Analytical Methods: Exercises 1

1. Try a regular perturbation expansion in the following differential equation:
$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1, \qquad y(0) = 0, \qquad y(\pi/2) = 0.$$
Calculate the first three terms, that is, up to order ε^2. Apply the boundary conditions at each order.

2. Calculate the first two nonzero terms of a regular expansion in ε for the following integral:
$$I = \int_0^\varepsilon \frac{\mathrm{d}x}{(\varepsilon^2 - x^2 + \cos\varepsilon - \cos x)^{1/2}}.$$
[Hint: you will need to keep terms of order ε^4 initially.]

3. Find the general solution to the PDE for $f(\theta, \phi)$:
$$\frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta f) + \frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}(v_\phi f) + \sin^2\theta\cos 2\phi = 0$$
in which
$$\begin{aligned} v_\theta &= a\sin\theta\cos\theta\cos 2\phi \\ v_\phi &= -a\sin\theta\sin 2\phi \end{aligned}$$

4. Consider the problem
$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0$$
in $x \geq 0$, $t \geq 0$, with initial and boundary conditions
$$u(x,0) = \sqrt{x} \qquad u(0,t) = 0.$$
Find the general solution implicitly and hence the specific solution in this case.

5. How would you expect the diameter of a spider's web to scale with L, the bodylength of the spider? Be clear about any assumptions you make.

Answers

1. $y = 1 - \cos x - \sin x + \varepsilon[(x - \pi/2)\sin x + x\cos x]$
$\quad - \varepsilon^2[1 + (x^2/2 - \pi x/2 - 1 + \pi^2/8)\sin x + (x^2/2 - 1)\cos x].$

2. $I = \dfrac{\pi}{\sqrt{2}}\left(1 - \dfrac{\varepsilon^2}{16} + O(\varepsilon^4)\right).$

3. $f(\theta,\phi) = F(\sin 2\phi\tan^2\theta)\sec^3\theta + \dfrac{1}{3}.$

4. Implicit solution $u = F(x - u^2 t)$, particular solution $u(x,t) = \sqrt{\dfrac{x}{(1+t)}}.$

5. If we assume that two quantities: the thickness of the spun fibre, and the hole-size in the finished web, are both independent of spider size, then the diameter scales as $L^{3/2}$.