

Analytical Methods: Exercises 3

1. Find where the following operators are hyperbolic, parabolic, and elliptic:

$$(a) \frac{\partial^2 u}{\partial t^2} + t \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} \quad (b) t \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.$$

2. Solve the following PDE with the boundary conditions given:

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{(t+1)^2} \frac{\partial^2 u}{\partial x^2} = 0 \quad u(x, 0) = u(1, t) = u(2, t) = 0.$$

3. Find the distinguished stretches, and the leading term of each solution:

$$\varepsilon^3 \frac{d^3 f}{dx^3} + \varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + f = 0.$$

4. $\varepsilon \frac{d^2 f}{dx^2} + f \frac{df}{dx} - f = 0.$

- (a) Find the scalings $f = \varepsilon^\alpha F$ and stretches $x = a + \varepsilon^\beta z$ at which two dominant terms balance, and sketch these scalings in the α - β plane.
 (b) Hence determine the critical α and β for all three terms to balance.
 (c) Give also the possible values of β if the boundary conditions fix $\alpha = 0$. Find the leading term in an expansion for f in each case.

5. Consider the following equation and boundary conditions:

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$

$$u(-1, y) = u(1, y) = 0 \quad u(x, 1) = 1 - x^2 \quad \varepsilon \frac{\partial u}{\partial y}(x, 0) + u(x, 0) = 0.$$

- (a) Calculate the first two terms of a regular perturbation expansion, ignoring the boundary condition at $y = 0$. Satisfy the other three boundary conditions at leading order only.
 (b) What scaling can be applied to y to find another solution? Calculate two terms of this solution, using only the $y = 0$ boundary condition.
 (c) Taking ε as a normal parameter (i.e. forgetting that it is small), find the full solution to the problem by separating variables. You need not determine all the coefficients in the sum; but find the general solution satisfying the x -boundary conditions.
 (d) Comment on the structure of your general solution when ε is small.

Answers [Note: questions 1 & 2 are from Weinberger]

1. (a) H $t^2 > 4x$; P $t^2 = 4x$; E $t^2 < 4x$. (b) H $xt < 1$; P $xt = 1$; E $xt > 1$.
2. $u(x, t) = \sum_n \alpha_n x^{1/2} (t+1)^{1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) \sin\left(\frac{n\pi \ln(t+1)}{\ln 2}\right).$
3. 1, ε , ε^2 . $f = be^{-x}$; $f = be^{-(x-a)/\varepsilon} + c$; $f = be^{-(x-a)/\varepsilon^2} + c(x-a)/\varepsilon^2 + d$.
4. (a) $\alpha + \beta = 1$, $\alpha < \beta$; $\beta = 1/2$, $\alpha > \beta$; $\alpha = \beta$, $\beta < 1/2$. (b) $\alpha = \beta = 1/2$.
 (c) $\beta = 0$: $f_0 = a + x$. $\beta = 1$: any of $F_0 = \text{constant}$, $F_0 = 2(z+b)^{-1}$, $F_0 = -2k \tan[k(z+b)]$, $2k \tanh[k(z+b)]$, or $2k \coth[k(z+b)]$.
5. (a) $u \sim 1 - x^2 + \varepsilon(2y + f_1(x)) + \dots$
 (b) The scaling is $y = a + \varepsilon Y$ giving $f \sim A_0(x)e^{-Y} + \varepsilon A_1(x)e^{-Y} + \dots$.
 (c) $u = \sum_n a_n \cos\left[\frac{(2n+1)\pi x}{2}\right] (c_n \exp[m_1 y] + d_n \exp[m_2 y])$ with
 $m_1, m_2 = [-1 \pm \sqrt{1 + (2n+1)^2 \pi^2 \varepsilon^2}]/2\varepsilon.$