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Finite Depth Stratified Flow over Topography on a Beta-plane

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The form of quasi-geostrophic flow past isolated topography on a beta plane is obtained for finite depth, linearly stratified flow. The response is shown to consist of an isotropic perturbation plus, in eastward flows, a single barotropic and a finite number of baroclinic waves. The form of the solution allows a rapid calculation of the Rossby wave drag on an axisymmetric obstacle in an eastward flow.

1. INTRODUCTION

In a previous paper (Johnson 1977, called I herein) the effect of bottom topography in flow on a beta plane was examined in the two extremes of homogeneous and strongly stratified flows. Solutions were also presented for intermediate values of stratification for westward currents. In the present work we obtain the solution for intermediate values of the stratification for eastward currents. The method used is an extension of that used already in I. The response in eastward currents is shown to consist of a barotropic wave and, if $b^{1/2}B>\pi$ (where b and B are the beta effect and stratification parameters respectively, see Section 2), a finite number of baroclinic waves.

The problem of flow past three dimensional obstacles has also been considered recently by Janowitz (1977) who included the effect of compressibility and vertical velocity shear in his derivation of a Green's function. However, the sole example he presents is for the far-field due to an obstacle in a uniform stream in the limit $b^{1/2}B<\pi$. There are no baroclinic waves present and, in the limit of incompressible flow, his results reduce to an isotropic perturbation plus a wake of the same form as has been obtained previously (e.g. Janowitz 1974 and I) for homogeneous flows. The present method for arbitrary b and B extends directly to the compressible, vertically sheared case.

The wave drag on an obstacle may be easily calculated. The expression reduces to those given in I in the limit of homogeneous flow and also for strongly stratified flows, once an error in the expression in I is removed. In flows where b is greater than order 4 and baroclinic waves are present, the drag is significantly larger than when solely the barotropic mode is present.

2. EQUATIONS OF MOTION

We consider the effect of an obstacle of horizontal scale L and height h_0 affixed to the lower boundary of an inviscid, incompressible fluid of depth H, confined between two horizontal planes. We assume the fluid velocity sufficiently far upstream from the obstacle to be uniform and of magnitude U. In order to model large scale terrestrial flows we use the beta plane approximation, taking Cartesian axes $Ox'y'z'$ with origin within the obstacle, Ox' parallel to the unperturbed flow and Oz' the vertical axis of rotation. The rotation rate is then equal to $\frac{1}{2}(f+\beta y)$ (Figure 1). We assume the oncoming flow has constant buoyancy frequency N. The

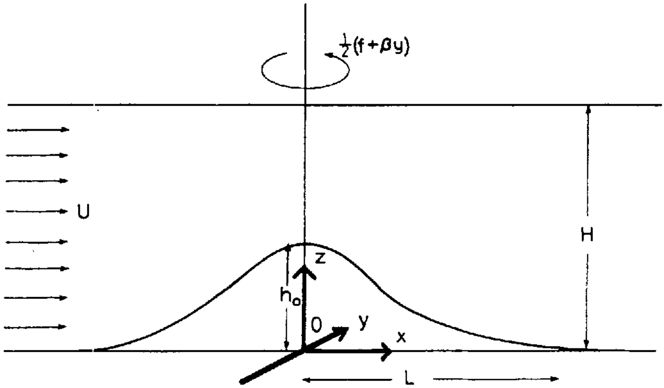

FIGURE 1 The coordinate system and scales of the motion.

equations for the non-diffusive, hydrostatic, quasi-geostrophic motion of the fluid (see I) are

$$D_t(\nabla^2 P + by) = 0, \tag{1.1}$$

$$D_t(P_z + \gamma h) = 0, \qquad \text{on } z = \gamma h Ro, \tag{1.2}$$

$$D_t(P_z) = 0, \qquad \text{on } z = B, \tag{1.3}$$

$$(P_x, P_y) \to (0, -1), \qquad \text{as } x^2 + y^2 \to \infty, \tag{1.4}$$

where P is the pressure, ∇^2 is the three dimensional Laplacian, and we have introduced the non-dimensional variables defined by

$$(x,y,z,P)=(x'/L,y'/L,Nz'/fL,P'/\rho_0 fUL).$$

where ρ_0 is the average upstream density. The non-dimensional groups are the Rossby number, $Ro=U/fL$, the obstacle height, $\gamma=Nh_0/U$, the stratification parameter $B=NH/fL$, and the beta effect parameter $b=\beta L^2/U$. The derivative is the quasi-geostrophic derivative $D_t=\partial_t+P_x\partial_y-P_y\partial_x$.

As in I we confine attention to γ sufficiently small so that all streamlines originate upstream, and for steady flow, linearizing the bottom boundary condition, have

$$\nabla^2\psi+b\psi=0, \tag{1.5}$$

$$\psi_z=-h, \qquad \text{on } z=0, \tag{1.6}$$

$$\psi_z=0, \qquad \text{on } z=B, \tag{1.7}$$

$$(\psi_x,\psi_y)\to(0,0), \qquad \text{as } x^2+y^2\to\infty, \tag{1.8}$$

where we have introduced the disturbance streamfunction defined by $P=-y+\gamma\psi$. The general solution to this system is derived in the appendix as

$$\psi(x,y,z;b,B)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,\xi,y,\eta,z;b,B)h(\xi,\eta)d\zeta dn,$$

where the Green's function is given by

$$G=(1/2\pi)⨍_0^\infty \kappa\cosh K(B-z)J_0(\kappa R)/(K\sinh KB)d\kappa$$

$$-(1/\pi B)h(b)[S(b^{1/2}R,\theta)+2\sum_{m=1}^{M}\cos(m\pi z/B)S(\kappa_m R,\theta)], \tag{1.9}$$

where M is the largest integer less than $b^{1/2}B/\pi$ (if $M=0$ the sum is absent), $h(b)$ is the Heaviside unit function (i.e. $h(b)$ is unity if b is positive and zero otherwise),

$$(R\cos\theta,R\sin\theta)=(x-\xi,y-\eta),$$

$$K=(\kappa^2-b)^{1/2},$$

$$\kappa_m=(b-m^2\pi^2/B^2)^{1/2}, \qquad m=1,2,\ldots,M,$$

$$S(\alpha,\rho)=\sum_{n=0}^{\infty}[\cos(2n+1)\alpha]J_{2n+1}(\rho)/(2n+1),$$

and J_n is the Bessel function of order n.

The integral in (1.9) is a principal value for positive b and gives the isotropic part of the disturbance due to the obstacle. It is the sole contribution to G for westward flows ($b<0$). However, standing Rossby waves may be present in eastward flows and the boundary condition (1.8) will not be sufficient to determine the flow uniquely. The ambiguity introduced by the possibility of these waves is removed by consideration of the linearized initial value problem in the appendix. The finite sum in the expression for G gives the form of a Rossby wave wake which cancels those waves otherwise present in front of the obstacle in eastward flows and reinforces those in the lee. The first term in this sum is independent of depth and corresponds to a barotropic wave. In the limit of homogeneous flow ($B \ll 1$), this is the sole wave present and (1.9) reduces to the homogeneous flow result given in I. The solution given for compressible flow by Janowitz (1977) corresponds to the case $M=0$. Only the barotropic mode is included in his solution and so in the limit of incompressible flow the solution reduces to an isotropic response given by the integral in (1.9) plus a wake of the same form and magnitude as in homogeneous flow. The value of M gives the number of baroclinic waves present in the flow. In the limit of strongly stratified flow or flow in a half plane ($B \gg 1$), M becomes indefinitely large and an infinite number of baroclinic modes are present as given by the expression in I for strongly stratified flows.

For axisymmetric obstacles, i.e. $h(x,y)=m(r)$ where $(x,y)=(r\cos\phi, r\sin\phi)$,

$$\psi(r,\phi,z;b,B)=\fint_0^{\infty} \kappa\hat{m}(\kappa)\cosh K(B-z)J_0(\kappa r)/(K\sinh KB)\,d\kappa$$

$$-(2/B)\not{k}(b)\Big[\hat{m}(b^{1/2})S(b^{1/2}r,\phi)+2\sum_{m=1}^{M}\cos(m\pi z/B)\hat{m}(\kappa_m)S(\kappa_m r,\phi)\Big], \qquad (1.10)$$

where $\hat{m}(\kappa)=\int_0^{\infty} rm(r)J_0(\kappa r)\,dr$ is the zero order Hankel transform.

3. THE ROSSBY WAVE DRAG

The presence of a wake in the lee of an obstacle in an eastward flow causes a drag to be exerted on the obstacle. For axisymmetric obstacles, expression (1.10) enables a rapid calculation of the drag, $\rho_0 f U L^2 h_0 \gamma \mathscr{D}$, where

$$\mathscr{D}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_x\psi\big|_{z=0}\,dx\,dy,$$

$$=(2\pi/b)\Big\{b^{1/2}[\hat{m}(b^{1/2})]^2+2\sum_{m=1}^{M}\kappa_m[\hat{m}(\kappa_m)]^2\Big\}. \qquad (2.1)$$

In weakly stratified flows only the first term is present and the drag is precisely that given in I for homogeneous flows. In the strongly stratified limit ($B \gg 1$) the sum approaches a value given by

$$\mathscr{D} = 4b \int_0^{\pi/2} \cos^2\theta [\hat{m}(b^{1/2}\cos\theta)]^2 d\theta, (B \gg 1). \qquad (2.2)$$

This expression is equal to that given in I for strongly stratified flows once an error in I is removed. In all expressions in I relating to axisymmetric obstacles in the limit $B \gg 1$, the coefficient of the wake term should be $\frac{1}{2}b^{1/2}$. This change does not affect any of the streamline patterns or blocking calculations but the expression for the drag must be multiplied by the factor $b^{1/2}$. The line (ii) in Figure 3 of I is thus also in error by a factor $b^{1/2}$.

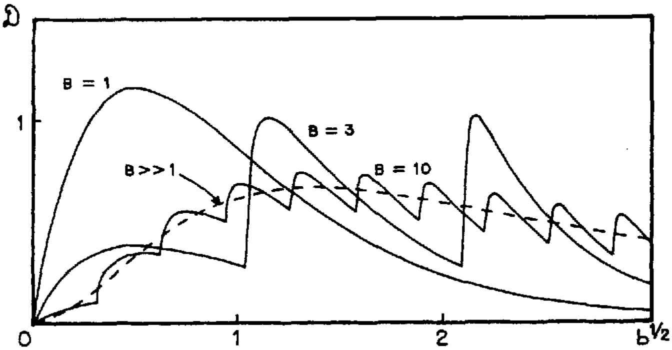

FIGURE 2 The non-dimensional wave drag $\mathscr{D}$ on the obstacle $m_1 = (1 + r^2)^{-\frac{3}{2}}$ in eastward flows plotted against the beta parameter for various values of stratification.

Figure 2 gives the correct plot of the drag for the obstacle considered in I, i.e. $m_1(r) = (1 + r^2)^{-3/2}$, $\hat{m}_1(\kappa) = \exp(-\kappa)$, against $b^{1/2}$ for varying stratifications B. The contribution from any particular mode decays exponentially with large $b^{1/2}$. If $b^{1/2} < \pi/B$ the drag is simply the homogeneous drag. As b increases additional, baroclinic, modes become possible, increasing the drag initially before their contribution also decays exponentially. The combined effects of these waves means that for large b the drag decays only algebraically,

$$\mathscr{D} = 4b^{-1/2} \int_0^\infty [u\hat{m}(u)]^2 du, \qquad (b, B \gg 1).$$

If b is greater than order 4, a far stronger wake is present in stratified flows than in homogeneous flows.

Figure 3 gives a similar plot for the obstacle $m_2(r)=\exp(-\frac{1}{2}r^2)$ $[\hat{m}_2(\kappa)=\exp(-\frac{1}{2}\kappa^2)]$. The drag due to the more confined obstacle m_2 is almost double that due to the more gradually sloping obstacle m_1.

$$\mathscr{D}(m_1)=b^{-1/2}, \qquad b,B\gg 1,$$

$$\mathscr{D}(m_2)=(\pi/b)^{1/2}, \qquad b,B\gg 1.$$

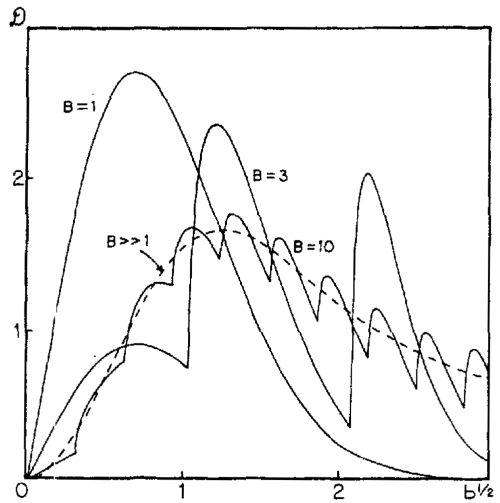

FIGURE 3 As for Figure 2 but for the obstacle $m_2=\exp(-\frac{1}{2}r^2)$.

For obstacles with vertical sides the drag assumes even larger values for large but finite b and B. However, the proof given in the appendix to Huppert (1975) may be extended to show that closed streamlines will then be present in the flow pattern for any non-zero obstacle height, γ. The assumption that all streamlines originate upstream is thus violated and the solutions given here are no longer unique solutions to the problem.

4. CONCLUSION

The form of quasi-geostrophic flow past isolated topography on a beta plane has been obtained for finite depth, linearly stratified flow. This reduces to that obtained previously in the limits of homogeneous or strongly stratified flow. The response in eastward currents consists of a barotropic wave and, if $b^{1/2}B>\pi$, a finite number of baroclinic waves. The drag on any axisymmetric obstacle may be calculated rapidly. The drag is significantly larger for steeply sloping obstacles and also when b is greater than order 4 and baroclinic modes are present.

Acknowledgement

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Appendix

THE DETERMINATION OF THE GREEN'S FUNCTION

We derive the required solution of (1.5) to (1.8) following Lighthill's (1965) method as in I. Consider an obstacle of the form

$$h^* = \gamma h(x,y)\exp(\varepsilon t), \text{ where } 0<\varepsilon\ll 1.$$

This corresponds to introducing, at $t=-\infty$, an obstacle whose height increases slowly to reach the value γ by $t=0$. The required steady solution is then given by the limit $\varepsilon\to 0$. We look for a solution of the form

$$P=-y+\gamma\psi_\varepsilon(x,y)\exp(\varepsilon t),$$

taking $\gamma\ll 1$, and retaining only those terms linear in γ. Thus

$$(\varepsilon+\partial_x)\nabla^2\psi_\varepsilon+b\partial_x\psi_\varepsilon=O,$$

$$\partial_z\psi_\varepsilon=O, \qquad \text{on } z=B,$$

$$\partial_z\psi_\varepsilon=-h, \qquad \text{on } z=O,$$

subject to the sole condition that ψ_ε remain bounded for all time at sufficiently large distances. The solution of this system may be written

$$\psi_\varepsilon(x,y,z;b,B)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta)G_\varepsilon(x,\xi,y,\eta,z;b,B)d\xi d\eta,$$

where

$$G_\varepsilon = (1/4\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\cosh K(B-z)/K \sinh BK]\exp[ik(x-\xi)+il(y-\eta)]\,dk\,dl,$$

and $K=[k^2+l^2-ikb/(\varepsilon+ik)]^{1/2}$. The required Green's function for (1.5) to (1.8) is given by $G=\lim_{\varepsilon\to 0}G_\varepsilon$. Let $(k,l)=(\kappa\cos\phi,\kappa\sin\phi)$ and $(x-\xi, y-\eta)=(R\cos\theta,R\sin\theta)$. Then

$$G=(1/4\pi^2)\lim_{\varepsilon\to 0}\int_0^{2\pi} d\phi \int_0^\infty [\cosh K(B-z)/K \sinh BK]\exp(i\kappa R\,\sin\phi)\kappa\,d\kappa$$

and $K=[\kappa^2-i\kappa b/(\varepsilon\,\mathrm{cosec}(\theta+\phi)+i\kappa)]^{1/2}$. The denominator vanishes whenever

$$BK=m\pi i, \quad \text{for} \quad m=0,1,2,\ldots.$$

When $\varepsilon=0$ this occurs when $\kappa=\kappa_m$ where

$$\kappa_m=(b-m^2\pi^2/B^2)^{1/2}, \quad m=0,1,2,\ldots.$$

If $b<0$, there are no zeroes for real non-negative κ. If $b>0$, there are $J+1$ zeroes where J is the integer part of $Bb^{1/2}/\pi$. These zeroes correspond to simple poles of the integrand. The branch points usually associated with the square root function are absent since the integrand is a single-valued function. If $Bb^{1/2}/\pi$ is an integer then the denominator vanishes at the origin. However, since the numerator vanishes there also, the origin is not a pole. Thus the poles which contribute are $\kappa_0,\kappa_1,\ldots,\kappa_M$ where M is the largest integer which is strictly less than $Bb^{1/2}/\pi$. For $0<\varepsilon\ll 1$ the pole corresponding to the zero at κ_m is at $\kappa_m+i\varepsilon\sigma$ where

$$\sigma=\tfrac{1}{2}b\,\mathrm{cosec}\,(\theta+\phi)/(b-m^2\pi^2/B^2)+O(\varepsilon^2).$$

Hence the pole is below the axis if $\pi-\theta\leqq\phi<2\pi-\theta$ and above otherwise. Thus

$$G=(1/4\pi^2)\int_0^{2\pi} d\phi \int_0^\infty F(R,z,\theta;\kappa,\phi)\,d\kappa, \tag{A1}$$

where

$$F=\cosh[(\kappa^2-b)^{1/2}(B-z)]\exp(i\kappa R\sin\phi)/\{(\kappa^2-b)^{1/2}\sinh[B(\kappa^2-b)^{1/2}]\},$$

and the path for the κ integration is a function of ϕ and θ, chosen so that the position of the pole for $0<\varepsilon\ll 1$ relative to the path is preserved in the limit $\varepsilon\to 0$. The paths for each ϕ for $0\leqq\theta<\pi$ are given in Figure 4. We

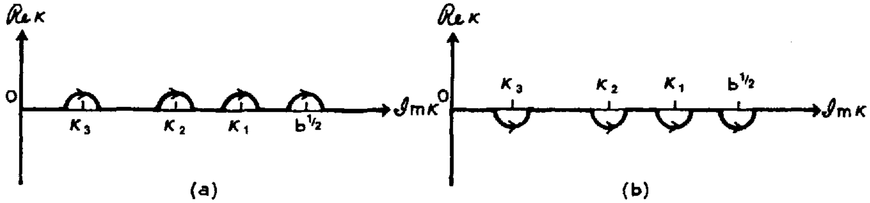

FIGURE 4 Integration paths for the integral (A1) for $0 \leqq \theta < \pi$. (a) $0 \leqq \phi < \pi - \theta$ or $2\pi - \theta < \phi < 2\pi$ (b) $\pi - \theta \leqq \phi < 2\pi - \theta$.

consider the indentations in the contour to be semi-circular, of vanishing radius, and centre κ_m. The contribution from an indentation is then given by

$$\pi i \,\text{residue}\, F(R,z,\theta;\kappa,\phi), \quad \text{if } 0 < \phi \leqq \pi - \theta \text{ or } 2\pi - \theta < \phi \leqq 2\pi,$$

$$-\pi i \,\text{residue}\, F(R,z,\theta;\kappa,\phi), \quad \text{if } \pi - \theta < \phi \leqq 2\pi - \theta.$$

Thus

$$G = (1/4\pi^2)\int_0^{2\pi} d\phi \int_0^{\infty} F(R,z,\theta;\kappa,\phi)d\kappa$$

$$+ (i/4\pi B)k(b) \sum_{m=0}^{M} a_m \cos(m\pi z/B)\left[\int_0^{\pi-\theta} - \int_{\pi-\theta}^{2\pi-\theta} + \int_{2\pi-\theta}^{2\pi}\right]\exp(i\kappa_m R \sin\phi)d\phi,$$

where $a_0 = \frac{1}{2}$, $a_m = 1$ for $m \geqq 1$ and the first integral is a principal value when $b > 0$. Changing the integration order in the first integral and evaluating the integrals with respect to ϕ, using the Fourier-Bessel series from I, gives the result quoted in (1.9).