^2_1-connection $\nabla' = \nabla + B$ (∇ is a smooth reference connection). The action $F : \mathcal{G}_{\mathbb{C}} \to \mathcal{A}$ (sending g to $g\nabla'$) of the L^2_2-complexified gauge transformations on the L^2_1-connections has the property that $d_1 F$ is Fredholm. Here $d_1 F : L^2_2(\Omega^0(M; \mathrm{ad}(P_{\mathbb{C}}))) \to L^2_1(\Omega^1(M; \mathrm{ad}(P)))$ denotes the derivative at $1 \in \mathcal{G}_{\mathbb{C}}$.

Proof.

We can think of $\mathcal{G}_{\mathbb{C}}$ just acting on the $(0,1)$-parts of connections via

$$(g\nabla')^{0,1} = (\nabla')^{0,1} - (\nabla'^{0,1} g)g^{-1}$$

hence we have $d_1 F(\epsilon) = -(\nabla')^{0,1}\epsilon = -\nabla^{0,1}\epsilon - [B, \epsilon]$. □

Proof, continued.

The first part is certainly Fredholm (ellipticity of $\nabla^{0,1}$) and the second part is compact because the action of an $\epsilon \in L^2_2$ on a $B \in L^2_1$ factors through the (compact) inclusion of $L^2_2 \subset L^2_{3/2}$ (or any other intermediate Sobolev space). If the concept of 3/2-differentiable disturbs you, join the club. However, one can make rigorous sense of this Sobolev space using Fourier analysis (where K derivatives corresponds under Fourier transform to multiplying with ξ^k, and k doesn't have to be an integer to use as an exponent). □

In particular we see (from the Banach space implicit function theorem) that there are neighbourhoods $\mathcal{G}_{\mathbb{C}} \supset U \ni 1$ and $\mathcal{A} \supset V \ni \nabla'$ such that for $\nabla'' \in V$, $U \cdot \nabla''$ is a smooth Banach submanifold of V with finite codimension (equal to $\mathrm{coker}(\nabla')^{0,1}$, which we can identify we a Dolbeaut cohomology group $H^{0,1}(\mathrm{End}(\mathcal{E}))$).

Lemma

Every L^2_2-complexified gauge orbit in the space of L^2_1-connections contains a smooth connection.

Proof.

Let N be a finite-dimensional subspace of $\mathcal{A}$ transversal to the L^2_2-complexified gauge orbit through the fixed L^2_1-connection ∇'. On some small neighbourhood V of ∇' there is a projection $\pi\colon V \to N$ with $\pi^{-1}(\nabla') = U(\nabla')$ for some open neighbourhood $U \ni 1$ in $\mathcal{G}_{\mathbb{C}}$. Given $r+1$ points $B_1, \ldots, B_{r+1}$ in V ($r = \dim(N)$) we define an affine linear map f_B from the r-simplex σ_r into V taking the vertices to the B_i. Then $\pi \circ f_B\colon \sigma_r \to N$ is a continuous map depending continuously on B. □

Proof, continued:

If we pick B so that ∇' is at their barycentre then the restriction of $\pi \circ f_B$ to the boundary represents the generator of $H_r(N \setminus A; \mathbb{Z})$ and hence $\pi \circ f_B$ must hit ∇' at some point in the interior of the simplex. This remains true (by continuity) when we replace B by a nearby collection of points C. We can assume that the C_i are smooth since smooth connections are dense in L^2_1-connections. Since linear combinations of smooth connections are again smooth, there is a point $p \in \sigma_r$ such that $f_C(p)$ is a smooth connection living in $\pi^{-1}(\nabla')$, which is a subset of the L^2_2-complexified gauge orbit of ∇'. □

Here again we have a gorgeous proof ripped straight out of Atiyah and Bott. You should go and read it.