8 Matching with logs

Consider the ODE:

$$x^2 f'' - \varepsilon f f' = 0$$

with boundary conditions $f(0) = 1$, $f(1) = 0$.

8.1 Outer expansion

Let's attempt a regular expansion first. We won't be throwing away the highest derivative so we could reasonably expect this to work.

The expansion $f = f_0 + \varepsilon f_1 + \cdots$ gives, to two terms,

$$\begin{array}{rcl} x^2 f_0'' & & = 0 \\ \varepsilon x^2 f_1'' & - \ \varepsilon f_0 f_0' & = 0 \end{array}$$

with solutions,

$$f_0 = A_0 x + B_0$$

$$f_1 = A_0^2(x \ln x - x) - A_0 B_0 \ln x + A_1 x + B_1$$

Suppose we start by satisfying the boundary condition at $x = 1$. Then we have

$$f_0 = A_0 x - A_0$$

$$f_1 = A_0^2(x \ln x - x + \ln x + 1) + A_1 x - A_1.$$

At leading order, the boundary condition at $x = 0$ may be satisfied by setting $A_0 = -1$; but now f_1 diverges as $x \to 0$ and we cannot satisfy the boundary condition with any choice of A_1. What has gone wrong?

Let us look again at the ODE, this time paying attention to small x (where we know the problems occur). Suppose we stretch $x = \delta z$. Then the two terms scale as, respectively, 1 and $\varepsilon \delta^{-1}$, which balance when $\delta = \varepsilon$. So we only expect the solution we found above to work for $x \gg \varepsilon$. We have found an outer solution:

$$f \sim A_0(x-1) + \varepsilon[A_0^2(x \ln x - x + \ln x + 1) + A_1 x - A_1].$$

Note that when $x \sim \varepsilon$ this solution becomes

$$f \sim -A_0 + A_0^2 \varepsilon \ln \varepsilon + \varepsilon[A_0^2 + A_0 - A_1] + \cdots$$

which suggests that the scaling for terms in the inner expansion should be a series 1, $\varepsilon \ln(1/\varepsilon)$, ε, ...

8.2 Inner expansion

Let's trying using this stretch for an inner:

$$f = F_0 + \varepsilon \ln(1/\varepsilon) F_1 + \varepsilon F_2 + \cdots$$

with the equation giving

$$z^2 f'' - f f' = 0.$$

Unfortunately we can't solve this in general... we are going to have to take some information from the boundary condition to inform an attempt at solution. Since we need $f = 1$ at $x = 0$, we try a solution

$$f = 1 + \varepsilon \ln(1/\varepsilon) F_1(z) + \varepsilon F_2(z) + \cdots$$

and see if it works. The leading order term is satisfied as all derivatives of our F_0 are zero; the next two equations become

$$z^2 F_1'' - F_1' = 0$$
$$z^2 F_2'' - F_2' = 0$$

which have solution

$$F_i' = a_i \exp[-1/z]$$
$$F_i = a_i \int_0^z \exp[-1/t]\,\mathrm{d}t + b_i = b_i + a_i \int_{1/z}^{\infty} \frac{e^{-\tau}}{\tau^2}\,\mathrm{d}\tau.$$

The conditions $F_i = 0$ at $z = 0$ give $b_1 = b_2 = 0$ but a_1 and a_2 are still undetermined.

8.3 Matching

Can we match this onto our outer? We have

$$f_{\text{outer}} \sim A_0(x-1) + \varepsilon[A_0^2(x\ln x - x + \ln x + 1) + A_1(x-1)] + \cdots$$
$$f_{\text{inner}} \sim 1 + \varepsilon \ln(1/\varepsilon) a_1 \int_{1/z}^{\infty} \frac{e^{-\tau}}{\tau^2}\,\mathrm{d}\tau + \varepsilon a_2 \int_{1/z}^{\infty} \frac{e^{-\tau}}{\tau^2}\,\mathrm{d}\tau + \cdots$$

We use an intermediate variable in the usual way: $x = \varepsilon^{\alpha}\eta$ and $z = \varepsilon^{\alpha-1}\eta$.

$$f_{\text{outer}} \sim -A_0 + \varepsilon^{\alpha} A_0 \eta - \varepsilon \ln(1/\varepsilon)\alpha A_0^2 + \varepsilon[A_0^2 - A_1 + A_0^2 \ln \eta] + O(\varepsilon^{1+\alpha}\ln(1/\varepsilon))$$

but for the inner, we need more information about our integral: for small ρ,

$$\int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau^2}\,\mathrm{d}\tau \sim \frac{1}{\rho} + \ln\rho + \gamma - 1 - \frac{\rho}{2} + O(\rho^2)$$

in which γ is Euler's constant $\gamma = 0.57721\,56649\ldots$. So, putting large z and hence small $1/z$ into our inner expansion, we have

$$\begin{aligned}
f_{\text{inner}} &\sim 1 + \varepsilon\ln(1/\varepsilon) a_1[\varepsilon^{\alpha-1}\eta + (\alpha-1)\ln(1/\varepsilon) - \ln\eta + \gamma - 1 + O(\varepsilon^{1-\alpha})] \\
&\quad + \varepsilon a_2[\varepsilon^{\alpha-1}\eta + (\alpha-1)\ln(1/\varepsilon) - \ln\eta + \gamma - 1 + O(\varepsilon^{1-\alpha})] + \cdots \\
&\sim 1 + \varepsilon^{\alpha}\ln(1/\varepsilon) a_1 \eta + \varepsilon^{\alpha} a_2 \eta + \varepsilon \ln^2(1/\varepsilon) a_1(\alpha-1) \\
&\quad + \varepsilon\ln(1/\varepsilon)(a_2(\alpha-1) - a_1[\ln\eta - \gamma + 1]) - \varepsilon a_2[\ln\eta - \gamma + 1] \\
&\quad + O(\varepsilon^{2-\alpha}\ln(1/\varepsilon))
\end{aligned}$$

Let us match at each order:

$O(1)$	$1 = -A_0$	$A_0 = -1$
$O(\varepsilon^{\alpha}\ln(1/\varepsilon))$	$a_1\eta = 0$	$a_1 = 0$
$O(\varepsilon^{\alpha})$	$a_2\eta = A_0\eta$	$a_2 = -1$
$O(\varepsilon\ln^2(1/\varepsilon))$	$0 = 0$	
$O(\varepsilon\ln(1/\varepsilon))$	$a_2(\alpha-1) = -\alpha A_0^2$	

and we cannot match at this order. We are missing a term $\varepsilon \ln(1/\varepsilon)$ from the outer: we now realise we should have used the same expansion series in the outer that we used in the inner. Ironically, as $a_1 = 0$ we didn't actually need to use the full expansion in the inner – but that's just the way the cookie crumbles.

8.4 Outer revisited

We pose a new outer expansion: $f = f_0 + \varepsilon \ln(1/\varepsilon) g_1 + \varepsilon f_1 + \cdots$, which gives:

$$\begin{aligned} x^2 f_0'' &= 0 \\ x^2 g_1'' &= 0 \\ x^2 f_1'' - f_0 f_0' - f_0 f_0' &= 0 \end{aligned}$$

The solution for f_0 is as before (with $f_0(1) = 0$):

$$f_0 = A_0 x + B_0 \qquad f_0 = A_0 x - A_0$$

and g_1 is the same:

$$g_1 = C_1 x - D_1 \qquad g_1 = C_1 x - C_1$$

and finally, f_1 is unchanged by the modification:

$$x^2 f_1'' = A_0^2 (x-1) \qquad f_1 = A_0^2 (x \ln x - x + \ln x - 1) + A_1 (x-1)$$

The full amended outer solution is

$$f = A_0(x-1) + \varepsilon \ln(1/\varepsilon) C_1 (x-1) + \varepsilon [A_0^2 (x \ln x - x + \ln x - 1) + A_1 (x-1)] + \cdots$$

8.5 Matching revisited

Using the intermediate variable $x = \varepsilon^\alpha \eta$, the inner expansion is as before (setting $a_1 = 0$ and $a_2 = -1$ from our first matching):

$$f_{\text{inner}} \sim 1 - \varepsilon^\alpha \eta + \varepsilon \ln(1/\varepsilon)(1-\alpha) + \varepsilon[\ln \eta - \gamma + 1] + O(\varepsilon^{2-\alpha} \ln(1/\varepsilon))$$

and the outer becomes (setting $A_0 = -1$ from our first attempt):

$$f_{\text{outer}} \sim 1 - \varepsilon^\alpha \eta - \varepsilon \ln(1/\varepsilon)[C_1 + \alpha] + \varepsilon[\ln \eta - 1 - A_1] + O(\varepsilon^{1+\alpha} \ln(1/\varepsilon))$$

The expressions match for the first two terms: moving further in we have:

$$\begin{aligned} O(\varepsilon \ln(1/\varepsilon)) &: \quad 1 - \alpha = -C_1 - \alpha \qquad C_1 = -1 \\ O(\varepsilon) &: \quad -\gamma + 1 = -1 - A_1 \qquad A_1 = \gamma - 2 \end{aligned}$$

To summarise, we have the outer solution

$$f = 1 - x + \varepsilon \ln(1/\varepsilon)(1-x) + \varepsilon[(x+1)\ln x + (\gamma - 3)x - \gamma + 1] + \cdots$$

and inner (with $x = \varepsilon z$)

$$f = 1 - \varepsilon \int_{1/z}^{\infty} \frac{e^{-\tau}}{\tau^2} \, d\tau + \cdots$$