

Analytical Methods: Exercises 5

1. [Blasius boundary layer] Consider the steady Navier-Stokes equations:

$$\underline{\nabla} \cdot \underline{u} = 0 \quad \underline{u} \cdot \underline{\nabla} \underline{u} = -\underline{\nabla} p + \nabla^2 \underline{u},$$

here made dimensionless using a typical velocity U , the fluid density ρ and the viscous lengthscale $L = \eta/\rho U$.

Investigate flow past a semi-infinite flat plate (no natural lengthscale):

$$\underline{u} \rightarrow \underline{e}_x \quad \text{at } \infty; \quad \underline{u} = 0 \quad \text{on } x \geq 0, y = 0.$$

A dilation transformation is appropriate; expect the vertical velocity to be a smaller scale than the horizontal. You may find it easiest to work with a streamfunction $u = \partial\psi/\partial y$. The ODE which results can only be solved numerically.

2. Find the asymptotic behaviour of

$$J_\nu(\nu z) = \frac{1}{2\pi i} \int_{\infty-i\pi}^{\infty+i\pi} \exp[\nu z \sinh t - \nu t] dt$$

for fixed real z with $0 < z < 1$ as $\nu \rightarrow \infty$.

Answers

1. The leading-order scalings are $u = U(\xi)$, $v = x^{-1/2}V(\xi)$ and $p = P_0$ (a constant), in which $\xi = x^{-1/2}y$. A streamfunction gives

$$\psi = x^{1/2}f(\xi) \quad \text{with } U(\xi) = f'(\xi), \quad V(\xi) = \frac{1}{2}[\xi f'(\xi) - f(\xi)].$$

and the resulting ODE is $2f'''(\xi) + f(\xi)f''(\xi) = 0$, with $f(0) = f'(0) = 0$ and $f'(\xi) \rightarrow 1$ as $\xi \rightarrow \infty$.

2. $J_\nu(\nu z) \sim \left(\frac{1}{2\pi\nu(1-z^2)^{1/2}} \right)^{1/2} \exp \left[\nu \left((1-z^2)^{1/2} - \operatorname{arccosh}(1/z) \right) \right].$