Analytical Methods: Exercises 2

1. Try a dilation transformation on the Burger's equation: $u_t + uu_x = 0$. Find the specific solution for initial conditions

$$u(x,1) = \frac{x + (x^2-1)^{1/2}}{2}$$

and show it matches that obtained by the method of characteristics.

2. Find the distinguished scalings, and the first two terms in the expansion of each root, for the following equation:

$$\varepsilon x^3 + x^2 + (2-\varepsilon)x + 1 = 0.$$

3. Find the first two terms of all four roots of $\varepsilon x^4 - x^2 - x + 2 = 0$.

4. Work out the first two terms in an expansion of each solution to $xe^{-x} = \varepsilon$.

5. Verify that the function

$$u = \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x',t')\,\mathrm{d}x'\,\mathrm{d}t'$$

satisfies the inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = F(x,t)$.

6. [Weinberger p.40] Find the characteristics through $(0,1)$ for the equation

$$\frac{\partial^2 u}{\partial t^2} - e^{2x}\frac{\partial^2 u}{\partial x^2} = 0.$$

7. Find two terms of a regular perturbation expansion for $f(x,t)$ in:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \varepsilon \cos x f = x$$

with boundary conditions $f(x,0) = \partial f/\partial t(x,0) = 0$. This particular problem can be solved in the same way even if $\varepsilon = 1$: this is the method of *successive approximations*. [Ref: Weinberger p. 384.]

Answers

1. $u = t^{m-1}f(\xi)$ with $\xi = t^{-m}x$ and $(f(\xi) - m\xi)f'(\xi) + (m-1)f(\xi) = 0$. Specific solution $u = (x/t + [(x/t)^2 - t^{-1}]^{1/2})/2$.

2. Scalings $x \sim 1$ and $x \sim \varepsilon^{-1}$; roots $x = -1-2\varepsilon+O(\varepsilon^2)$, $x = -\varepsilon^{-1}+2+O(\varepsilon)$, $x = -1$ (exact solution, no further terms).

3. $x \sim 1 + \varepsilon/3$; $x \sim -2 - 16\varepsilon/3$; $x \sim \varepsilon^{-1/2} + 1/2$; $x \sim -\varepsilon^{-1/2} + 1/2$.

4. $xe^{-x} = \varepsilon$. There are two roots: $x \sim \varepsilon + \varepsilon^2$ and $x \sim \ln(1/\varepsilon) - \ln(\ln(1/\varepsilon))$.

6. $t = e^x$ and $t = 2 - e^x$.

7. $f(x,t) = \frac{1}{2}xt^2 + \varepsilon\left[\frac{1}{2}t^2(x\cos x - 2\sin x) - x\cos x + 4\sin x\right.$ $\left. + \frac{1}{2}(x+t)\cos(x+t) + \frac{1}{2}(x-t)\cos(x-t) - 2\sin(x+t) - 2\sin(x-t)\right]$