

Analytical Methods: Solutions 4

1. The two distinguished stretches are $\delta = 1$ and $\delta = \varepsilon$. $\delta = 1$ gives us a regular expansion:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

$$\frac{df_0}{dx} = \cos x \quad \frac{df_1}{dx} = -\frac{d^2 f_0}{dx^2} = \sin x \quad \frac{df_2}{dx} = -\frac{d^2 f_1}{dx^2} = -\cos x$$

$$f = c_0 + \sin x + \varepsilon[c_1 - \cos x] + \varepsilon^2[c_2 - \sin x] + \dots$$

in which the boundary condition at $x = \pi$ gives:

$$1 = c_0 + \varepsilon[c_1 + 1] + \varepsilon^2[c_2] + \dots$$

$$f(x) = 1 + \sin x - \varepsilon[1 + \cos x] - \varepsilon^2 \sin x + \dots$$

For the stretch $\delta = \varepsilon$, if $z = x/\varepsilon$ then

$$\frac{d^2 f}{dx^2} + \frac{df}{dz} = \varepsilon \cos \varepsilon z = \varepsilon(1 - \varepsilon^2 z^2/2 + \dots)$$

$$f = F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \dots$$

$$\frac{d^2 F_0}{dz^2} + \frac{dF_0}{dz} = 0 \quad \frac{d^2 F_1}{dz^2} + \frac{dF_1}{dz} = 1 \quad \frac{d^2 F_2}{dz^2} + \frac{dF_2}{dz} = 0$$

$$f(z) = a_0 + b_0 e^{-z} + \varepsilon[a_1 + b_1 e^{-z} + z] + \varepsilon^2[a_2 + b_2 e^{-z}] + \dots$$

which becomes (using the BC at $z = 0$):

$$f(z) = a_0 - a_0 e^{-z} + \varepsilon[a_1 - a_1 e^{-z} + z] + \varepsilon^2[a_2 - a_2 e^{-z}] + \dots$$

Now we match, using an intermediate variable $x = \varepsilon^\alpha \xi$. The expansion of the outer solution is

$$f(x) = 1 + \varepsilon^\alpha \xi - 2\varepsilon - \varepsilon^{3\alpha} \xi^3/6 + \varepsilon^{1+2\alpha} \xi^2/2 - \varepsilon^{2+\alpha} \xi + O(\varepsilon^3, \varepsilon^{5\alpha})$$

The inner expansion, with $z = \varepsilon^{\alpha-1} \xi$, becomes

$$f(z) = a_0 + \varepsilon^\alpha \xi + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$$

Matching the two gives $a_0 = 1$, $a_1 = -2$ and $a_2 = 0$, so the matched inner form is

$$f(z) = 1 - e^{-z} + \varepsilon[2e^{-z} - 2 + z] + O(\varepsilon^3).$$

2. $(1 + \varepsilon)x^2 y' = \varepsilon((1 - \varepsilon)xy^2 - (1 + \varepsilon)x + y^3 + 2\varepsilon y^2)$ with $y(1) = 1$.

Outer: set $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$ to have (rows being order 1, ε , ε^2):

$$\begin{array}{rcl} x^2 y_0' & = & 0 \\ x^2 y_1' + x^2 y_0' & = & x y_0^2 - x + y_0^3 \\ x^2 y_2' + x^2 y_1' & = & 2x y_0 y_1 - x y_0^2 - x + 3y_0^2 y_1 + 2y_0^2 \end{array}$$

Order 1 At leading order, $y_0' = 0$ gives $y_0 = a_0$ and, using the boundary condition, $y_0 = 1$.

Order ε The equation becomes $x^2 y_1' = 1$ so $y_1 = a_1 - x^{-1}$. The boundary condition fixes $y_1 = 1 - x^{-1}$.

Order ε^2 The equation is $x^2 y_2' = 2 - 3x^{-1}$ so $y_2 = a_2 - 2x^{-1} + (3/2)x^{-2}$. With the boundary condition we have $y_2 = 1/2 - 2x^{-1} + (3/2)x^{-2}$.

The outer expansion is

$$y \sim 1 + \varepsilon \left(1 - \frac{1}{x}\right) + \varepsilon^2 \left(\frac{1}{2} - \frac{2}{x} + \frac{3}{2x^2}\right) + \dots$$

which ceases to be uniformly asymptotic when $x \sim \varepsilon$.

We rescale (noting that y is still order 1) by putting $x = \varepsilon z$ and the original equation becomes:

$$(1 + \varepsilon)z^2 y' = \varepsilon(1 - \varepsilon)zy^2 - \varepsilon(1 + \varepsilon)z + y^3 + 2\varepsilon y^2$$

In the inner, we pose $y = f_0 + \varepsilon f_1 + \dots$ to have

$$\begin{aligned} z^2 f_0' &= f_0^3 \\ \varepsilon z^2 f_1' + \varepsilon z^2 f_0' &= \varepsilon z f_0^2 - \varepsilon z + 3\varepsilon f_0^2 f_1 + 2\varepsilon f_0^2 \end{aligned}$$

Order 1: $z^2 f_0' = f_0^3$ has solution $f_0 = (A_0 + 2/z)^{-1/2}$.

Matching: We use an intermediate variable $\eta = \varepsilon^{-\alpha} x = \varepsilon^{1-\alpha} z$. The outer becomes

$$y \sim 1 - \varepsilon^{1-\alpha} \frac{1}{\eta} + \varepsilon^{2-2\alpha} \frac{3}{2\eta^2} + \varepsilon - \varepsilon^{2-\alpha} \frac{2}{\eta} + \frac{\varepsilon^2}{2} + \dots$$

and the inner,

$$y \sim A_0^{-1/2} \left(1 - \varepsilon^{1-\alpha} \frac{1}{A_0 \eta} + \varepsilon^{2-2\alpha} \frac{3}{2A_0^2 \eta^2} + \dots\right)$$

which matches the first three terms if we set $A_0 = 1$. So we have $f_0 = (1 + 2/z)^{-1/2}$.

Order ε : The equation becomes $z^2 f_1' - 3(1 + 2/z)^{-1} f_1 = -(1 + 2/z)^{-3/2}$ so (using the integrating factor $(1 + 2/z)^{3/2}$) we have

$$f_1 = \left(A_1 + \frac{1}{z}\right) \left(1 + \frac{2}{z}\right)^{-3/2}$$

Matching The outer is unchanged from before: the inner now becomes

$$y \sim 1 - \varepsilon^{1-\alpha} \frac{1}{\eta} + \varepsilon^{2-2\alpha} \frac{3}{2\eta^2} + \varepsilon A_1 + \varepsilon^{2-\alpha} \left(\frac{1 - 3A_1}{\eta}\right) + \dots$$

which matches the next two terms of the outer if we set $A_1 = 1$.

The inner expansion is

$$y \sim \left(1 + \frac{2}{z}\right)^{-1/2} + \varepsilon \left(1 + \frac{1}{z}\right) \left(1 + \frac{2}{z}\right)^{-3/2} + \dots$$

3. The image of $|z - 1| \leq 1$ under $w = 1/z$.

The boundary of the domain may be parametrised as

$$|z - 1| = 1 \quad z = 1 + e^{i\theta} \quad 0 \leq \theta < 2\pi$$

which transforms to

$$w = \frac{1}{1 + \cos \theta + i \sin \theta} = \frac{1 + \cos \theta - i \sin \theta}{(1 + \cos \theta)^2 + \sin^2 \theta} = \frac{1 + \cos \theta - i \sin \theta}{2 + 2 \cos \theta}$$

$$w = \frac{1}{2} - i \frac{\sin \theta}{2 + 2 \cos \theta}$$

The real part of w is always $1/2$; the imaginary part spans the whole line from negative infinity at $\theta = \pi$ to positive infinity at $\theta = -\pi$.

To complete the mapping we simply need to know which side of the boundary our domain lies. The point $z = 1$ is in the original domain: therefore the point $w = 1$ is in the image domain, which is therefore given by

$$\text{Real}(w) \geq \frac{1}{2}.$$

4. The image of $-\pi/2 < x < \pi/2, 0 < y < 1$ under $w = \sin z$.

We look at each boundary in turn, writing $w = \eta + i\xi$ where necessary.

Bottom edge $y = 0$: $w = \sin x, -\pi/2 < x < \pi/2$.

$$-1 < \eta < 1, \quad \xi = 0.$$

Top edge $y = 1$: $w = \sin(x + i) = \cosh 1 \sin x + i \sinh 1 \cos x$.

$$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1, \quad \xi > 0.$$

Left edge $x = -\pi/2$: $w = \sin(-\pi/2 + iy) = -\cosh y, 0 < y < 1$.

$$-\cosh 1 < \eta < -1 \quad \xi = 0.$$

Right edge $x = \pi/2$: $w = \sin(\pi/2 + iy) = \cosh y, 0 < y < 1$.

$$1 < \eta < \cosh 1 \quad \xi = 0.$$

The top edge is half an ellipse; the other three form the straight line $-\cosh 1 < \eta < \cosh 1, \xi = 0$. We now need to check whether the interior or exterior of the half-ellipse is our image. Take a point from the interior of the rectangle – say, $z = i/2$. Then $w = \sin i/2 = i \sinh(1/2)$, which is inside our half ellipse. The image domain is

$$w = \eta + i\xi \quad \frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} \leq 1, \quad \xi > 0.$$

5. The image of $-\pi/4 < x < \pi/4, -1 < y < 1$ under $w = \sin z$.

Again, we find the image of each edge in turn, putting $z = x + iy, w = \eta + i\xi$.

Bottom edge $y = -1$: $w = \sin(x - i) = \cosh 1 \sin x - i \sinh 1 \cos x$.

$$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1 \quad -\frac{\cosh 1}{\sqrt{2}} < \eta < \frac{\cosh 1}{\sqrt{2}} \quad \xi < -\frac{\sinh 1}{\sqrt{2}}.$$

Top edge $y = 1$: As in 4, but with a reduced range of x :

$$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1 \quad -\frac{\cosh 1}{\sqrt{2}} < \eta < \frac{\cosh 1}{\sqrt{2}} \quad \xi > \frac{\sinh 1}{\sqrt{2}}.$$

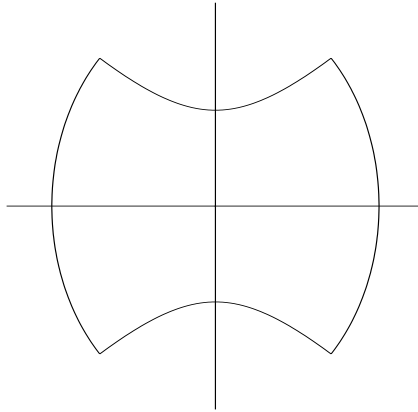
Left edge $x = -\pi/4$: $w = \sin(iy - \pi/4) = (-\cosh y + i \sinh y)/\sqrt{2}$.

$$\eta^2 - \xi^2 = \frac{1}{2} \quad -\frac{\cosh 1}{\sqrt{2}} < \eta < -\frac{1}{\sqrt{2}} \quad -\frac{\sinh 1}{\sqrt{2}} < \xi < \frac{\sinh 1}{\sqrt{2}}.$$

Right edge $x = \pi/4$: $w = \sin(iy + \pi/4) = (\cosh y + i \sinh y)/\sqrt{2}$.

$$\eta^2 - \xi^2 = \frac{1}{2} \quad \frac{1}{\sqrt{2}} < \eta < \frac{\cosh 1}{\sqrt{2}} \quad -\frac{\sinh 1}{\sqrt{2}} < \xi < \frac{\sinh 1}{\sqrt{2}}.$$

This curvilinear rectangle looks like:



and is bounded by the hyperbola $\eta^2 - \xi^2 = 1/2$ and the ellipse

$$\frac{\eta^2}{\cosh^2 1} + \frac{\xi^2}{\sinh^2 1} = 1$$

6. $\nabla^2 u = 0$ in the domain $1 < r < e^\alpha$, $0 < \alpha < \pi$ with boundary conditions $\partial u / \partial r(1, \theta) = 0$, $\partial u / \partial r(e^\alpha, \theta) = \sin \theta$, $u(r, 0) = u(r, \pi) = 0$.

(a) The geometry of the domain suggests polar coordinates, in which

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

and the three types of separable solution are

$$u = (a \cos[\lambda\theta] + b \sin[\lambda\theta])(cr^\lambda + dr^{-\lambda})$$

$$u = (A \exp [\mu \theta] + B \exp [-\mu \theta])(C \cos [\mu \ln r] + D \sin [\mu \ln r])$$

$$u = (\alpha + \beta \ln r)(\gamma + \delta \theta).$$

The boundary conditions $u(r, 0) = u(r, \pi) = 0$ impose $a = A = B = \gamma = \delta = 0$ and $\lambda = n$ so

$$u = \sum_n \sin [n\theta](c_n r^n + d_n r^{-n}).$$

Then

$$\frac{\partial u}{\partial r} = \sum_n \sin [n\theta](ncr^{n-1} - ndr^{-(n+1)})$$

and the boundary conditions $\partial u/\partial r(1, \theta) = 0$ and $\partial u/\partial r(e^\alpha, \theta) = \sin \theta$ give $n = 1$ and finally

$$u(r, \theta) = \frac{(r + r^{-1}) \sin \theta}{(1 - e^{-2\alpha})}.$$

- (b) Under $w = \ln z$ with $w = \eta + i\xi$, the region $0 < \theta < \pi$, $1 < r < e^\alpha$ becomes $0 < \eta < \alpha$, $0 < \xi < \pi$. The three zero boundary conditions become

$$u(\eta, 0) = 0 \quad u(\eta, \pi) = 0 \quad \frac{\partial u}{\partial \eta}(0, \xi) = 0$$

and since $|dw/dz| = |1/z| = e^{-\alpha}$ on $r = e^\alpha$, the final boundary condition becomes

$$\frac{\partial u}{\partial \eta}(\alpha, \xi) = e^\alpha \sin \xi.$$

It is clear that the solution is

$$u = \sin \xi(a \cosh \eta + b \sinh \eta) \quad \frac{\partial u}{\partial \eta} = \sin \xi(a \sinh \eta + b \cosh \eta)$$

$$u = \frac{e^\alpha \cosh \eta \sin \xi}{\sinh \alpha} = \frac{2 \cosh \eta \sin \xi}{(1 - e^{-2\alpha})}$$

Now if $\ln z = \eta + i\xi$ then $\eta = \ln r$ and $\xi = \theta$ so our solution is

$$u(z) = \frac{2 \cosh [\ln r] \sin \theta}{(1 - e^{-2\alpha})} = \frac{(r + r^{-1}) \sin \theta}{(1 - e^{-2\alpha})}.$$

Note the analytic function of which u is the real part is

$$f = \frac{2 \sin (-iw)}{(1 - e^{-2\alpha})} = \frac{2 \sin (-i \ln z)}{(1 - e^{-2\alpha})}.$$