7 More matching!

Last lecture we looked at matched asymptotic expansions in the situation where we found all the possible underlying scalings first, located where to put the boundary later from the direction of the exponential decay, applied all sets of boundary conditions and finally matched our two expansions. That's a good generic picture but there are more possibilities.

7.1 Another way to find scalings: breakdown of ordering

Way back when we looked at regular expansions, I mentioned that one possible warning sign was that the ordering of terms in our expansion could break down. This can be used as an alternative method of seeking out new scalings and stretches, particularly for complex problems and when the outer scale and stretch are fixed by the boundary conditions.

This example comes from Hinch exercise 5.12 (and originally Van Dyke):

$$x^3 \frac{\mathrm{d}y}{\mathrm{d}x} = \varepsilon((1+\varepsilon)x + 2\varepsilon^2)y^2 \qquad \text{in } \ 0 < x < 1$$

with boundary condition $y(1) = 1 - \varepsilon$.

We start with the obvious expansion:

$$y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

and substitute to have

$$\begin{array}{rcl} x^3 y_0' & = & 0 \\ \varepsilon x^3 y_1' & = & \varepsilon x y_0^2 \\ \varepsilon^2 x^3 y_2' & = & 2\varepsilon^2 x y_0 y_1 \quad + \quad \varepsilon^2 x y_0^2 \end{array}$$

We will be applying the boundary condition to this solution: $y_0(1) = 1$, $y_1(1) = -1$, $y_2(1) = 0$, and so on.

Order 1 $y_0' = 0$ gives $y_0 = a_0$ and hence $y_0 = 1$.

Order ε $x^3 y_1' = x$ gives $y_1 = a_1 - 1/x$ and hence $y_1 = -1/x$.

Order ε^2 $x^3 y_2' = x - 2$ gives $y_2 = a_2 - 1/x + 1/x^2$ and hence $y_2 = 1/x^2 - 1/x$.

Our outer solution begins

$$y \sim 1 - \frac{\varepsilon}{x} + \varepsilon^2 \left(\frac{1}{x^2} - \frac{1}{x} \right) + \cdots$$

Now both the (nominally) order ε and order ε^2 terms become order 1 when $x \sim \varepsilon$. The function value is still order 1 (pick, for instance, $x = 2\varepsilon$ to see this) and so we look for an inner expansion with $x = \varepsilon z$ and put

$$y = f_0(z) + \varepsilon f_1(z) + \varepsilon^2 f_2(z) + \cdots$$

The differential equation transforms to

$$z^3 \frac{\mathrm{d}y}{\mathrm{d}z} = ((1+\varepsilon)z + 2\varepsilon)y^2$$

which then gives

$$\begin{array}{rcccc} z^3 f_0' &=& z f_0^2 && \\ \varepsilon z^3 f_1' &=& 2\varepsilon z f_0 f_1 &+& \varepsilon(z+2) f_0^2 \\ \varepsilon^2 z^3 f_2' &=& \varepsilon^2 z(f_1^2 + 2 f_0 f_2) &+& 2\varepsilon^2 (z+2) f_0 f_1 \end{array}$$

We will solve for two terms before matching with the outer.

Order 1 $z^3 f_0' = z f_0^2$ gives $1/f_0 = A_0 + 1/z$ and $f_0 = z/(1 + A_0 z)$.

Order ε $z^3 f_1' - 2z f_0 f_1 = (z+2) f_0^2$ becomes

$$f_1' - 2f_1/z(1 + A_0 z) = (z+2)/z(1 + A_0 z)^2$$

and hence (using an integrating factor of $(1 + A_0 z)^2/z^2$) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(1 + A_0 z)^2}{z^2} f_1\right) = \frac{1}{z^2} + \frac{2}{z^3}$$

$$f_1 = \frac{A_1 z^2}{(1 + A_0 z)^2} - \frac{1 + z}{(1 + A_0 z)^2}.$$

So now our two solutions are

$$y_{\text{outer}} = 1 - \frac{\varepsilon}{x} + \varepsilon^2\left(\frac{1}{x^2} - \frac{1}{x}\right) + \cdots$$

$$y_{\text{inner}} = \frac{z}{(1 + A_0 z)} + \varepsilon\left(\frac{A_1 z^2}{(1 + A_0 z)^2} - \frac{1 + z}{(1 + A_0 z)^2}\right) + \cdots$$

related by $x = \varepsilon z$. Introducing $x = \varepsilon^\alpha \eta$ and $z = \varepsilon^{\alpha - 1}\eta$ and expanding (noting that z is large and so z^{-1} is small) gives

$$\begin{aligned} y_{\text{outer}} &\sim 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{2-\alpha}\frac{1}{\eta} + \cdots \\ y_{\text{inner}} &= \frac{1}{A_0} - \frac{\varepsilon^{1-\alpha}}{A_0^2 \eta} + \frac{\varepsilon^{2-2\alpha}}{\eta^2 A_0^3} + \cdots \\ &\quad + \varepsilon\left(\frac{A_1}{A_0^2(1 + (A_0 \varepsilon^{\alpha-1}\eta)^{-1})^2} - \frac{1 + \varepsilon^{\alpha-1}\eta}{(1 + A_0(\varepsilon^{\alpha-1}\eta))^2}\right) + \cdots \end{aligned}$$

Clearly to match the order 1 term we need $A_0 = 1$; then the comparison becomes

$$\begin{aligned} y_{\text{outer}} &\sim 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{2-\alpha}\frac{1}{\eta} + \cdots \\ y_{\text{inner}} &= 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{3-3\alpha}\frac{1}{\eta^3} + \cdots \\ &\quad + \varepsilon A_1(1 + \varepsilon^{1-\alpha}\eta^{-1})^{-2} - \varepsilon^{2-\alpha}\eta^{-1}(1 + \varepsilon^{1-\alpha}\eta^{-1})^{-1} + \cdots \\ &= 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{3-3\alpha}\frac{1}{\eta^3} + \cdots \\ &\quad + (A_1\varepsilon - 2A_1\varepsilon^{2-\alpha}\eta^{-1} + \cdots) - \varepsilon^{2-\alpha}\eta^{-1}(1 - \varepsilon^{1-\alpha}\eta^{-1} + \cdots) \end{aligned}$$

There is nothing in the outer solution to match the $A_1\varepsilon$ term so we need $A_1 = 0$; the other unmatched terms all have powers like $\varepsilon^{3-n\alpha}$ so would match the third term of the outer, which we have not calculated. Our inner solution is therefore

$$y_{\text{inner}} = \frac{z}{(1+z)} - \frac{\varepsilon}{(1+z)} + \cdots$$

with $x = \varepsilon z$.

This problem has hidden depths though: the first two terms of our inner expansion break order when z is order ε. At that point the function value is also order ε, so we look for an inner-inner expansion $y = \varepsilon F(X)$ with $z = \varepsilon X$. The governing equation:

$$z^3 \frac{\mathrm{d}y}{\mathrm{d}z} = ((1+\varepsilon)z + 2\varepsilon)y^2$$

becomes:

$$X^3 \frac{\mathrm{d}F}{\mathrm{d}X} = ((1+\varepsilon)X + 2)F^2.$$

Here we will only look for the leading order term:

$$X^3 \frac{\mathrm{d}F_0}{\mathrm{d}X} = (X+2)F_0^2 \qquad \int \frac{\mathrm{d}F_0}{F_0^2} = \int \frac{1}{X^2} + \frac{2}{X^3}\,\mathrm{d}X$$

$$\frac{-1}{F_0} = \frac{-1}{X} - \frac{1}{X^2} - C_0 \qquad F_0 = \frac{X^2}{C_0X^2 + X + 1}.$$

Now we need to compare the inner and double-inner expansions:

$$\begin{aligned} y_{\text{inner}} &= \frac{z}{(1+z)} - \frac{\varepsilon}{(1+z)} + \cdots \\ y_{\text{double}} &\sim \varepsilon \frac{X^2}{C_0X^2 + X + 1} \end{aligned}$$

with $z = \varepsilon X$. We set $z = \varepsilon^\alpha \xi$ and $X = \varepsilon^{\alpha-1}\xi$ to have

$$\begin{aligned} y_{\text{inner}} &= \varepsilon^\alpha \xi(1 - \varepsilon^\alpha \xi + \cdots) - \varepsilon(1 + \cdots) \\ y_{\text{double}} &\sim \frac{\varepsilon}{C_0}(1 + \varepsilon^{1-\alpha}C_0^{-1}\xi^{-1} + \varepsilon^{2-2\alpha}C_0^{-1}\xi^{-2})^{-1} \\ &\sim \frac{\varepsilon}{C_0}(1 - \varepsilon^{1-\alpha}C_0^{-1}\xi^{-1} + \cdots) \end{aligned}$$

The leading order terms here simply don't balance. There is nothing in the double-inner that gets as large as the $\varepsilon^\alpha\xi$ term in the inner. However, in expanding our double-inner solution, we did assume that C_0 was nonzero. If we try the case where it is zero, we get:

$$\begin{aligned} y_{\text{double}} &= \varepsilon \frac{X^2}{X+1} + \cdots = \varepsilon X \frac{1}{1 + X^{-1}} + \cdots \\ &\sim \varepsilon^\alpha \xi(1 - \varepsilon^{1-\alpha}\xi^{-1} + \cdots) \end{aligned}$$

which now matches the leading term from the inner. To match any more terms we would need to go to higher order in both expansions.

In summary, this ODE has three layers of asymptotic solution:

$$\begin{aligned}
y_{\text{outer}} &= 1 - \frac{\varepsilon}{x} + \varepsilon^2\left(\frac{1}{x^2} - \frac{1}{x}\right) + \cdots \\
y_{\text{inner}} &= \frac{z}{(1+z)} - \frac{\varepsilon}{(1+z)} + \cdots \qquad \text{with} \quad x = \varepsilon z \\
y_{\text{double}} &= \varepsilon\frac{X^2}{X+1} + \cdots \qquad \text{with} \quad z = \varepsilon X.
\end{aligned}$$

This three-layered structure is known as a triple-deck problem.

7.2 A worse example

This example comes from the book by Cole. The governing equation is

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$

with boundary conditions $f(0) = -1$, $f(1) = 1$.

These boundary conditions fix f to be strictly order 1, so we cannot scale f and can only consider stretching x. Note that you have seen this equation before in exercise 4 of sheet 3. In the case of no scaling ($\alpha = 0$) you should have found two possible stretches: $x \sim 1$ and $x \sim \varepsilon$. You will also have found all the possible leading-order outer and inner solutions, but I didn't give you any boundary conditions and you hadn't learnt about matching yet, so you couldn't determine any of the constants.

Outer

Let us look first at the outer solution. We pose $f = f_0(x)+\varepsilon f_1(x)+\varepsilon^2 f_2(x)+\cdots$. The leading-order equation is

$$f_0\frac{\mathrm{d}f_0}{\mathrm{d}x} - f_0 = 0 \implies f_0\left(\frac{\mathrm{d}f_0}{\mathrm{d}x} - 1\right) = 0$$

which has two solutions, $f_0(x) \equiv 0$ and $f_0(x) = x + C$. Note that for both of these, $\mathrm{d}^2 f_0/\mathrm{d}x^2 = 0$ and so f_0 is an exact solution of the equation, and $f_1 = f_2 = \cdots = 0$.

Clearly the branch $f_0 = 0$ can't match either of the boundary conditions, so we know our outer solution must be

$$f(x) = x + C.$$

We have not yet found where the boundary layer will be; since the outer is so simple, we might as well work out the constant for both possibilities now.

If the outer meets $x = 1$ then we have $C = 0$ and so $f_{\text{outer},1}(x) = x$.
If the outer meets $x = 0$ then instead we have $C = -1$ and $f_{\text{outer},0}(x) = x - 1$.

Inner

What stretch do we expect for the inner? Note that the boundary conditions mean we can't scale f, we can only stretch x. We found in your exercise that we should stretch $x = a + \varepsilon z$.

We introduce $z = (x - a)/\varepsilon$ and rewrite our differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f\frac{\mathrm{d}f}{\mathrm{d}z} - \varepsilon f = 0$$

Now we pose an inner expansion: $f \sim F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \cdots$, and at leading order the governing equation is

$$\frac{\mathrm{d}^2 F_0}{\mathrm{d}z^2} + F_0\frac{\mathrm{d}F_0}{\mathrm{d}z} = 0.$$

We can integrate this directly once:

$$\frac{\mathrm{d}F_0}{\mathrm{d}z} + \frac{1}{2}F_0^2 = C.$$

Now remember that for a boundary layer solution, we are going to need solutions which *decay* to some fixed value out of the layer. This means that as $z \to \pm\infty$ (but not necessarily both), we need $\mathrm{d}F_0/\mathrm{d}z \to 0$ and so $C \geq 0$. (This already eliminates some of the possible solutions you found.) Let us set $C = 2k^2$ for convenience.

This ODE for F_0 has three different possible forms of solution. If $k = 0$ the solution is

$$F_0 = \frac{2}{z + C},$$

if we are to use this solution the point $z = -C$ must not lie within our domain.

For $k > 0$ there are three solutions, two of which work.

First we look at the possibility that $|F_0| = 2k$. In that case

$$\frac{\mathrm{d}F_0}{\mathrm{d}z} = 0 \qquad F_0 = \pm 2k.$$

This is not a true inner solution: it does not depend on z, so it doesn't vary quickly w.r.t. x. In fact, it is just a regular outer solution expanded in terms of the inner variable. So we move on to the two other cases: $|F_0| < 2k$ and $|F_0| > 2k$.

In both of these cases we can solve the ODE by partial fractions:

$$2\frac{\mathrm{d}F_0}{\mathrm{d}z} = 4k^2 - F_0^2.$$

$$\int 2k \,\mathrm{d}z = \int \frac{4k}{4k^2 - F_0^2}\,\mathrm{d}F_0 = \int \left(\frac{1}{2k - F_0} + \frac{1}{2k + F_0}\right)\mathrm{d}F_0$$

$$2kz + 2B = -\ln|2k - F_0| + \ln|2k + F_0| = \ln\left|\frac{2k + F_0}{2k - F_0}\right|$$

$$\frac{2k + F_0}{2k - F_0} = \pm\exp\left[2(kz + B)\right]$$

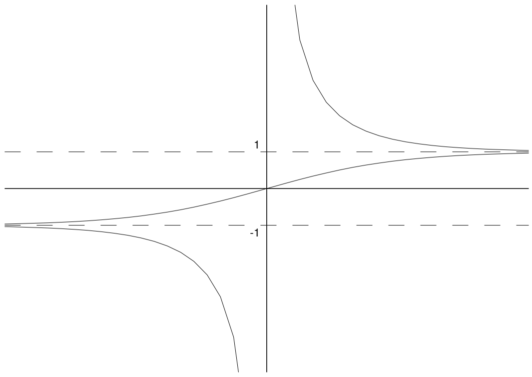
$$F_0 = 2k\frac{\exp[(kz+B)] \mp \exp[-(kz+B)]}{\exp[(kz+B)] \pm \exp[-(kz+B)]}$$

which has two solutions,

$$F_0 = 2k\tanh[(kz+B)] \qquad F_0 = 2k\coth[(kz+B)],$$

both of which decay exponentially to some limit as $z \to \infty$.

Look at the forms of the tanh and coth curves:



We can see that the tanh solution moves smoothly from one value to another over the width of the boundary layer, whereas the coth profile cannot be given a value $z = 0$. This means that the coth profile can only be used if the boundary layer is at one end or other of the region, whereas the tanh profile can be used anywhere.

Matching with a single boundary layer

Let us try first to put the boundary layer near $x = 0$. The outer solution must match the boundary condition at $x = 1$ so

$$f_{\text{outer}} = x.$$

Now in the inner region we can either have

$$F(z) = 2k\tanh[kz+B] \quad \text{or} \quad F(z) = 2k\coth[kz+B] \quad \text{or} \quad F(z) = 2/(z+C).$$

In each case we need $F(z=0) = -1$ and $F(z \to \infty) = 0$. The second of these gives $k = 0$ both the first two cases, and then we cannot match the other boundary condition for any B. For the third function, we have the right result as $z \to \infty$, but to match the condition at $z = 0$ gives $C = -2$ and the forbidden point $z = -C = 2$ lies within our domain. **FAILED.**

Now we try with a boundary layer near $x = 1$. This time the outer solution must match the boundary condition at $x = 0$ so

$$f_{\text{outer}} = x - 1.$$

In the inner region the possibilities are

$$F(z) = 2k \tanh [kz + B] \quad \text{or} \quad F(z) = 2k \coth [kz + B] \quad \text{or} \quad F(z) = 2/(z + C).$$

The boundary conditions are $F(z = 0) = 1$ and $F(z \to -\infty) = 0$. We have the same problem again: we need both $k \neq 0$ and $k = 0$, or $z = -C$ lies within our domain. **FAILED.**

Finally, let us try having the "boundary layer" in the middle, at some general position a between 0 and 1. This time we have two different branches of the outer solution:

$$f_{\text{outer},1}(x) = x \qquad f_{\text{outer},1}(a) = a.$$

$$f_{\text{outer},0}(x) = x - 1 \qquad f_{\text{outer},0}(a) = a - 1.$$

Our inner solution will then have boundary conditions

$$F(z \to -\infty) = a - 1 \qquad F(z \to \infty) = a.$$

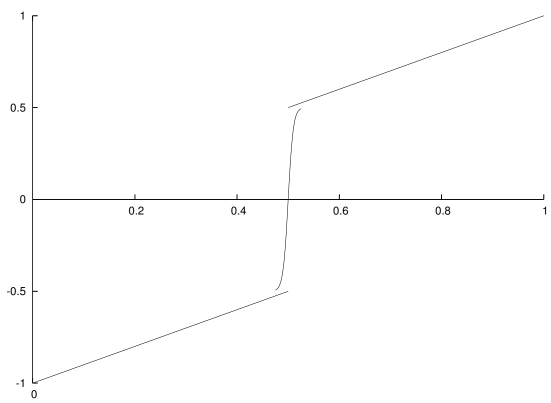
The only profile we are allowed is the tanh profile, which goes from $-2k$ to $2k$ over the width of the layer. This fixes

$$a - 1 = -2k \qquad a = 2k \qquad \Longrightarrow a = 1/2, \ k = 1/4.$$

Our leading-order inner solution is

$$F(z) = \frac{1}{2} \tanh [z/4]$$

and $z = (x - \frac{1}{2})/\varepsilon$. The complete solution looks like this:



Note: It is also possible to construct a solution having more than one boundary layer: for example, try putting a tanh boundary layer at each end. However, a single localised region of "failure" is more physically realistic.

Further expansion

Since the solution we have found in the inner is not an exact solution, we could continue to higher orders. Often you will find that the later equations are easier to solve than the first because the new terms come in linearly. Although the equation becomes linear, it's not really easier in this case; but let us try calculate one more term. Recall we had

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f\frac{\mathrm{d}f}{\mathrm{d}z} - \varepsilon f = 0$$

with

$$f \sim \frac{1}{2}\tanh\left[z/4\right] + \varepsilon F_1(z) + \cdots$$

At order ε this gives

$$\frac{\mathrm{d}^2 F_1}{\mathrm{d}z^2} + F_0\frac{\mathrm{d}F_1}{\mathrm{d}z} + F_1\frac{\mathrm{d}F_0}{\mathrm{d}z} - F_0 = 0$$

$$\frac{\mathrm{d}^2 F_1}{\mathrm{d}z^2} + \frac{1}{2}\tanh\left[\frac{z}{4}\right]\frac{\mathrm{d}F_1}{\mathrm{d}z} + \frac{1}{8}\mathrm{sech}^2\left[\frac{z}{4}\right]F_1 = \frac{1}{2}\tanh\left[\frac{z}{4}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{\mathrm{d}F_1}{\mathrm{d}z} + \frac{1}{2}\tanh\left[\frac{z}{4}\right]F_1\right\} = \frac{1}{2}\tanh\left[\frac{z}{4}\right]$$

$$\frac{\mathrm{d}F_1}{\mathrm{d}z} + \frac{1}{2}\tanh\left[\frac{z}{4}\right]F_1 = 2\ln\cosh\left[\frac{z}{4}\right] + C_1$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\cosh^2\left[\frac{z}{4}\right]F_1\right\} = 2\cosh^2\left[\frac{z}{4}\right]\ln\cosh\left[\frac{z}{4}\right] + C_1\cosh^2\left[\frac{z}{4}\right]$$

which may be integrated to give the solution:

$$F_1 = C_1\left(\frac{z}{4} + \sinh\left[\frac{z}{4}\right]\right)\mathrm{sech}^2\left[\frac{z}{4}\right] + C_2\mathrm{sech}^2\left[\frac{z}{4}\right] + 2\mathrm{sech}^2\left[\frac{z}{4}\right]\int\cosh^2\left[\frac{z}{4}\right]\ln\cosh\left[\frac{z}{4}\right]\,\mathrm{d}z$$

Unfortunately the final integral is only available in terms of the polylogarithm function:

$$\int\cosh^2\left[\frac{z}{4}\right]\ln\cosh\left[\frac{z}{4}\right]\,\mathrm{d}z = \frac{z}{4} - \frac{z^2}{16} - \frac{1}{2}\sinh\left[\frac{z}{2}\right] - \frac{z}{2}\ln\left(1 + \exp\left[-\frac{z}{2}\right]\right) + \mathrm{Li}_2\left(-e^{-z/2}\right) + \ln\cosh\left[\frac{z}{4}\right]\left(\frac{z}{2} + \sinh\left[\frac{z}{2}\right]\right)$$

in which

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^2},$$

but if we had been able to find F_1 in terms of more useful functions, the matching procedure would have continued as before.