Analytical Methods: Exercises 4

1. Look at the problem
$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}+\frac{\mathrm{d}f}{\mathrm{d}x}=\cos x$$
with boundary conditions $f(0)=0$, $f(\pi)=1$. Find the two distinguished stretches for this equation. Calculate the first three terms of the regular expansion, and apply the boundary condition at π to determine the constants.

   Now apply your stretch near $x=0$. Find the first three terms of the inner solution, and apply the boundary condition at $x=0$ to determine some of the constants in this expansion.

   Finally use an intermediate variable to match your two expressions and determine the remaining constants.

2. Calculate three terms of the outer solution of
$$(1+\varepsilon)x^2y'=\varepsilon((1-\varepsilon)xy^2-(1+\varepsilon)x+y^3+2\varepsilon y^2) \quad \text{in} \quad 0<x<1$$
with $y(1)=1$. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Thence find two terms for this inner solution.

3. Find the image of the unit disc $|z-1|\le 1$ under the mapping $w=1/z$.

4. Find the image of $-\pi/2<x<\pi/2$, $0<y<1$ under $w=\sin z$.

5. Find the image of $-\pi/4<x<\pi/4$, $-1<y<1$ under $w=\sin z$.

6. Solve the problem $\nabla^2 u=0$ for $1<r<e^\alpha$, $0<\alpha<\pi$ with boundary conditions
$$\partial u/\partial r(1,\theta)=0 \qquad \partial u/\partial r(e^\alpha,\theta)=\sin\theta \qquad u(r,0)=0 \qquad u(r,\pi)=0$$
   (a) by separation of variables, and
   (b) using the transformation $w=\ln z$.

Answers [Note: question 2 is from Hinch; 3–6 are from Weinberger]

1. $\delta=1$, $\delta=\varepsilon$. Outer: $f=1+\sin x-\varepsilon[1+\cos x]-\varepsilon^2\sin x+\cdots$
   Inner ($x=\varepsilon z$): $f=a_0-a_0e^{-z}+\varepsilon[a_1-a_1e^{-z}+z]+\varepsilon^2[a_2-a_2e^{-z}]+\cdots$
   After matching: $f(z)=1-e^{-z}+\varepsilon[2e^{-z}-2+z]+O(\varepsilon^3)$.

2. Outer $y\sim 1+\varepsilon[1-1/x]+\varepsilon^2[1/2-2/x+3/(2x^2)]+\cdots$.
   Inner (with $x=\varepsilon z$): $y\sim(1+2/z)^{-1/2}+\varepsilon[(1+1/z)(1+2/z)^{-3/2}]+\cdots$

3. $\mathrm{Real}\,(w)\ge 1/2$.

4. Putting $w=\eta+i\xi$, the image is $(\eta/\cosh 1)^2+(\xi/\sinh 1)^2\le 1$, $\xi\ge 0$.

5. Putting $w=\eta+i\xi$, the image is the curvilinear rectangle bounded by the hyperbola $\eta^2-\xi^2=1/2$ and the ellipse $(\eta/\cosh 1)^2+(\xi/\sinh 1)^2=1$.

6. $u(r,\theta)=(r+r^{-1})\sin\theta/(1-e^{-2\alpha})$.