A priori error estimates for the finite element discretization of optimal distributed control problems governed by the biharmonic operator

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Abstract

In this article a priori error estimates are derived for the finite element discretization of optimal distributed control problems governed by the biharmonic operator. The state equation is discretized in primal mixed form using continuous piecewise biquadratic finite elements, while piecewise constant approximations are used for the control. The error estimates derived for the state variable as well as that for the control are order-optimal on general unstructured meshes. However, on uniform meshes not all error estimates are optimal due to the low-order control approximation. All theoretical results are confirmed by numerical tests.

1 Introduction

During the last decade the discretization of optimal control problems involving *second-order* elliptic partial differential equations with additional inequality constraints has been extensively studied. First L^2-error estimates for the simpler case of pure control-constraints have been obtained by Falk [19] and Geveci [21] for distributed controls. An overview including Neumann control can be found in Malanowski [29]. All these papers consider a linear state equation and discretization using a piecewise constant approximation of the control and continuous piecewise linear finite elements for the state. These results have been extended to semilinear equations in Arada et al. [2] and Casas et al. [10] for the case of Neumann boundary control. For Dirichlet boundary control there has up to now only been given an analysis in the unconstrained case by May et al. [30] on polygonal domains and by Deckelnick et al. [17] on domains with curved boundaries. Continuous piecewise linear finite elements for approximating the control space has been considered in Rösch [41] and Casas & Tröltzsch [9]. Convergence results in L^∞ have been derived by Meyer & Rösch [35]. Further, it could be shown that certain post-processing enhances the convergence of the discrete control, see Meyer & Rösch [34] for a scalar state equation, and Rösch & Vexler [42] for the Stokes equation. By Hinze [25] a so called *variational discretization* has been introduced where the discretization for the control variable is implicitly induced by the discretization of the adjoint state through the necessary optimality conditions. Recently, an analysis of mixed finite elements has been considered for the discretization of the state equation in Chen et al. [12] and Xing et al. [45].

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In the case of semilinear state equations with pointwise state constraints Casas & Mateos [8] have shown asymptotic convergence, followed by results of Deckelnick & Hinze [15], which also yield convergence rates for the variational discretization. For piecewise constant control approximations convergence rates have been obtained in Deckelnick & Hinze [14]. The case of piecewise linear control approximation has been discussed in Meyer [33]. These results yield optimal convergence rates for the control while the rates for the state are only suboptimal. The first optimal result on the convergence rate for the state has been obtained in Merino et al. [32] for the case of a finite dimensional control space.

For the case of constraints on the gradient of the state variable, we refer to Deckelnick et al. [16] who have shown convergence for a variational discretization in combination with a mixed discretization of the state equation and additional control constraints. In Günther & Hinze [22] and Ortner & Wollner [37] this analysis has been extended to the case of piecewise constant control approximation. The case of piecewise linear continuous control approximation has been considered in Ortner & Wollner [37].

So far not much attention has been paid to state equations of higher order, relevant references are Bégis & Glowinski [3], Di Iorio & Toscano [18], Krabs [27], Outrata et al. [38], and Adams et al. [1], and also He & Moroşanu [23], for biharmonic variational inequalities. However, none of these papers contain error estimates for the approximate solutions. As a first step in this direction, we derive a priori error estimates for the finite element approximation of an optimal distributed control problem governed by the fourth-order biharmonic operator. The discretization is based on a primal mixed formulation due to Herrmann [24] and Miyoshi [36]. In particular, we derive error estimates for the approximate controls and states both on arbitrary unstructured meshes as well as on certain uniform meshes that allow for some superconvergence results.

From an application point of view the interest in higher order elliptic equations as constraints in optimization problems is two fold. First, of course, the use of plate models as constraints in optimization and similarly identification problems is of interest of its own as indicated by the above references. Further applications occur in fluid mechanics in the context of the stream function formulation of the Navier-Stokes equations. Second, mixed formulations of fourth order problems have a close connection to necessary conditions for optimization problems governed by second order partial differential equations. Thus, the results here may be of further use in the analysis of bi-level optimization problems.

The choice of a mixed discretization of the biharmonic problem is thus on one hand motivated by the connection to the bi-level optimization case and on the other hand by the possibility to approximate this using H^1-conforming finite elements instead of the more expensive H^2-conforming elements or the use of a non conforming method. The choice for the particular mixed discretization is rather arbitrary and other mixed methods can be analyzed quite similar as we will show at the end of the article for the Ciarlet-Raviart mixed scheme see Ciarlet & Raviart [13], or the simultaneous work of Mercier [31].

This article is structured as follows. Next, in Section 2, we formulate the model problem and recall several known results on the well-posedness of the state equation. In Section 3, we discuss the discretization of the optimal control problem and derive some new error estimates for the approximation of the state equation. The main results of this article, the a priori error estimates for the approximate controls and states, are derived in Section 4 accompanied by the results of some numerical tests. We conclude this article with a sketch of an application in 2d fluid mechanics in Section 5.

2 Problem

We consider the following optimal control problem governed by a fourth-order equation in (primal) mixed formulation:

$$\min_{\substack{\{u,\sigma\}\in U\times M\\ q\in Q^{\rm ad}}} J(q,u) = \tfrac{1}{2}\|u-u^D\|^2 + \tfrac{\alpha}{2}\|q\|^2, \tag{1a}$$

$$\text{subject to} \quad \begin{cases} (\sigma,\psi)+(\nabla u,\operatorname{div}\psi)=0 & \forall\psi\in M,\\ (\nabla\varphi,\operatorname{div}\sigma)=-(q,\varphi) & \forall\varphi\in U. \end{cases} \tag{1b}$$

The spaces M and U are defined as

$$M=\left\{v=(v_{ij})_{1\le i,j\le 2}\,\middle|\, v_{ij}\in H^1(\Omega);\ v_{12}=v_{21}\right\},\quad U=H_0^1(\Omega).$$

The domain $\Omega\subset\mathbb{R}^2$ is assumed to be convex polygonal and $H_0^1(\Omega)\subset H^1(\Omega)$ and $H^k(\Omega)$ are the usual Sobolev spaces over Ω. The further notation of function spaces is standard and self-explanatory. Throughout, $(\cdot,\cdot)$ denotes the L^2 or the L^2-tensor scalar product, respectively, depending on the case at hand, and $\|\cdot\|$ the corresponding norm. For any measurable subset $S\subset\Omega$, we write $(\cdot,\cdot)_S:=(\cdot,\cdot)_{L^2(S)}$ and analogously for the corresponding norm.

Let $\alpha>0$ and u^D a given function in $L^2(\Omega)$. The set $Q^{\rm ad}\subset Q:=L^2(\Omega)$ of admissible controls is defined by pointwise constraints, i.e.,

$$Q^{\rm ad}=\left\{q\in Q\,\middle|\, q^a\le q(x)\le q^b \text{ a.e. in }\Omega\right\}, \tag{2}$$

where $q^a,q^b\in\mathbb{R}\cup\{\pm\infty\}$, $q^a\le q^b$, are given.

An example for the above setting is plate bending. In this case u describes the vertical deflection of a thin clamped plate, σ is the tensor of bending moments and q a force density acting vertically on the plate. The state equation (1b) has been extensively studied in the literature and is mostly referred to as *Hermann-Miyoshi mixed formulation*, see for example Brezzi & Raviart [7] or Rannacher [39]. The original references are Herrmann [24] and Miyoshi [36].

Since (1b) has a unique solution $\{u,\sigma\}$ for every right-hand side $q\in H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$), we can write $u(q)$ and $\sigma(q)$. Using this notation, we define the reduced cost functional $j:L^2(\Omega)\to\mathbb{R}$ by

$$j(q):=J(q,u(q)).$$

Then, the formulation of problem (1) simplifies to

$$\min_{q\in Q^{\rm ad}} j(q). \tag{3}$$

Since $\alpha>0$, the reduced cost functional is strictly convex on $Q=L^2(\Omega)$. Thus, existence and uniqueness of a solution to (1) can be shown by standard arguments, see for example Lions [28] or Tröltzsch [44].

2.1 Optimality conditions

By standard arguments one sees that for a solution $\{\bar q,\bar u,\bar\sigma\}\in Q^{\rm ad}\times U\times M$ of (1) there exists an adjoint state $\{\bar\lambda^u,\bar\lambda^\sigma\}\in U\times M$, which solves

$$\begin{aligned}(\psi,\bar\lambda^\sigma)+(\nabla\bar\lambda^u,\operatorname{div}\psi)&=0 && \forall\psi\in M,\\ (\nabla\varphi,\operatorname{div}\bar\lambda^\sigma)&=-(\bar u-u^D,\varphi) && \forall\varphi\in U.\end{aligned} \tag{4}$$

Since the above equation defines a mapping $u \mapsto \{\lambda^u, \lambda^\sigma\}$, we will sometimes denote the tuple $\{\lambda^u, \lambda^\sigma\}$ by $\{\lambda^u(q), \lambda^\sigma(q)\}$, in order to indicate the dependence of $\{\lambda^u, \lambda^\sigma\}$ on a given control q through to the mapping $q \mapsto u(q) \mapsto \{\lambda^u(q), \lambda^\sigma(q)\}$. To summarize, by convexity an element $\{\bar q, \bar u, \bar\sigma\} \in Q^{\mathrm{ad}} \times U \times M$ solves (1) if and only if there exists $\{\bar\lambda^u, \bar\lambda^\sigma\} \in U \times M$, such that the element $\{\bar q, \bar u, \bar\sigma, \bar\lambda^u, \bar\lambda^\sigma\} \in Q^{\mathrm{ad}} \times U \times M \times U \times M$ solves the so-called *Karush-Kuhn-Tucker (KKT) system*

$$
\begin{aligned}
(\bar\sigma, \psi) + (\nabla \bar u, \operatorname{div} \psi) &= 0 && \forall \psi \in M, && \text{(5a)}\\
(\nabla \varphi, \operatorname{div} \bar\sigma) &= -(\bar q, \varphi) && \forall \varphi \in U, && \text{(5b)}\\
(\bar\lambda^\sigma, \psi) + (\nabla \bar\lambda^u, \operatorname{div} \psi) &= 0 && \forall \psi \in M, && \text{(5c)}\\
(\nabla \varphi, \operatorname{div} \bar\lambda^\sigma) &= -(\bar u - u^D, \varphi) && \forall \varphi \in U, && \text{(5d)}\\
(\bar\lambda^u + \alpha \bar q, p - \bar q) &\ge 0 && \forall p \in Q^{ad}. && \text{(5e)}
\end{aligned}
$$

Here, since (5e) holds in L^2, it holds pointwise a.e.. Thus, using the projection operator

$$P_{[a,b]}(f) := \min\{b, \max\{a, f\}\},$$

we have the following representation for the optimal control $\bar q$ a.e. in $x \in \Omega$:

$$\bar q(x) = P_{[q^a, q^b]}\Big(-\frac{1}{\alpha}\bar\lambda^u(x)\Big). \tag{6}$$

For clarity of notation, we emphasize that the solution $\{\bar q, \bar u, \bar\sigma, \bar\lambda^u, \bar\lambda^\sigma\}$ of the fully coupled KKT-system (5), including the coupling condition (5e), is indicated by bars. In contrast to that, we use the notation $\{u, \sigma\} = \{u(q), \sigma(q)\}$, without bars, for the solution of the state equation (5a), (5b), corresponding to the right-hand side $-(q, \cdot)$, and $\{u, \sigma, \lambda^u, \lambda^\sigma\} = \{u(q), \sigma(q), \lambda^u(q), \lambda^\sigma(q)\}$ for the solution of the coupled system (5a)–(5d) also corresponding to the right-hand side $-(q, \cdot)$. For the derivative of the reduced (quadratic) functional $j(\cdot)$ we find

$$j'(q)(r) := \lim_{t \to 0} t^{-1}\{j(q + tr) - j(q)\} = (u(q) - u^D, u(r)) + \alpha(q, r). \tag{7}$$

Using in the following order the second equation in (4), the first equation in (1b), the first equation in (4), and the second equation in (1b), we obtain

$$
\begin{aligned}
(u(q) - u^D, u(r)) &= -(\nabla u(r), \operatorname{div} \lambda^\sigma(q)) = (\sigma(r), \lambda^\sigma(q))\\
&= -(\nabla \lambda^u(q), \operatorname{div} \sigma(r)) = (\lambda^u(q), r).
\end{aligned}
$$

This implies that

$$j'(q)(r) = (\lambda^u(q) + \alpha q, r). \tag{8}$$

2.2 Regularity of solutions

On a convex polygonal domain Ω, for given $q \in H^{-1}(\Omega)$, the solution $u(q)$ of (1b) belongs to $H^3(\Omega)$ and even to $H^4(\Omega)$ if $q \in L^2(\Omega)$ and all interior angles of Ω are of less than $126.283°$ (see Blum & Rannacher [5]). In particular the following a priori estimate holds:

$$\|u(q)\|_{H^k} + \|\sigma(q)\|_{H^{k-2}} \le C\|q\|_{H^{k-4}}, \quad k \in \{2,3,4\}. \tag{9}$$

The same holds for the adjoint variables $\lambda^u(q)$ and $\lambda^\sigma(q)$:

$$\|\lambda^u(q)\|_{H^k} + \|\lambda^\sigma(q)\|_{H^{k-2}} \le C\|u(q) - u^D\|_{H^{k-4}}, \quad k \in \{2,3,4\}. \tag{10}$$

In view of (6) and Kinderlehrer & Stampacchia [26] this immediately implies that the optimal control satisfies $\bar{q} \in W^{1,\infty}(\Omega)$ because $\bar{\lambda}^u(q) \in H^3(\Omega) \subset W^{1,\infty}(\Omega)$. In the above a priori estimates, as well as in those occurring below, the generic constant C does not depend on the functions involved.

3 Discretization

Let $(\mathcal{T}_h)_{h>0}$ be a quasi-uniform family of regular decompositions of $\overline{\Omega}$, in the sense of Brenner & Scott [6], into closed triangles or quadrilaterals T, of maximum width $h \in \mathbb{R}_+$. Such meshes have a *superconvergence property* if all their edges are parallel to any one of three resp. two fixed coordinate directions. If this superconvergence property holds for a mesh we refer to this as *sc-mesh*.

On these meshes, we define spaces $\mathcal{Q}_h^{(2)}$ of H^1-conforming P_2 resp. Q_2-finite elements (*biquadratic* elements). Then, we define the spaces for the approximation of the state equation by

$$M_h = M \cap (\mathcal{Q}_h^{(2)})^{2\times 2}, \quad U_h = U \cap \mathcal{Q}_h^{(2)}.$$

As in the continuous case, for given $q \in Q^{\mathrm{ad}}$ let $\{u_h(q), \sigma_h(q)\} \in U_h \times M_h$ be the solution to the discrete state equation

$$(\sigma_h, \psi_h) + (\nabla u_h, \operatorname{div} \psi_h) = 0 \qquad \forall \psi_h \in M_h, \tag{11a}$$

$$(\nabla \varphi_h, \operatorname{div} \sigma_h) = -(q, \varphi_h) \quad \forall \varphi_h \in U_h. \tag{11b}$$

The control space is discretized by cell wise constant P_0-finite elements resulting in the discrete admissible sets

$$Q_h := \{q_h \in Q,\ q_h|_T \in P_0,\ T \in \mathcal{T}_h\}, \quad Q_h^{\mathrm{ad}} := Q_h \cap Q^{\mathrm{ad}}.$$

We note that the choice for quadratic/biquadratic elements is due to the fact, that linear/bilinear finite elements would require sc-meshes for the convergence of the Herrmann/Miyoshi scheme, see [36]. The Ciarlet-Raviart method to be used in Section 5 would work on arbitrary meshes, but with reduced order, see [43].

On the other hand, higher order elements can easily be analyzed by techniques analog to those presented here.

In what follows, we will analyze the convergence of the discretization for the state problem. We will consider only the case when the state is sufficiently regular which is asserted considering only convex domains. For an analysis of the convergence on non-convex domains as well as for other (mixed) boundary conditions we refer to the forthcoming publication of Blum & Rannacher [4].

3.1 A priori error estimates for the Herrmann-Miyoshi scheme

From Rannacher [39] and Chen [11], we recall the following a priori error estimates for the Herrmann-Miyoshi scheme applied to the plate bending problem. In these error estimates, as well as in those occurring below, the generic constant C does not depend on the mesh-size parameter h and the functions involved.

Lemma 3.1. *Let $\{u(q), \sigma(q)\} \in U \times M$ and $\{u_h(q), \sigma_h(q)\} \in U_h \times M_h$ be the solutions of the state equation* (1b) *and its discrete analog* (11)*, respectively. The following estimates hold.*
(i) On convex polygonal domains and general quasi-uniform meshes:

$$\|u(q) - u_h(q)\|_{H^1} + h\|\sigma(q) - \sigma_h(q)\| \le C h^2 \|u(q)\|_{H^3}. \tag{12}$$

(ii) On polygonal domains with maximum interior angle of less than $126,283°$ *and uniform sc-meshes:*

$$\|u(q) - u_h(q)\| + h\|\sigma(q) - \sigma_h(q)\| \le Ch^3\|u(q)\|_{H^4}. \tag{13}$$

For the derivation of corresponding optimal-order error estimates for the approximation of the optimization problem, we will need further error estimates of the form (12) and (13) for the bending moment variables in the weaker H^{-1}-norm. Since such estimates do not seem to be available in the literature, we provide a sketch of the proof.

Lemma 3.2. *Under the conditions of Lemma 3.1, the following estimates hold.*
(i) On convex polygonal domains and general quasi-uniform meshes:

$$\|\sigma(q) - \sigma_h(q)\|_{H^{-1}} \le Ch^2\|u(q)\|_{H^3}. \tag{14}$$

(ii) On polygonal domains with maximum interior angle of less than $126,283°$ *and uniform sc-meshes:*

$$\|\sigma(q) - \sigma_h(q)\|_{H^{-2}} \le Ch^3\|u(q)\|_{H^4}. \tag{15}$$

Proof. For simplicity, we set $\sigma = \sigma(q)$ and $\sigma_h = \sigma_h(q)$. We use a duality argument. Let $\eta \in M$ be arbitrary and let $\{z^u, z^\sigma\} \in U \times M$ be the unique solution of the problem

$$(z^\sigma, \psi) + (\nabla z^u, \operatorname{div}\psi) = (\eta, \psi) \quad \forall \psi \in M, \tag{16}$$
$$(\nabla\varphi, \operatorname{div} z^\sigma) = 0 \qquad \forall \varphi \in U. \tag{17}$$

We test (16) by $\psi = \sigma - \sigma_h$ obtaining

$$(\sigma - \sigma_h, \eta) = (z^\sigma, \sigma - \sigma_h) + (\nabla z^u, \operatorname{div}(\sigma - \sigma_h)). \tag{18}$$

To get an estimate for the right-hand side of (18), we need an interpolation operator $I_h : H^1(\Omega) \to \mathcal{Q}^{(2)}$, which satisfies

$$\begin{aligned} \|v - I_h v\| &\le Ch^m\|v\|_{H^m}, \\ \|v - I_h v\|_{H^1} &\le Ch^{m-1}\|v\|_{H^m}, \end{aligned} \tag{19}$$

for $1 \le m \le 3$ and $v \in H^m(\Omega)$. Such an operator is provided, for example, by the Scott-Zhang operator (cf. Brenner & Scott [6], Section 4.8). For $v \in H^1(\Omega)^{2\times 2}$, we define I_h component wise. Now, we use the Galerkin orthogonality property

$$(\sigma - \sigma_h, \psi_h) + (\nabla(u - u_h), \operatorname{div}\psi_h) = 0 \quad \forall \psi_h \in M_h,$$
$$(\nabla\varphi_h, \operatorname{div}(\sigma - \sigma_h)) = 0 \quad \forall \varphi_h \in U_h.$$

Taking $\psi_h = I_h z^\sigma$ and $\varphi_h = I_h z^u$, we obtain from (18) that

$$\begin{aligned} (\sigma - \sigma_h, \eta) = (z^\sigma - I_h z^\sigma, \sigma - \sigma_h) &+ (\nabla(u - u_h), \operatorname{div} I_h z^\sigma) \\ &+ (\nabla(z^u - I_h z^u), \operatorname{div}(\sigma - \sigma_h)). \end{aligned}$$

For the second term on the right, we use (17) to obtain

$$(\nabla(u - u_h), \operatorname{div} I_h z^\sigma) = (\nabla(u - u_h), \operatorname{div}(I_h z^\sigma - z^\sigma)).$$

Hence by the Cauchy-Schwarz inequality,

$$\begin{aligned} (\sigma - \sigma_h, \eta) \le \|z^\sigma - I_h z^\sigma\|\|\sigma - \sigma_h\| &+ C\|I_h z^\sigma - z^\sigma\|_{H^1}\|u - u_h\|_{H^1} \\ &+ C\|z^u - I_h z^u\|_{H^1}\|\sigma - \sigma_h\|_{H^1}. \end{aligned} \tag{20}$$

(i) We derive the error bound (14) an general quasi-uniform meshes. To this end, we estimate the terms in (20) one by one. For the first and second term, we obtain

$$\|z^\sigma - I_h z^\sigma\| \|\sigma - \sigma_h\| \le Ch^2 \|z^\sigma\|_{H^1} |u|_{H^3} \le Ch^2 \|\eta\|_{H^1} \|u\|_{H^3}, \tag{21}$$

and

$$\|I_h z^\sigma - z^\sigma\|_{H^1} \|u - u_h\|_{H^1} \le Ch^2 \|z^\sigma\|_{H^1} \|u\|_{H^3} \le Ch^2 \|\eta\|_{H^1} \|u\|_{H^3}. \tag{22}$$

Splitting the third term as

$$\|\sigma - \sigma_h\|_{H^1} \le \|\sigma - I_h\sigma\|_{H^1} + \|I_h\sigma - \sigma_h\|_{H^1},$$

we obtain from (19)

$$\|\sigma - I_h\sigma\|_{H^1} \le C\|\sigma\|_{H^1} \le C\|u\|_{H^3}, \tag{23}$$

and using an inverse estimate for finite elements

$$\begin{aligned} \|I_h\sigma - \sigma_h\|_{H^1} &\le Ch^{-1}\|I_h\sigma - \sigma_h\| \le Ch^{-1}\{\|I_h\sigma - \sigma\| + \|\sigma - \sigma_h\|\} \\ &\le C\{\|\sigma\|_{H^1} + \|u\|_{H^3}\} \le C\|u\|_{H^3}. \end{aligned} \tag{24}$$

Combing (23) and (24) implies

$$\|\sigma - \sigma_h\|_{H^1} \le C\{\|\sigma\|_{H^1} + \|u\|_{H^3}\} \le C\|u\|_{H^3}. \tag{25}$$

Finally, we conclude by (19)

$$\|z^u - I_h z^u\|_{H^1} \le Ch^2 \|z^u\|_{H^3} \le Ch^2 \|\eta\|_{H^1},$$

and thus

$$\|z^u - I_h z^u\|_{H^1} \|\sigma - \sigma_h\|_{H^1} \le Ch^2 \|z^u\|_{H^3} \|u\|_{H^3} \le Ch^2 \|\eta\|_{H^1} \|u\|_{H^3}. \tag{26}$$

Altogether, we showed in (20)–(26) that

$$(\sigma - \sigma_h, \eta) \le Ch^2 \|u\|_{H^3} \|\eta\|_{H^1}.$$

Now, taking the supremum over $\eta \in M$, we obtain the asserted estimate (14).
(ii) To get the improved convergence rate on uniform meshes, we assume that $\eta \in H^2(\Omega)^{2\times 2}$. In this case all the estimates (21), (22), (23), and (24) can be improved by one order yielding

$$\begin{aligned} \|z^\sigma - I_h z^\sigma\| \|\sigma - \sigma_h\| &\le Ch^3 \|\eta\|_{H^2} \|u\|_{H^3}, \\ \|I_h z^\sigma - z^\sigma\|_{H^1} \|u - u_h\|_{H^1} &\le Ch^3 \|\eta\|_{H^2} \|u\|_{H^3}, \\ \|z^u - I_h z^u\|_{H^1} \|\sigma - \sigma_h\|_{H^1} &\le Ch^3 \|\eta\|_{H^1} \|u\|_{H^4}. \end{aligned}$$

Using these estimates in (20) and taking the supremum over $\eta \in H^2(\Omega)^{2\times 2}$ yields the asserted superconvergence estimate (15). □

3.2 Discrete optimal control problem

Finally, we introduce the discrete optimization problem

$$\min_{\substack{q_h \in Q_h^{\text{ad}} \\ \{u_h, \sigma_h\} \in U_h \times M_h}} J(q_h, u_h) = \tfrac{1}{2}\|u_h - u^D\|^2 + \tfrac{\alpha}{2}\|q_h\|^2, \tag{27a}$$

$$\text{subject to} \quad \left\{ \begin{aligned} (\sigma_h, \psi_h) + (\nabla u_h, \operatorname{div}\psi_h) &= 0 && \forall \psi_h \in M_h \\ (\nabla \varphi_h, \operatorname{div}\sigma_h) &= -(q_h, \varphi_h) && \forall \varphi_h \in U_h. \end{aligned} \right. \tag{27b}$$

With the corresponding solution operator, we can define the discrete reduced cost functional $j_h : L^2(\Omega) \to \mathbb{R}$ by

$$j_h(q) := J(q, u_h(q)).$$

As in the continuous case, a point $\{\bar{q}_h, \bar{u}_h, \bar{\sigma}_h\} \in Q_h^{ad} \times U_h \times M_h$ is a solution to the optimization problem (27) if and only if there exists an adjoint state $\{\bar{\lambda}_h^u, \bar{\lambda}_h^\sigma\} \in U_h \times M_h$, such that $\{\bar{q}_h, \bar{u}_h, \bar{\sigma}_h, \bar{\lambda}_h^u, \bar{\lambda}_h^\sigma\} \in Q_h^{ad} \times U_h \times M_h \times U_h \times M_h$ solves the discrete KKT-system:

$$(\bar{\sigma}_h, \psi_h) + (\nabla \bar{u}_h, \operatorname{div} \psi_h) = 0 \qquad \forall \psi_h \in M_h, \tag{28a}$$
$$(\nabla \varphi_h, \operatorname{div} \bar{\sigma}_h) = -(\bar{q}_h, \varphi_h) \qquad \forall \varphi_h \in U_h, \tag{28b}$$
$$(\psi_h, \bar{\lambda}_h^\sigma) + (\nabla \bar{\lambda}_h^u, \operatorname{div} \psi_h) = 0 \qquad \forall \psi_h \in M_h, \tag{28c}$$
$$(\nabla \varphi_h, \operatorname{div} \bar{\lambda}_h^\sigma) = -(\bar{u}_h - u^D, \varphi_h) \qquad \forall \varphi_h \in U_h, \tag{28d}$$
$$(\bar{\lambda}_h^u + \alpha \bar{q}_h, p_h - \bar{q}_h) \geq 0 \qquad \forall p_h \in Q_h^{ad}. \tag{28e}$$

Since we consider piecewise constant control functions, inequality (28e) is also valid element wise and we can express the discrete optimal control via the cell wise projection

$$\bar{q}_h|_T = P_{[q^a, q^b]}\Big(-\frac{1}{\alpha |T|} \int_T \bar{\lambda}_h^u \, dx\Big). \tag{29}$$

As on the continuous level, bars are used to indicate the solution of the full discrete KKT system (28), including the coupling condition (28e). The notation without bars is used for the solutions of the discrete primal state equations and the coupled discrete primal/dual state equations corresponding to the common right-hand side $(-q, \cdot)$.
As on the continuous level, for the derivative of the discrete reduced (quadratic) functional $j_h(\cdot)$, we have the representation

$$j_h'(q_h)(r_h) = (u_h(q_h) - u^D, u_h(r_h)) + \alpha(q_h, r_h) = (\lambda_h^u(q_h) + \alpha q_h, r_h), \tag{30}$$

and also the identity

$$(\lambda_h^u(q_h), r_h) = (u_h(q_h) - u^D, u_h(r_h)), \quad q_h, r_h \in Q_h. \tag{31}$$

4 A priori error estimation

In this section, we derive estimates for the error between the solutions to the optimization problems (1) and (27). To this end, we follow the lines of the standard argument by firstly analyzing the error in the control variable using the strict convexity of the cost functional.

4.1 Error estimates for the control variable

We start with proving two simple lemmas.

Lemma 4.1. *For any $q, r \in L^2(\Omega)$ there holds the estimate*

$$|j'(q)(r) - j_h'(q)(r)| \leq C \|\lambda^u(q) - \lambda_h^u(q)\| \|r\|. \tag{32}$$

Proof. By the representations (8) and (30) it follows that

$$|j'(q)(r) - j_h'(q)(r)| = |(r, \lambda^u(q)) - \big(r, \lambda_h^u(q)\big)| \le C\|r\|\|\lambda^u(q) - \lambda_h^u(q)\|,$$

which proves the assertion. □

Lemma 4.2. *For any $p, q, r \in L^2(\Omega)$ there holds the estimate*

$$\big|j'(q)(r) - j'(p)(r)\big| \le C\|q-p\|\|r\|.$$

Proof. By (8), we have

$$\begin{aligned}\big|j'(q)(r) - j'(p)(r)\big| &= \big|(r, \lambda^u(q) - \lambda^u(p)) + \alpha(q-p, r)\big| \\ &\le C\|r\|\{\|\lambda^u(q) - \lambda^u(p)\| + \|q-p\|\}.\end{aligned}$$

Now, $\{\lambda^u(q) - \lambda^u(p), \lambda^\sigma(q) - \lambda^\sigma(p)\}$ is the solution of the adjoint problem with right-hand side $(u(q) - u(p), \cdot)$. Using (9) and (10), we conclude that

$$\|\lambda^u(q) - \lambda^u(p)\| \le C\|u(q) - u(p)\| \le C\|q-p\|,$$

which proves the assertion. □

Now, we are able to prove our first result for the error in the control approximation.

Theorem 4.3. *Let $\bar q$ and $\bar q_h$ be the optimal controls of problems* (1) *and* (27)*, respectively. Further, let $\lambda^u(\bar q)$ and $\lambda_h^u(\bar q)$ be the corresponding adjoint variables. Then, there holds*

$$\|\bar q - \bar q_h\| \le C\{\|\lambda^u(\bar q) - \lambda_h^u(\bar q)\| + \inf_{p_h \in \widehat{Q}_h} \|\bar q - p_h\|\}, \tag{33}$$

where $C = \mathcal{O}(\alpha^{-1})$. The set $\widehat{Q}_h \subset Q_h^{\mathrm{ad}}$ is defined as

$$\widehat{Q}_h = \{p_h \in Q_h^{\mathrm{ad}} \mid j'(\bar q)(\bar q_h - p_h) \ge 0\}. \tag{34}$$

Proof. Let $p_h \in \widehat{Q}_h$. From (30), we conclude that

$$\begin{aligned}\alpha\|\bar q_h - p_h\|^2 &\le (u_h(\bar q_h - p_h), u_h(\bar q_h - p_h)) + \alpha\,(\bar q_h - p_h, \bar q_h - p_h) \\ &= j_h'(\bar q_h)(\bar q_h - p_h) - j_h'(p_h)(\bar q_h - p_h).\end{aligned}$$

In view of the discrete optimality condition (28e) the first term on the right is non-positive, which implies that

$$\alpha\|\bar q_h - p_h\|^2 \le -j_h'(p_h)(\bar q_h - p_h).$$

By definition of $\widehat{Q}_h$, we can insert the non-negative term $j'(\bar q)(\bar q_h - p_h)$ obtaining

$$\begin{aligned}\alpha\|\bar q_h - p_h\|^2 &\le j'(\bar q)(\bar q_h - p_h) - j_h'(p_h)(\bar q_h - p_h) \\ &= j'(\bar q)(\bar q_h - p_h) - j'(p_h)(\bar q_h - p_h) + j'(p_h)(\bar q_h - p_h) - j_h'(p_h)(\bar q_h - p_h).\end{aligned}$$

Applying Lemma 4.2 and Lemma 4.1, we further deduce

$$\alpha\|\bar q_h - p_h\|^2 \le C\|\bar q_h - p_h\|\big(\|\bar q - p_h\| + \|\lambda^u(\bar q) - \lambda_h^u(\bar q)\|\big),$$

and, consequently,

$$\alpha\|\bar q_h - p_h\| \le C\{\|\bar q - p_h\| + \|\lambda^u(\bar q) - \lambda_h^u(\bar q)\|\}.$$

Then, applying the triangle inequality, we obtain

$$\begin{aligned}\|\bar q-\bar q_h|&\le\|\bar q-p_h\|+\|p_h-\bar q_h\|\\&\le\|\bar q-p_h\|+C\alpha^{-1}\big(\|\bar q-p_h\|+\|\lambda^u(\bar q)-\lambda^u_h(\bar q)\|\big)\\&\le C\alpha^{-1}\|\lambda^u(\bar q)-\lambda^u_h(\bar q)\|+(C\alpha^{-1}+1)\|\bar q-p_h\|,\end{aligned}$$

which completes the proof. □

It remains to estimate the two terms on the right of (33). First, we consider the term $\|\lambda^u(\bar q)-\lambda^u_h(\bar q)\|$. By definition $\lambda^u(\bar q)$ is the solution of the adjoint equation with the state variable $u(\bar q)$ on the right-hand side, while $\lambda^u_h(\bar q)$ is the solution of the discrete adjoint equation with the discrete state variable $u_h(\bar q)$ on the right-hand side.
We introduce new variables for splitting the error into two parts, which can be estimated separately. Let $\{\hat\lambda^u(\bar q),\hat\lambda^\sigma(\bar q)\}\in U\times M$ be the solution of the auxiliary problem

$$\begin{aligned}(\psi,\hat\lambda^\sigma(\bar q))+(\nabla\hat\lambda^u(\bar q),\operatorname{div}\psi)&=0 &&\forall\psi\in M,\\(\nabla\varphi,\operatorname{div}\hat\lambda^\sigma(\bar q))&=-(u_h(\bar q)-u^D,\varphi)&&\forall\varphi\in U.\end{aligned}\tag{35}$$

Clearly, the pair $\{\lambda^u(\bar q)-\hat\lambda^u(\bar q),\lambda^\sigma(\bar q)-\hat\lambda^\sigma(\bar q)\}$ is solution of the adjoint system (35) with right-hand side $-(u(\bar q)-u_h(\bar q),\cdot)$. Hence, using (10), we deduce

$$\|\lambda^u(\bar q)-\hat\lambda^u(\bar q)\|\le C\|u(\bar q)-u_h(\bar q)\|.$$

Then, Lemma 3.1 and (9) give us

$$\|\lambda^u(\bar q)-\hat\lambda^u(\bar q)\|\le Ch^2\|u(\bar q)\|_{H^3}\le Ch^2\|\bar q\|_{H^{-1}}.\tag{36}$$

It remains to estimate the term $\|\hat\lambda^u(\bar q)-\lambda^u_h(\bar q)\|$, which represents the discretization error of the adjoint equation with given right-hand side $(u_h(\bar q)-u^D,\cdot)$. Hence, we can apply Lemma 3.1 obtaining

$$\begin{aligned}\|\lambda^u(\bar q)-\lambda^u_h(\bar q)\|&\le\|\lambda^u(\bar q)-\hat\lambda^u(\bar q)\|+\|\hat\lambda^u(\bar q)-\lambda^u_h(\bar q)\|\\&\le Ch^2\{\|\bar q\|_{H^{-1}}+\|\lambda^u(\bar q)\|_{H^3}\}.\end{aligned}\tag{37}$$

Finally, we want to find an upper bound for $\inf_{p_h\in\widehat Q_h}\|\bar q-p_h\|$. To this end, we give an explicit cell wise definition of a function $p_h\in\widehat Q_h$ (compare Arada et al.[2]), which will turn out to be an optimal-order approximation to $\bar q$.

Lemma 4.4. *The approximation $p_h\in Q_h^{\mathrm{ad}}$ to $\bar q$ cell wise defined by*

$$p_h|_T=\begin{cases}q^a, & \text{if }\int_T\{\lambda^u(\bar q)+\alpha\bar q\}\,dx>0,\\ q^b, & \text{if }\int_T\{\lambda^u(\bar q)+\alpha\bar q\}\,dx<0,\\ |T|^{-1}\int_T\bar q\,dx, & \text{if }\int_T\{\lambda^u(\bar q)+\alpha\bar q\}\,dx=0,\end{cases}\tag{38}$$

satisfies

$$j'(\bar q)(\bar q_h-p_h)\ge0,\tag{39}$$

i.e., $p_h\in\widehat Q_h$, and

$$\|\bar q-p_h\|\le Ch\|\bar q\|_{W^{1,\infty}}.\tag{40}$$

Proof. (i) We show that the inequality (39) holds even cell wise,

$$j'(\bar q)(\bar q_h - p_h) = (\lambda^u(\bar q) + \alpha \bar q, \bar q_h - p_h) = \sum_{T\in\mathscr{T}_h} (\lambda^u(\bar q) + \alpha \bar q, \bar q_h - p_h)_T.$$

We consider the three cases in the definition of p_h separately.

a) $p_h = q^a$ on T:
By definition of p_h, we have $\int_T \{\lambda^u(\bar q) + \alpha \bar q\}\, dx \geq 0$ and therefore

$$(\lambda^u(\bar q) + \alpha \bar q, \bar q_h - p_h)_T = (\bar q_h|_T - q^a) \int_T \{\lambda^u(\bar q) + \alpha \bar q\}\, dx \geq 0.$$

b) $p_h = q^b$ on T:
By definition of p_h, we have $\int_T \{\lambda^u(\bar q) + \alpha \bar q\}\, dx \leq 0$ and therefore

$$(\lambda^u(\bar q) + \alpha \bar q, \bar q_h - p)_T = (\bar q_h|_T - q^b) \int_T \{\lambda^u(\bar q) + \alpha \bar q\}\, dx \geq 0.$$

c) $p_h = |T|^{-1} \int_T \bar q\, dx$ on T:
By definition of p_h, we have $\int_T \{\lambda^u(\bar q) + \alpha \bar q\}\, dx = 0$ and therefore

$$(\lambda^u(\bar q) + \alpha \bar q, \bar q_h - p_h)_T = (\bar q_h|_T - p_h) \int_T \{\lambda^u(\bar q) + \alpha \bar q\}\, dx = 0.$$

This proves the relation (39).

(ii) We show that the error estimate (40) again holds cell wise.

a) $p_h = q^a$ on T:
By definition of p_h, there is a point $x_0 \in T$, at which $(\lambda^u(\bar q) + \alpha \bar q)(x_0) > 0$. In view of the projection formula (6), we conclude $\bar q(x_0) = q^a$. It follows that

$$\begin{aligned} \|\bar q - p_h\|_T^2 &= \int_T |\bar q(x) - \bar q(x_0)|^2 dx \\ &\leq \int_T \Big(\int_0^1 |\nabla \bar q(x_0 + t(x - x_0))(x - x_0)|\, dt \Big)^2 dx \leq h^2 |T| \|\bar q\|_{W^{1,\infty}}^2. \end{aligned}$$

b) $p_h = q^b$ on T:
By definition of p_h there is an $x_0 \in T$ such that $q(x_0) = q^b$. Hence, using the same argument as above, we find

$$\|\bar q - p_h\|_T^2 \leq h^2 |T| \|\bar q\|_{W^{1,\infty}}^2.$$

c) $p_h = |T|^{-1} \int_T \bar q\, dx$ on T:
In this case $p_h|_T$ is the L^2-projection of $\bar q|_T$ and we can apply a standard error estimate to get

$$\|\bar q - p_h\|_T^2 \leq C h^2 |\nabla \bar q|_T^2 \leq C h^2 |T| \|\bar q\|_{W^{1,\infty}}^2.$$

We conclude that

$$\|\bar q - p_h\|^2 = \sum_{T\in\mathscr{T}_h} \|\bar q - p_h\|_T^2 \leq C h^2 \sum_{T\in\mathscr{T}_h} |T| \|\bar q\|_{W^{1,\infty}}^2 = C h^2 \|\bar q\|_{W^{1,\infty}}^2.$$

This completes the proof. □

Combining the foregoing results, i.e. the estimates (33), (37), and (40), and observing that $\|\bar q\|_{H^{-1}} + \|\lambda^u(\bar q)\|_{H^3} \le C(1+\|\bar q\|_{W^{1,\infty}})$, we have proven the following theorem.

Theorem 4.5. *Let $\bar q$ and $\bar q_h$ be the optimal controls of problems* (1) *and* (27)*, respectively. Then, there holds*

$$\|\bar q - \bar q_h\| \le Ch(1+\|\bar q\|_{W^{1,\infty}}), \tag{41}$$

where $C = \mathcal{O}(\alpha^{-1})$.

4.2 Error estimates for the state variable

We will now derive estimates for the errors $u(\bar q) - u_h(\bar q_h)$ and $\sigma(\bar q) - \sigma_h(\bar q_h)$ using appropriate norms. We follow the approach of Meyer & Rösch [34] who have employed these techniques for the case of second-order elliptic operators.

As in the last section, we introduce an auxiliary cell wise constant function $p_h \in Q_h^{\mathrm{ad}}$ for splitting the error into three parts, which will be estimated separately:

$$\|u(\bar q) - u_h(\bar q_h)\| \le \|u(\bar q) - u(p_h)\| + \|u(p_h) - u(\bar q_h)\| + \|u(\bar q_h) + u_h(\bar q_h)\|. \tag{42}$$

The third term on the right-hand side is again the discretization error for the state equation for which Lemma 3.1 yields

$$\|u(\bar q_h) - u_h(\bar q_h)\|_{H^1} \le Ch^2\|u(\bar q_h)\|_{H^3} \le Ch^2\|\bar q_h\|_{H^{-1}} \le Ch^2\|\bar q\|_{W^{1,\infty}}, \tag{43}$$

where the boundedness of $\bar q_h$ follows from the already proven error estimate (41) for the control variable. Hence, we will only be concerned with the first two terms on the right of (42).

For any cell $T \in \mathcal{T}_h$ let m_T denote its midpoint. We define the projection $\mathcal{M}_h : C(\overline{\Omega}) \to Q_h$ cell wise by

$$\mathcal{M}_h(v)|_T := v(m_T), \quad T \in \mathcal{T}_h.$$

We note that for any $v \in H^2(T)$, $T \in \mathcal{T}_h$, there holds (see, e.g., Meyer & Rösch [34])

$$\left| \int_T \{v(x) - v(m_T)\}\,dx \right| \le Ch^2|T|^{1/2}\|v\|_{H^2(T)}. \tag{44}$$

Using this notation, we introduce the auxiliary function

$$p_h := \mathcal{M}_h(\bar q),$$

where $\bar q$ is again the optimal control of problem (1). Then, by standard scaling arguments, we obtain

$$\|\bar q - p_h\|_{L^\infty(T)} \le Ch\|\bar q\|_{W^{1,\infty}(T)}, \tag{45}$$

$$\|\bar q - p_h\|_T \le Ch\|\bar q\|_{H^2(T)}, \tag{46}$$

provided $\bar q \in W^{1,\infty}(T)$ or $\bar q \in H^2(T)$.

For the following argument, we split the set of cells $T \in \mathcal{T}_h$ into an uncritical part $\mathcal{T}_h^1$ where $\bar q$ is smooth, and a remaining critical part $\mathcal{T}_h^2$.

Definition 4.1. *By $\mathcal{T}_h^1$, we denote the set of all cells T in $\mathcal{T}_h$, where the optimal control $\bar q \in Q^{ad}$ fulfills one of the following conditions:*

$$(a)\ \bar q \equiv q^a \ \ on\ T, \quad (b)\ \bar q \equiv q^b \ \ on\ T, \quad (c)\ q^a < \bar q < q^b \ \ on\ T.$$

We denote $\mathcal{T}_h^2 = \mathcal{T}_h \setminus \mathcal{T}_h^1$, i.e., $\mathcal{T}_h^2$ denotes the set of all cells where $\bar{q}$ takes on the values q^a (resp. q^b) as well as values bigger than q^a (resp. smaller than q^b). For the unions of these cells, we will use the following notation:

$$\Omega_h^1 = \mathrm{int}\big(\cup_{T\in\mathcal{T}_h^1} T\big), \quad \Omega_h^2 = \mathrm{int}\big(\cup_{T\in\mathcal{T}_h^2} T\big).$$

To be able to show optimal-order error estimates for the state variable, we will need the following assumption.

Assumption 1. *The mesh domain Ω_h^2 of critical cells is sufficiently small in the following sense:*

$$|\Omega_h^2| = \sum_{T\in\mathcal{T}_h^2} |T| \le Ch. \tag{47}$$

Remark 4.6. *Assumption 1 is crucial for the optimal-order error estimates for the state variable to be derived below. It is clearly satisfied if the boundaries of the sets $-\alpha^{-1}\bar{\lambda}^u = q^a$ and $-\alpha^{-1}\bar{\lambda}^u = q^b$ consist of finitely many smooth curves or points. Since $\bar{\lambda}^u$ is a solution of the biharmonic equation, this property is likely to be true. In our test calculations this conditions has always been satisfied.*

According to the splitting $\bar{\Omega} = \bar{\Omega}_h^1 \cup \bar{\Omega}_h^2$, we introduce the following norm:

$$\|\bar{q}\|_h := \|\bar{q}\|_{H^2(\bar{\Omega}_h^1)} + \|\bar{q}\|_{W^{1,\infty}(\bar{\Omega}_h^2)}.$$

Lemma 4.7. *Suppose that Assumption 1 is satisfied. Then, for $p_h := \mathcal{M}_h(\bar{q})$ the following error estimate holds:*

$$\|u(\bar{q}) - u(p_h)\|_{H^2} + \|\sigma(\bar{q}) - \sigma(p_h)\| \le Ch^2 \|\bar{q}\|_h. \tag{48}$$

Proof. The pair $\{u(\bar{q}) - u(p_h), \sigma(q) - \sigma(p_h)\}$ is the solution of the system

$$\begin{aligned}(\sigma(\bar{q}) - \sigma(p_h), \psi) + (\nabla(u(\bar{q}) - u(p_h)), \operatorname{div}\psi) &= 0 && \forall \psi \in M,\\ (\nabla\varphi, \operatorname{div}(\sigma(\bar{q}) - \sigma(p_h))) &= -(q - p_h, \varphi) && \forall \varphi \in U.\end{aligned}$$

Testing here with $\varphi = u(\bar{q}) - u(p_h)$ and $\psi = \sigma(\bar{q}) - \sigma(p_h)$, we obtain

$$\begin{aligned}\|\sigma(\bar{q}) - \sigma(p_h)\|^2 &= -(\nabla(u(\bar{q}) - u(p_h)), \operatorname{div}(\sigma(\bar{q}) - \sigma(p_h)))\\ &= (\bar{q} - p_h, u(\bar{q}) - u(p_h))\\ &= (\bar{q} - p_h, u(\bar{q}) - u(p_h))_{\bar{\Omega}_h^1} + (\bar{q} - p_h, u(\bar{q}) - u(p_h))_{\bar{\Omega}_h^2}.\end{aligned} \tag{49}$$

For abbreviation, we set $v := u(\bar{q}) - u(p_h)$. First, we estimate on the critical cells in $\mathcal{T}_h^2$. In view of Assumption 1, there holds

$$\begin{aligned}|(\bar{q} - p_h, v)_{\bar{\Omega}_h^2}| &\le \sum_{T\in\mathcal{T}_h^2} |(\bar{q} - p_h, v)_T|\\ &\le \|\bar{q} - p_h\|_{L^\infty(\bar{\Omega}_h^2)} \|v\|_{L^\infty(\bar{\Omega}_h^2)} \sum_{T\in\mathcal{T}_h^2} |\mathcal{T}_h^2|\\ &\le Ch\|\bar{q} - p_h\|_{L^\infty(\bar{\Omega}_h^2)} \|v\|_{L^\infty(\bar{\Omega}_h^2)}.\end{aligned}$$

Hence by the continuous embedding $H^2(\bar{\Omega}) \subset L^\infty(\bar{\Omega}_h^2)$ and the estimate (45), we conclude that

$$|(\bar{q} - p_h, v)_{\bar{\Omega}_h^2}| \le Ch^2 \|\bar{q}\|_{W^{1,\infty}(\bar{\Omega}_h^2)} \|v\|_{H^2(\bar{\Omega}_h^2)}. \tag{50}$$

Next, we estimate on the uncritical cells in $\mathcal{T}_h^1$. There the optimal control $\bar{q}$ belongs to $H^2(\bar{\Omega}_h^1)$. Recalling that $p_h = \mathcal{M}_h(\bar{q})$, we obtain

$$\begin{aligned}
|(\bar{q} - p_h, v)_{\bar{\Omega}_h^1}| &\leq \sum_{T \in \mathcal{T}_h^1} |(\bar{q} - \mathcal{M}_h(\bar{q}), v)_T| \\
&= \sum_{T \in \mathcal{T}_h^1} |(\bar{q} - \mathcal{M}_h(\bar{q}), v - \mathcal{M}_h(v))_T + (\bar{q} - \mathcal{M}_h(\bar{q}), \mathcal{M}_h(v))_T| \\
&\leq \sum_{T \in \mathcal{T}_h^1} \Big\{ \|(\bar{q} - \mathcal{M}_h(\bar{q})\|_T \|v - \mathcal{M}_h(v)\|_T \\
&\qquad + \big|\mathcal{M}_h(v)|_T\big| \Big| \int_T \{\bar{q} - \mathcal{M}_h(\bar{q})\}\, dx \Big| \Big\},
\end{aligned}$$

and, consequently, in view of the estimates (44) and (46),

$$\begin{aligned}
|(\bar{q} - p_h, v)_{\hat{\Omega}_h^1}| &\leq C h^2 \sum_{T \in \mathcal{T}_h^1} \Big\{ \|\bar{q}\|_{H^2(T)} \|v\|_{H^2(T)} + |T|^{1/2} \big|\mathcal{M}_h(v)|_T\big| \|\bar{q}\|_{H^2(T)} \Big\} \\
&= C h^2 \sum_{T \in \mathcal{T}_h^1} \Big\{ \|\bar{q}\|_{H^2(T)} \|v\|_{H^2(T)} + \|\mathcal{M}_h(v)\|_T \|\bar{q}\|_{H^2(T)} \Big\} \\
&\leq C h^2 \sum_{T \in \mathcal{T}_h^1} \|\bar{q}\|_{H^2(T)} \|v\|_{H^2(T)}.
\end{aligned}$$

Thus, we have proved

$$|(\bar{q} - p_h, v)_{\hat{\Omega}_h^1}| \leq C h^2 \|\bar{q}\|_{H^2(\Omega_h^1)} \|v\|_{H^2(\Omega_h^1)}. \tag{51}$$

Altogether, we showed that

$$\|\sigma(\bar{q}) - \sigma(p_h)\|^2 \leq |(\bar{q} - p_h, u(\bar{q}) - u(p_h))| \leq C h^2 \|\bar{q}\|_h \|u(\bar{q}) - u(p_h)\|_{H^2}. \tag{52}$$

Since $u(\bar{q}) - u(p_h) \in H_0^2(\Omega)$, we can apply Poincaré's inequality to obtain

$$\|u(\bar{q}) - u(p_h)\|_{H^2} \leq C|\nabla^2(u(\bar{q}) - u(p_h))| \leq C\|\sigma(\bar{q}) - \sigma(p_h)\|.$$

This together with (52) implies that

$$\|u(\bar{q}) - u(p_h)\|_{H^2} \leq C\|\sigma(\bar{q}) - \sigma(p_h)\| \leq C h^2 \|\bar{q}\|_h,$$

which completes the proof. □

We will need the following consequence of Lemma 4.7

Corollary 4.8. *With the assumptions and notation of Lemma 4.7, there holds the error estimate*

$$\|\lambda^u(\bar{q}) - \lambda_h^u(p_h)\|_{H^1} \leq C h^2 (1 + \|\bar{q}\|_h). \tag{53}$$

Proof. From the a priori estimate (10) and Lemma 4.7, we obtain

$$\|\lambda^u(\bar{q}) - \lambda^u(p_h)\|_{H^1} \leq C\|u(\bar{q}) - u(p_h)\| \leq C h^2 \|\bar{q}\|_h.$$

Further, as in (37) it follows that

$$\|\lambda^u(p_h) - \lambda_h^u(p_h)\|_{H^1} \leq C h^2 \{\|p_h\|_{H^{-1}} + \|\lambda^u(p_h)\|_{H^3}\} \leq C h^2 \{\|\bar{q}\|_h + \|\lambda^u(p_h)\|_{H^3}\}.$$

Combining the foregoing estimates yields

$$\|\lambda^u(\bar q)-\lambda_h^u(p_h)\|_{H^1}\le Ch^2\{\|\bar q\|_h+\|\lambda^u(p_h)\|_{H^3}\}.$$

Finally, using (9), (10), and (45), (46), we conclude that

$$\begin{aligned}\|\lambda^u(p_h)\|_{H^3}&\le\|\lambda^u(\bar q)\|_{H^3}+\|\lambda^u(\bar q)-\lambda^u(p_h)\|_{H^3}\\&\le\|\lambda^u(\bar q)\|_{H^3}+\|\bar q-p_h\|_{H^{-1}}\\&\le\|\lambda^u(\bar q)\|_{H^3}+Ch\|\bar q\|_h\le C(1+h)(1+\|\bar q\|_h),\end{aligned}$$

which completes the proof. □

Next, we want to estimate $\|\bar q_h-p_h\|$. From the optimality condition (6), we see that at the midpoint m_T of a cell $T\in\mathcal{T}_h$, there must hold

$$\{\lambda^u(\bar q)(m_T)+\alpha\bar q(m_T)\}\{\bar q_h(m_T)-\bar q(m_T)\}\ge 0. \tag{54}$$

Recalling $p_h|_T=\mathcal{M}_h(\bar q)|_T\equiv\bar q(m_T)$, (54) becomes

$$\{\lambda^u(\bar q)(m_T)+\alpha p_h(m_T)\}\{\bar q_h(m_T)-p_h(m_T)\}\ge 0.$$

Integrating this over T and summing up over all cells $T\in\mathcal{T}_h$, we obtain

$$(\mathcal{M}_h(\lambda^u(\bar q))+\alpha p_h,\bar q_h-p_h)\ge 0.$$

In addition, testing the discrete optimality condition (5e) with p_h yields

$$(\lambda_h^u(\bar q_h)+\alpha\bar q_h,p_h-\bar q_h)\ge 0.$$

Adding the last two inequalities gives us

$$(\mathcal{M}_h(\lambda^u(\bar q))-\lambda_h^u(\bar q_h)+\alpha(p_h-\bar q_h),\bar q_h-p_h)\ge 0,$$

or, equivalently,

$$\alpha\|\bar q_h-p_h\|^2\le(\mathcal{M}_h(\lambda^u(\bar q))-\lambda_h^u(\bar q_h),\bar q_h-p_h). \tag{55}$$

With these preparations, we are now able to prove the following lemma.

Lemma 4.9. *With the assumption and notation of Lemma 4.7, there holds the superconvergence result*

$$\|\bar q_h-p_h\|\le Ch^2(1+\|\bar q\|_h). \tag{56}$$

Proof. We start from inequality (55). Its right-hand side can be estimated as follows:

$$\begin{aligned}(\mathcal{M}_h(\lambda^u(\bar q))-\lambda_h^u(\bar q_h),\bar q_h-p_h)=&(\mathcal{M}_h(\lambda^u(\bar q))-\lambda^u(\bar q),\bar q_h-p_h)\\&+(\lambda^u(\bar q)-\lambda_h^u(p_h),\bar q_h-p_h)\\&+(\lambda_h^u(p_h)-\lambda_h^u(\bar q_h),\bar q_h-p_h).\end{aligned} \tag{57}$$

For the first term on the right in (57), we use the fact that $\bar q_h-p_h$ is cell wise constant to obtain

$$\begin{aligned}(\mathcal{M}_h(\lambda^u(\bar q))-\lambda^u(\bar q),\bar q_h-p_h)&=\sum_{T\in\mathcal{T}_h}\int_T\{\lambda^u(\bar q)(m_T)-\lambda^u(\bar q)\}\{\bar q_h-p_h\}\,dx\\&=\sum_{T\in\mathcal{T}_h}\{\bar q_h(m_T)-p_h(m_T)\}\int_T\{\lambda^u(\bar q)(m_T)-\lambda^u(\bar q)\}\,dx\\&\le Ch^2\sum_{T\in\mathcal{T}_h}|\bar q_h(m_T)-p_h(m_T)||T|^{1/2}\|\lambda^u(\bar q)\|_{H^2(T)}\\&=Ch^2\sum_{T\in\mathcal{T}_h}\|\bar q_h-p_h\|_T\|\lambda^u(\bar q)\|_{H^2(T)}.\end{aligned}$$

This implies that

$$\big(\mathcal{M}_h(\lambda^u(\bar q)) - \lambda^u(\bar q), \bar q_h - p_h\big) \le Ch^2\|\bar q_h - p_h\|\|\lambda^u(\bar q)\|_{H^2}. \tag{58}$$

For the second term, we get from Corollary 4.8 that

$$\begin{aligned}(\lambda^u(\bar q) - \lambda^u_h(p_h), \bar q_h - p_h) &\le \|\lambda^u(\bar q) - \lambda^u_h(p_h)\|\|\bar q_h - p_h\| \\ &\le Ch^2\{\|\bar q\|_h + \|\lambda^u(\bar q)\|_{H^3}\}\|\bar q_h - p_h\|.\end{aligned} \tag{59}$$

For the third term, in view of (31) and the linear dependence of $u_h(q_h)$ on q_h, we have

$$(\lambda^u_h(p_h) - \lambda^u_h(\bar q_h), \bar q_h - p_h) = (u_h(p_h) - u_h(\bar q_h), u_h(\bar q_h) - u_h(p_h)) \le 0. \tag{60}$$

Altogether, (55)–(60) and (9), (10), yield

$$\|\bar q_h - p_h\|^2 \le Ch^2\{\|\bar q\|_h + \|\lambda^u(\bar q)\|_{H^3}\}\|\bar q_h - p_h\| \le Ch^2\{1 + \|\bar q\|_h\}\|\bar q_h - p_h\|,$$

with a constant $C \sim \alpha^{-1}$. Division by $\|\bar q_h - p\|$ completes the proof. □

Now, we combine the results of Lemma 4.7 and Lemma 4.9 to obtain the following error estimate for the deflection variable:

$$\begin{aligned}\|u(\bar q) - u(\bar q_h)\|_{H^2} &\le \|u(\bar q) - u(p_h)\|_{H^2} + \|u(p_h) - u(\bar q_h)\|_{H^2} \\ &\le \|u(\bar q) - u(p_h)\|_{H^2} + C\|p_h - \bar q_h\| \\ &\le Ch^2(1 + \|\bar q\|_h).\end{aligned} \tag{61}$$

A corresponding error estimate follows for the bending moment variable,

$$\begin{aligned}\|\sigma(\bar q) - \sigma(\bar q_h)\| &\le \|u(\bar q) - u(p_h)\|_{H^2} + \|u(p_h) - u(\bar q_h)\|_{H^2} \\ &\le \|u(\bar q) - u(p_h)\|_{H^2} + C\|p_h - \bar q_h\| \\ &\le Ch^2(1 + \|\bar q\|_h).\end{aligned} \tag{62}$$

With these preparations, we are now able to show the main result of this section, which is mainly a recollection of what we have already shown.

Theorem 4.10. *Suppose that Assumption 1 is satisfied and let* $\{u(\bar q), \sigma(\bar q)\}$ *be the optimal states of problem* (1) *and* $\{u_h(\bar q_h), \sigma_h(\bar q_h)\}$ *those of problem* (27)*, respectively. Then, on regular quasi-uniform meshes there holds*

$$\|u(\bar q) - u_h(\bar q_h)\|_{H^1} + \|\sigma(\bar q) - \sigma_h(\bar q_h)\|_{H^{-1}} + h\|\sigma(\bar q) - \sigma_h(\bar q_h)\| \le Ch^2(1 + \|\bar q\|_h). \tag{63}$$

Further, if $u(\bar q) \in H^4(\Omega)$*, on uniform sc-meshes there holds*

$$\|\sigma(\bar q) - \sigma_h(\bar q_h)\| \le Ch^2(1 + \|\bar q\|_h). \tag{64}$$

Proof. In view of (61) and (62) it remains to estimate the error terms $\|u(\bar q_h) - u_h(\bar q_h)\|_{H^1}$, $\|\sigma(\bar q_h) - \sigma_h(\bar q_h)\|_{H^{-1}}$, and $\|\sigma(\bar q_h) - \sigma_h(\bar q_h)\|$, on quasi-uniform regular meshes, and $\|\sigma(\bar q_h) - \sigma_h(\bar q_h)\|$ on uniform sc-meshes. On quasi-uniform regular meshes Lemma 3.1 and Lemma 3.2 yield

$$\begin{aligned}\|u(\bar q_h) - u_h(\bar q_h)\|_{H^1} + \|\sigma(\bar q_h) - \sigma_h(\bar q_h)\|_{H^{-1}} + h\|\sigma(\bar q_h) - \sigma_h(\bar q_h)\| \\ \le Ch^2\|u(\bar q_h)\|_{H^3} \le Ch^2\|\bar q_h\|.\end{aligned}$$

On uniform sc-meshes Lemma 3.1 yields

$$\|\sigma(\bar q_h) - \sigma_h(\bar q_h)\| \le Ch^2\|\bar q_h\|.$$

Now, the result of Theorem 4.5 completes the proof by the estimate

$$\|\bar q_h\| \le C\|q\|_h + Ch\|\lambda(\bar q)\|_{H^3} \le C(1 + \|\bar q\|_h),$$

for $h \le 1$. □

4.3 Numerical results

In this chapter, we present the results of numerical tests, which confirm the theoretical predictions of our a priori error analysis. The computations have been done using the software packages RoDoBo [40] and Gascoigne [20].

The first test is on a family of uniform sc-meshes whose edges are all parallel to the Cartesian axes. Starting from the unit square, in every step the next finer mesh has been obtained by dividing every element into four sub-elements of equal size (global uniform refinement). Afterwards these uniform meshes have been systematically distorted in order to eliminate superconvergence effects. Every node of the uniform mesh (except the nodes on the boundary) has been randomly moved, between $0 < \varepsilon_x < 0.02 \cdot h$ in horizontal direction and $0 < \varepsilon_y < 0.02 \cdot h$ in vertical direction. On the obtained family of distorted meshes the same test problem has been solved.

The test problem is as follows:

$$\min_{\substack{\{u,\sigma\}\in U\times M\\ q\in Q^{\mathrm{ad}}}} J(q,u) = \tfrac{1}{2}\|u-u^D\|^2 + \tfrac{\alpha}{2}\|q\|^2, \tag{65a}$$

$$\text{subject to} \quad \begin{cases} (\sigma,\psi)+(\nabla u,\operatorname{div}\psi)=0 & \forall \psi\in M,\\ (\nabla\varphi,\operatorname{div}\sigma)=-(f+q,\varphi) & \forall\varphi\in U. \end{cases} \tag{65b}$$

Here, we chose $\Omega=(0,1)^2$, $\alpha=10^{-3}$, and

$$Q^{\mathrm{ad}} = \left\{q\in L^2(\Omega),\ -750\le q\le -50 \text{ a.e. in } \Omega\right\}.$$

Further, we define

$$f(x,y) = \Delta^2\left(\sin^2(\pi x)\sin^2(\pi y)\right) - P_{[-750,-50]}(-\alpha^{-1}\sin^2(\pi x)\sin^2(\pi y))$$

and

$$u^D(x,y) = \Delta^2\left(\sin^2(\pi x)\sin^2(\pi y)\right) + \sin^2(\pi x)\sin^2(\pi y).$$

Then, for these data the solution of the KKT system is given by

$$\begin{aligned} \bar u(x,y) &= \sin^2(\pi x)\sin^2(\pi y),\\ \bar\lambda^u(x,y) &= \sin^2(\pi x)\sin^2(\pi y),\\ \bar q(x,y) &= P_{[-750,-50]}(-\alpha^{-1}\sin^2(\pi x)\sin^2(\pi y)). \end{aligned}$$

The errors of the numerical approximations to control and state variables on uniform meshes are listed in Table 1. The observed orders of convergence confirm the theoretical predictions. The results of the computations on the distorted meshes are shown in Table 2. The observed orders of convergence for control variable and deflection largely agree with the predicted ones. On coarser meshes, a slightly faster convergence can be seen for the bending moments than predicted. This phenomenon can already be observed in computations solving only the state equation on the same meshes. However, in comparison to the order of convergence on the uniform meshes a significant difference can be observed in accordance with the theoretical results.

5 An application in 2d fluid mechanics

In this chapter, we consider another optimal control problem arising in fluid mechanics. An efficient way to compute a solution for the Stokes problem in two dimensional approximation

Table 1. Convergence results for control and state variables on uniform meshes

	$\|\|\bar{q}-\bar{q}_h\|\|$		$\|\|\bar{\sigma}-\bar{\sigma}_h\|\|$		$\|\|\bar{u}-\bar{u}_h\|\|_{H^1}$	
N	error	order	error	order	error	order
81	$1.10\cdot 10^{2}$	-	$8.44\cdot 10^{-1}$	-	$1.94\cdot 10^{-1}$	-
289	$6.27\cdot 10^{1}$	0.81	$2.79\cdot 10^{-1}$	1.60	$5.58\cdot 10^{-2}$	1.79
1089	$3.20\cdot 10^{1}$	0.97	$5.85\cdot 10^{-2}$	2.25	$1.27\cdot 10^{-2}$	2.13
4225	$1.62\cdot 10^{1}$	0.99	$1.76\cdot 10^{-2}$	1.74	$3.48\cdot 10^{-3}$	1.87
16641	$8.08\cdot 10^{0}$	1.00	$3.76\cdot 10^{-3}$	2.22	$8.04\cdot 10^{-4}$	2.12
66049	$4.04\cdot 10^{0}$	1.00	$1.05\cdot 10^{-3}$	1.84	$2.12\cdot 10^{-4}$	1.92
predicted		1.00		2.00		2.00

Table 2. Convergence results for control and state variables on distorted meshes

	$\|\|\bar{q}-\bar{q}_h\|\|$		$\|\|\bar{\sigma}-\bar{\sigma}_h\|\|$		$\|\|\bar{u}-\bar{u}_h\|\|_{H^1}$	
N	error	order	error	order	error	order
81	$1.10\cdot 10^{2}$	-	$8.50\cdot 10^{-1}$	-	$1.94\cdot 10^{-1}$	-
289	$6.25\cdot 10^{1}$	0.82	$2.80\cdot 10^{-1}$	1.60	$5.58\cdot 10^{-2}$	1.80
1089	$3.20\cdot 10^{1}$	0.97	$6.01\cdot 10^{-2}$	2.22	$1.28\cdot 10^{-2}$	2.13
4225	$1.61\cdot 10^{1}$	0.99	$1.88\cdot 10^{-2}$	1.67	$3.49\cdot 10^{-3}$	1.87
16641	$8.08\cdot 10^{0}$	1.00	$4.92\cdot 10^{-3}$	1.94	$8.05\cdot 10^{-4}$	2.12
66049	$4.05\cdot 10^{0}$	1.00	$1.92\cdot 10^{-3}$	1.36	$2.12\cdot 10^{-4}$	1.92
predicted		1.00		1.00		2.00

is to reformulate the problem in terms of the *stream function*. This results in a state equation, which is also governed by the biharmonic operator and can be solved again using a mixed finite element scheme. In this way the *second-order* Stokes system, which involves the incompressibility constraint, is replaced by a scalar but *fourth-order* equation.

The 2-dimensional Stokes problem under consideration reads as follows:

$$-\Delta v+\nabla p=f \quad \text{in } \Omega, \tag{66}$$

$$\operatorname{div} v=0 \quad \text{in } \Omega, \tag{67}$$

$$v=0 \quad \text{on } \partial\Omega, \tag{68}$$

where v is the velocity, p the pressure, and f the driving volume force. We introduce the stream function ψ as a potential of the velocity field v given by $v=\operatorname{rot}\psi=(\partial_2\psi,-\partial_1\psi)^T$. Inserting this in the above system, (67) is automatically fulfilled, since $\operatorname{div}\operatorname{rot}\psi\equiv 0$. The rotation operator acting on a vector valued function is defined by $\operatorname{rot} v=\partial_1 v_2-\partial_2 v_1$. Using this notation, we introduce the vorticity $\omega=\operatorname{rot} v=-\Delta\psi$. Applying the rotation operator to (66), we end up with the system

$$\begin{aligned} -\Delta\omega &= \operatorname{rot} f && \text{in } \Omega,\\ -\Delta\psi &= \omega && \text{in } \Omega,\\ \psi=\partial_n\psi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{69}$$

In this case one is not interested in the whole bending moments but rather in the vorticity $\omega=-\Delta\psi$. Hence, we will consider a less costly mixed method, the so-called Ciarlet-Raviart mixed scheme (see Ciarlet & Raviart [13], or the the simultaneous work of Mercier [31]). It consists of finding $\psi \in U := H_0^1(\Omega)$ and $\omega \in M := H^1(\Omega)$, such that

$$\begin{aligned} (\omega, \eta) - (\nabla\psi, \nabla\eta) &= 0 && \forall \eta \in M, \\ -(\nabla\varphi, \nabla\omega) &= -(f, \operatorname{rot}\varphi) && \forall \varphi \in U. \end{aligned}$$

We consider the following optimal distributed control problem

$$\min_{\substack{q \in Q^{\mathrm{ad}} \\ \{\psi,\omega\} \in H_0^1(\Omega) \times H^1(\Omega)}} J(q, \psi) = \tfrac{1}{2}\|\operatorname{rot}\psi - v^D\|^2 + \tfrac{\alpha}{2}\|q\|^2, \tag{70a}$$

$$\text{subject to} \quad \left\{ \begin{aligned} (\omega, \eta) - (\nabla\psi, \nabla\eta) &= 0 && \forall \eta \in M, \\ -(\nabla\varphi, \nabla\omega) &= -(q, \operatorname{rot}\varphi) && \forall \varphi \in U, \end{aligned} \right. \tag{70b}$$

which is a tracking problem for a given flow field $v^D \in H^1(\Omega)^2$. The control variable q is some force acting on the whole domain, e.g., an electromagnetic field. For the space of admissible control functions, we again impose pointwise constraints, i.e.,

$$Q^{\mathrm{ad}} = \{ q \in (L^2(\Omega))^2 \,|\, q_i^a \le q_i(x) \le q_i^b \text{ a.e. in } \Omega,\ i = 1, 2 \},$$

where $q^a, q^b \in (\mathbb{R} \cup \{\pm\infty\})^2$, $q_i^a \le q_i^b$. Furthermore, let $\alpha > 0$, and Ω be a convex polygonal domain. The corresponding adjoint system takes the form

$$(\eta, \lambda^\omega) - (\nabla\lambda^\psi, \nabla\eta) = 0 \qquad \forall \eta \in M, \tag{71a}$$

$$-(\nabla\varphi, \nabla\lambda^\omega) = -(\operatorname{rot}\psi - v^D, \operatorname{rot}\varphi) \qquad \forall \varphi \in U. \tag{71b}$$

The discretization is analogous to that in Section 3 by choosing

$$U_h = \mathcal{Q}_h^{(2)} \cap U, \quad M_h = \mathcal{Q}_h^{(2)} \cap M, \quad Q_h^{\mathrm{ad}} = Q_h \cap Q^{\mathrm{ad}}.$$

Then the discrete optimal control problem reads as follows:

$$\min_{\substack{q_h \in Q_h^{ad} \\ (\psi_h,\omega_h) \in (U_h \times M_h)}} J(q_h, \psi_h) = \tfrac{1}{2}\|\operatorname{rot}\psi_h - v^D\|^2 + \tfrac{\alpha}{2}\|q_h\|^2, \tag{72a}$$

$$\text{subject to} \quad \left\{ \begin{aligned} (\omega_h, \eta_h) - (\nabla\psi_h, \nabla\eta_h) &= 0 && \forall \eta_h \in M_h, \\ -(\nabla\varphi_h, \nabla\omega_h) &= -(q_h, \operatorname{rot}\varphi_h) && \forall \varphi_h \in U_h. \end{aligned} \right. \tag{72b}$$

Defining the adjoint variables $\{\lambda_h^\psi, \lambda_h^\omega\} \in U_h \times M_h$ as in the continuous case, we end up with the following optimality system:

$$(\omega_h, \eta_h) - (\nabla\psi_h, \nabla\eta_h) = 0 \qquad \forall \eta_h \in M_h, \tag{73a}$$

$$-(\nabla\varphi_h, \nabla\omega_h) = -(q_h, \operatorname{rot}\varphi_h) \qquad \forall \varphi_h \in U_h, \tag{73b}$$

$$(\eta_h, \lambda_h^\omega) - (\nabla\lambda_h^\psi, \nabla\eta_h) = 0 \qquad \forall \eta_h \in M_h, \tag{73c}$$

$$-(\nabla\varphi_h, \nabla\lambda_h^\omega) = -(\operatorname{rot}\psi_h - v^D, \operatorname{rot}\varphi_h) \qquad \forall \varphi_h \in U_h, \tag{73d}$$

$$(\operatorname{rot}\lambda_h^\psi + \alpha q_h, p_h - q_h) \ge 0 \qquad \forall p_h \in Q_h^{ad}. \tag{73e}$$

5.1 Error estimates

As in the proofs in Section 4, we can derive the following error estimates for the present situation.

Lemma 5.1. *Let $\bar q$ and $\bar q_h$ be the optimal controls of problems* (70) *and* (72)*, respectively. Then, there holds*

$$\|\bar q - \bar q_h\| \leq Ch(1+\|\bar q\|_{W^{1,\infty}}). \tag{74}$$

where $C = \mathcal{O}(\alpha^{-1})$.

Proof. The proof is completely analogous to that of Theorem 4.5. □

When deriving error estimates for the state variable, a minor modification has to be taken into account. Due to the rotation operator on the right-hand side of the state equation, we cannot use an analogue to Lemma 4.7 to get a bound for $\|\omega(\bar q) - \omega(p_h)\|$. Instead, we need an error bound in the H^{-1}-norm. This may be accomplished by another duality argument. For lack of space the rather standard technical details of this argument are omitted. Then, we immediately obtain the following result.

Theorem 5.2. *Suppose that Assumption 1 is satisfied and let $\{\bar\psi(\bar q), \omega(\bar q)\}$ be the optimal states of problem* (1) *and $\{\psi_h(\bar q_h), \omega(\bar q_h)\}$ those to problem* (27)*, respectively. Then, on regular quasi-uniform meshes there holds:*

$$\|\bar\psi(\bar q) - \bar\psi_h(\bar q_h)\|_{H^1} + \|\bar\omega(\bar q) - \bar\omega_h(\bar q_h)\|_{H^{-1}} \leq Ch^2(1+\|\bar q\|_h). \tag{75}$$

5.2 Numerical example

We consider the following problem:

$$\min_{\substack{\{\psi,\omega\}\in U\times M\\ q\in Q^{\mathrm{ad}}}} J(q,u) = \tfrac{1}{2}\|\operatorname{rot}\psi - v^D\|^2 + \tfrac{\alpha}{2}\|q\|^2, \tag{76a}$$

$$\text{subject to} \quad \begin{cases} (\omega,\eta) + (\nabla\psi,\nabla\eta) = 0 & \forall \eta\in M,\\ -(\nabla\varphi,\nabla\omega) = -(f+q,\varphi) & \forall\varphi\in U, \end{cases} \tag{76b}$$

where $\Omega = (0,1)^2$ and $\alpha = 10^{-3}$. Let

$$Q^{\mathrm{ad}} = \{q \in L^2(\Omega)^2,\ -700 \leq q_i \leq 700 \text{ a.e. in } \Omega,\ i=1,2\},$$

and

$$\begin{aligned}
f(x,y) :=& -\operatorname{rot}\Delta\big(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y))\big)\\
&- P_{[-700,700}\big(-\alpha^{-1}\operatorname{rot}(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y)))\big),\\
v_1^D(x,y) :=& \,\partial_y\left(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x))\sin(\pi y)\right)\\
&- \partial_y\Delta\left(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y))\right),\\
v_2^D(x,y) :=& -\partial_x\left(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y)\right)\\
&+ \partial_x\Delta\left(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y)\right).
\end{aligned}$$

For these data the analytical solution of the optimality system is given by

$$\begin{aligned}
\bar\psi(x,y) &= \bar\lambda^{\psi}(x,y) = \sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y)),\\
\bar\omega(x,y) &= \bar\lambda^{\omega}(x,y) = -\Delta\psi(x,y),\\
\bar q(x,y) &= -P_{[-700,700]}\big(-\alpha^{-1}\operatorname{rot}(\sin^2(\pi x)\sin^2(\pi y)(1-\sin(\pi x)\sin(\pi y)))\big).
\end{aligned}$$

Table 3. Convergence results for control and state variables on distorted meshes

	$\|\|\bar{q}-\bar{q}_h\|\|$		$\|\|\bar{\psi}-\bar{\psi}_h\|\|_{H^1}$		$\|\|\bar{\omega}-\bar{\omega}_h\|\|$	
N	error	order	error	order	error	order
81	$1.96 \cdot 10^{2}$	-	$6.41 \cdot 10^{-1}$	-	$8.08 \cdot 10^{-0}$	-
289	$1.59 \cdot 10^{2}$	0.30	$4.28 \cdot 10^{-1}$	0.58	$6.29 \cdot 10^{-0}$	0.36
1089	$8.91 \cdot 10^{1}$	0.84	$1.24 \cdot 10^{-1}$	1.78	$1.95 \cdot 10^{-0}$	1.69
4225	$4.56 \cdot 10^{1}$	0.97	$3.10 \cdot 10^{-2}$	2.00	$4.74 \cdot 10^{-1}$	2.04
16641	$2.30 \cdot 10^{1}$	0.99	$8.04 \cdot 10^{-3}$	1.95	$1.27 \cdot 10^{-1}$	1.91
66049	$1.15 \cdot 10^{1}$	1.00	$2.07 \cdot 10^{-3}$	1.95	$3.22 \cdot 10^{-2}$	1.98
predicted		1.00		2.00		2.00

The results of our computations are shown in Table 3 and confirm the theoretical findings. Except that for the vorticity ω, we proved second-order convergence only with respect to the weaker H^{-1}-norm, while it is observed also in the L^2-norm. The theoretical explanation of this superconvergence behavior has to be left as an open problem.

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