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Topographically Bound Vortices

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A variational principle is presented which characterises steady motions of inviscid, rotating, stratified fluids. This is used to construct nonlinear solutions corresponding to stationary cold-core and warm-core eddies above topography in a uniform stream. Numerical results are presented for the specific case of a right circular cylinder. The results show that below a critical free-stream velocity a circular, cold, anticyclonic eddy is present above the cylinder and a warm, cyclonic eddy with the same cross-sectional area is present to the right (looking downstream) of the cylinder.

1. INTRODUCTION

Isolated regions of concentrated vorticity or temperature perturbation are common in the atmosphere and are an everyday feature of weather maps. However, observation of such eddy-like regions in the ocean has only recently become feasible. Both the POLYGON and MODE experiments produced evidence of these isolated regions and more recently, Vastano and Warren (1976) reported warm-core and cold-core eddies in the lee of the Atlantis II Seamount.

A possible explanation of the formation of such eddies in the neighbourhood of an isolated topographic feature can be obtained by considering starting flow over an obstacle in a stratified rotating fluid bounded above by a rigid lid. As the flow starts from rest those vertical filaments of fluid initially above an obstacle move off downstream and, in the absence of vertical shear, remain vertical. Vortex lines are stretched and a cyclonic vortex is formed on the downstream side. The vortex is relatively lighter (warmer) than its surroundings as the fluid has originated from a higher level. Simultaneously, compression of filaments moving over the obstacle means that an anticyclonic vortex is formed above the obstacle. This vortex consists of relatively heavier (cooler) fluid which has been raised from a lower level.

The formation and subsequent interaction of the cooler anticyclonic topographic vortex and the warmer cyclonic vortex has been discussed in detail by Huppert and Bryan (1976) who integrated numerically the equations of motion for starting flow over a seamount. They found that when the oncoming flow exceeds a certain critical velocity, the warm vortex eventually moves off downstream with a velocity which approaches asymptotically that of the free stream. However, for oncoming flow with less than this critical velocity the interaction between the vortices is strong enough to counteract the tendency of the free stream to sweep the warm vortex away. The warm vortex slowly moves to take up a position to the right (looking downstream) of the obstacle. The aim of the present work is to show that there exists a steady solution of the quasigeostrophic equations of motion corresponding to this latter situation.

The existence of a steady bound vortex motion has been demonstrated previously (Johnson 1978, called I herein) for fluids of constant density using a variational principle based on work by Benjamin (1975). By extending the ideas introduced in I to flows with arbitrary (but statistically stable) vertical distributions of density, we obtain the shape and position of a trapped warm eddy.

In Section 2 it is shown that a knowledge of the perturbation buoyancy acceleration on the lower boundary is sufficient to determine the flow field at any instant. The evolution of this perturbation acceleration is governed by an equation which expresses the conservation of a quantity, which we shall call the "potential buoyancy", closely analogous to the potential vorticity of a homogeneous fluid. The conservation of potential buoyancy means that potential buoyancy distributions which evolve in any time interval are rearrangements of the initial potential buoyancy distribution. This restricts the number of possible steady distributions which can evolve from a given initial distribution. We are thus able, in Section 3, to characterise the steady-state potential buoyancy distribution above arbitrary topography in a uniform stream as that rearrangement of the original potential buoyancy which maximises a functional related to the total kinetic energy of the motion. The considerable simplification which occurs when attention is restricted to low, flat-topped obstacles is discussed in Section 4. Streamlines are presented for strongly stratified flow over a low right circular cylinder. The results are closely related to those obtained in I for homogeneous flows and demonstrate that steady, trapped vortices are possible in stratified flows, although the vertical extent of a vortex is limited by the stratification.

2. THE TIME DEPENDENT EQUATIONS

We commence with a brief discussion of the time dependent equations to motivate the procedure used in Section 3 to obtain steady solutions. We consider the motion of a layer of depth H, of incompressible, inviscid fluid confined between two horizontal planes. We assume the system to be initially in a state of solid-body rotation with angular velocity $\frac{1}{2}f$ about a vertical axis, $0z$, and to be stably stratified with buoyancy frequency given by $N(z)$. We take Cartesian axes $0xyz$ as in Figure 1, with corresponding

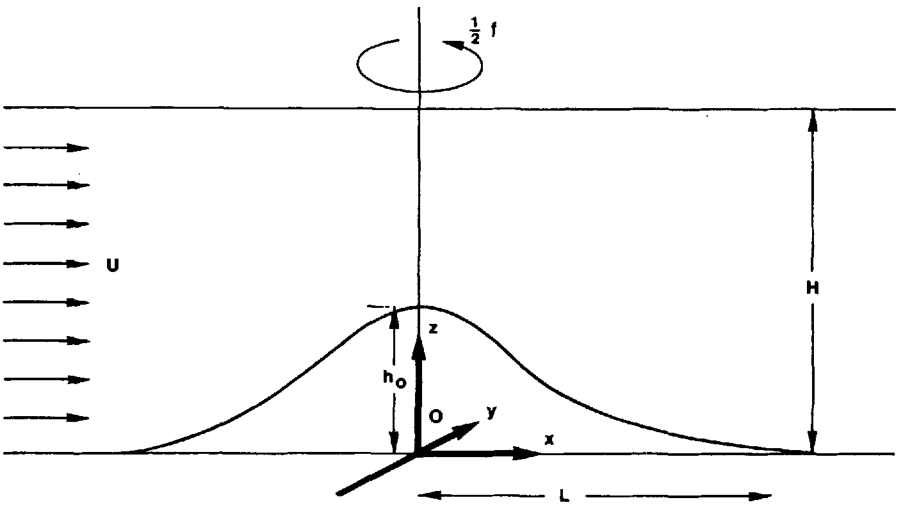

FIGURE 1 The coordinate system and scales of the motion.

velocity components (u, v, w) and time, t. We are concerned with the effect of an obstacle affixed to the lower boundary on the evolution of this system when an applied pressure gradient accelerates the fluid at infinity from rest to uniform motion in the positive x-direction with velocity U. We assume that the rotation is sufficiently rapid that the motion is quasigeostrophic, i.e. the Rossby number, $Ro = U/fL \ll 1$, where L is a typical horizontal length scale. The potential vorticity equation may then be written (Huppert, 1975),

$$\frac{D}{Dt}\left[(\zeta + f)\frac{\partial \tilde{\rho}}{\partial z}\right] = 0, \tag{2.1}$$

where $D/Dt = \partial/\partial t + \partial/\partial x + v\,\partial/\partial y$ is the quasi-geostrophic time-rate-ofchange following the motion, $\zeta = v_x - u_y$ is the vertical component of relative vorticity and $\tilde{\rho}$ is the total density. We assume the motion to be hydrostatic [i.e. $(U/LN_0)^2 \ll 1$, where N_0 is a typical value of $N(z)$] and non-diffusive. We introduce the buoyancy acceleration defined by

$$\sigma_1 = (\tilde{\rho} - \rho_0)/\rho_0 = \bar{\sigma}(z) + \sigma(x, y, z, t),$$

for some reference density ρ_0, where $d\bar{\sigma}/dz = N^2(z)$, and write the total pressure as

$$\tilde{P} = \rho_0 \int_0^z \bar{\sigma}(z')dz' + \rho_0 P(x, y, z, t).$$

The quasigeostrophic relationships give

$$(u, v) = (-f^{-1}P_y, f^{-1}P_x). \tag{2.2}$$

The conditions at the solid boundaries are

$$w = 0, \quad \text{at} \quad z = H, \tag{2.3a}$$

$$w = h_0(uh_x + vh_y), \quad \text{at} \quad z = h_0 h(x/L, y/L), \tag{2.3b}$$

where the function h has a maximum of unity and bounded support. The vertical momentum and density conservation equations may be written as

$$D\sigma/Dt = -N^2 w, \quad \text{where} \quad \sigma = P_z. \tag{2.4}$$

In terms of P alone, (2.1) to (2.4) give

$$(D/Dt)[P_{xx} + P_{yy} + [(f/N)^2 P_z]_z = 0, \tag{2.5}$$

$$(D/Dt)[P_z] = 0, \quad \text{on} \quad z = H, \tag{2.6a}$$

$$(D/Dt)[P_z + N^2 h_0 h] = 0, \quad \text{on} \quad z = h_0 h. \tag{2.6b}$$

For flow started from rest the potential vorticity, $P_{xx} + P_{yy} + [(f/N)^2 P_z]_z$, of each particle is initially zero. At succeeding times (2.5) states that the potential vorticity remains zero. Similarly (2.6a) states that P_z remains zero on the upper boundary. Non-dimensionalising (x, y) on L, z on fN/N_0, t on $L/N_0 h_0$ and P on $h_0 LN_0$ gives

$$P_{xx} + P_{yy} + (n^{-2} P_z)_z = 0, \tag{2.7}$$

$$P_z=0, \quad \text{on} \quad z=B, \tag{2.8a}$$

$$Q_t+P_xQ_y-Q_xP_y=0, \quad \text{on} \quad z=bh, \tag{2.8b}$$

where $n(z)=N(z)/N_0$, $B=HN_0/fL$, $b=N_0h_0/fL$, and

$$Q(x, y, t)=P_z(x, y, bh, t)+n^2(bh)h. \tag{2.9}$$

By analogy with the role of potential vorticity in I, Q may be described as a "potential buoyancy". The perturbation buoyancy at any point on the lower boundary is given by the weighted integral of the vorticity in the fluid filament above the point,

$$P_z(x, y, bh, t)=\int_{bh}^{B} \zeta \, d\tilde{\sigma},$$

and Q is the weighted integral of the homogeneous potential vorticity,

$$Q=\int_{bh}^{B} (\zeta+h)\, d\tilde{\sigma}+\text{constant}.$$

Initially Q is non-zero only above the topography h, i.e.

$$Q=\begin{cases} n^2(bh)h, & \text{at} \quad t=0, \\ Q_i, & \text{(say)}. \end{cases}$$

The conservation relation (2.8b) then implies that potential buoyancy distributions evolved during any time interval are rearrangements of some initial distribution which we take to be Q_i.

A crucial point, which enables the consideration of the bounded, two-dimensional support of Q instead of the infinite, three-dimensional fluid domain, is the existence for a wide range of topography $h(x, y)$ and density distributions $n(z)$, of a linear operator relating Q and P. There exists an operator K such that

$$\begin{aligned} P(x, y, z, t)&=K[Q(x, y, t)-Q_i(x, y)], \\ &=\iint_{\text{supp}(Q-Q_i)} k(x, y, z, \hat{x}, \hat{y})[Q(\hat{x}, \hat{y}, t)-Q_i(\hat{x}, \hat{y})]\, d\hat{x}\, d\hat{y}, \end{aligned}$$

where the Green's function, k, satisfies

$$k_{xx}+k_{yy}+(n^{-2}k_z)_z=0,$$

$$k_z=0, \quad \text{on} \quad z=B,$$

$$k_z=\delta(x-\hat{x})\delta(y-\hat{y}), \quad \text{on} \quad z=bh,$$

$$(k_x, k_y)\rightarrow(0,0), \quad \text{as} \quad x^2+y^2\rightarrow\infty.$$

A knowledge of the potential buoyancy at any instant is sufficient to determine the complete flow field.

3. THE STEADY SOLUTION

We may now pose the steady problem. We wish to find that streamfunction P_0, such that

$$Q_0=\partial P_0(x,y,bh)/\partial z+Q_i$$

is a rearrangement of Q_i and P_0 satisfies

$$\frac{\partial^2 P_0}{\partial x^2}+\frac{\partial^2 P_0}{\partial y^2}+\frac{\partial}{\partial z}\left(n^{-2}\frac{\partial P_0}{\partial z}\right)=0, \tag{3.1}$$

$$\partial P_0/\partial z=0, \quad \text{on} \quad z=B, \tag{3.2a}$$

$$\frac{\partial P_0}{\partial x}\frac{\partial Q_0}{\partial y}-\frac{\partial Q_0}{\partial y}\frac{\partial P_0}{\partial x}=0, \quad \text{on} \quad z=bh, \tag{3.2b}$$

$$(\partial P_0/\partial x, \partial P_0/\partial y)\rightarrow(0,-\alpha), \quad \text{as} \quad x^2+y^2\rightarrow\infty, \tag{3.2c}$$

where $\alpha=U/N_0h_0$.

From the definition of the operator K, it is sufficient to find a rearrangement Q_0 of Q_i which satisfies (3.2b) with P_0 defined by

$$P_0=-\alpha y+K(Q_0-Q_i). \tag{3.3}$$

We are now in a position to characterise Q_0 as that rearrangement of Q_i which maximises the functional

$$F(Q)=-\int_{\text{supp}(Q-Q_i)}(Q-Q_i)\{-\alpha y+\tfrac{1}{2}K^*(Q-Q_i)\}\,dx\,dy, \tag{3.4}$$

where

$$[K^*Q](x,y)=[KQ](x,y,bh).$$

Calculation of the first variation of F (analogously to the similar calculation in I) shows that Q_0 does indeed satisfy (3.2b) with P_0 defined by (3.3). Solving the system (3.1), (3.2) is reduced to maximising F over the class of rearrangements of Q_i.

The functional may be written

$$F(Q)=\alpha\int y(Q-Q_i)\,dx\,dy-\tfrac{1}{2}\int (Q-Q_i)K^*(Q-Q_i)\,dx\,dy,$$

$$=\alpha\int y\zeta\,dM-\tfrac{1}{2}\int P\zeta\,dM,$$

where $dM=dx\,dy\,d\bar{\sigma}$ is the elemental unit of mass. This relation shows F to be a linear combination of the impulse, i.e. the momentum of the fluid in the x-direction, and the kinetic energy of the perturbation. Substitution in (2.8b) shows that F is a constant of the motion.

To within a constant, we may write

$$F(Q)=-\int\{-\alpha y+\Phi\,|_{z=bh}+\tfrac{1}{2}K^*Q\}\,Q\,dx\,dy,$$

where $\Phi=-KQ_i$ is the streamfunction contribution from the anticyclonic vortex above the topography. The combination $\psi=-\alpha y+\Phi$ gives the flow field in the absence of the warm vortex. The streamfunction ψ is the finite-b, nonaxisymmetric form of the zero-potential-buoyancy streamfunction given by Huppert (1975) for axisymmetric obstacles in linearly stratified fluids.

4. LOW, FLAT-TOPPED OBSTACLES

The characterisation obtained in Section 3 requires no assumption about the relative height of the obstacle save that required for the flow to remain quasigeostrophic. However, in order to obtain a simple Green's function, independent of obstacle shape, we consider in this section obstacles of height small compared with the Prandtl height associated with their horizontal length-scale. We assume $b\ll 1$ and thus (following Huppert, 1975) linearise the bottom boundary condition by applying it on $z=0$.

Observed profiles of buoyancy frequency vary significantly with depth. In terms of the present variables, the profile proposed by Garrett and Munk (1972) for N below the mixed layer may be written

$$N(z)=N_0\exp(\mu z), \tag{4.1}$$

where $N_0=\hat{N}\exp(-H/Z)$ and $\mu=fL/ZN_0$. They give typical values as $\hat{N}=3$ cph and $Z=1.3$ km. The Green's function corresponding to this distribution can be written

$$k(x,y,z,\hat{x},\hat{y}) = -\frac{1}{2\pi} e^{\mu z} \int_0^\infty \frac{\left\{K_0\left(\frac{\xi}{\mu}e^{\mu B}\right)I_1\left(\frac{\xi}{\mu}e^{\mu z}\right) + I_0\left(\frac{\xi}{\mu}e^{\mu B}\right)K_1\left(\frac{\xi}{\mu}e^{\mu z}\right)\right\}}{\left\{K_0\left(\frac{\xi}{\mu}\right)I_0\left(\frac{\xi}{\mu}e^{\mu B}\right) - I_0\left(\frac{\xi}{\mu}\right)K_0\left(\frac{\xi}{\mu}e^{\mu B}\right)\right\}} \times J_0(\xi R)\, d\xi, \tag{4.2}$$

where I_0, I_1, K_0, K_1, J_0 are Bessel functions and $R^2 = (x-\hat{x})^2 + (y-\hat{y})^2$. The kernel of the functional (3.4) thus depends solely on the distance between (x, y) and $(\hat{x}, \hat{y})$

$$k(x,y,0,\hat{x},\hat{y}) = G(R \mid B, \mu),$$

for the function G defined by (4.2) with $z = 0$.

A further appreciable simplification occurs when attention is restricted to obstacles which are of uniform non-zero height within some given region, $\mathscr{A}$ (say), and of zero height elsewhere. The constraint that Q remain a rearrangement of Q_i becomes simply the requirement that the area of the support of Q remain a constant and equal to that of the support of Q_i. The warm and cold eddies should have the same cross-sectional area. The determination of the maximising function Q_0 is thus reduced to finding that region $\mathscr{B}$, of the same area as $\mathscr{A}$, which maximises

$$F_1(\mathscr{B}) = -\int_{\mathscr{B}} \{-\alpha y + \Phi_1 + \tfrac{1}{2}\int_{\mathscr{B}} G(R \mid B, \mu)\, d\hat{x}\, d\hat{y}\}\, dx\, dy,$$

where $\Phi_1 = -\int_{\mathscr{A}} G(R \mid B, \mu)\, d\hat{x}\, d\hat{y}$. Moreover, the dynamic equation (3.2b) reduces to the simple condition that the boundary of the warm eddy must be a streamline in steady flow.

The method of numerical maximisation is described in I. Once the Green's function has been tabulated, the amount of computation in the maximisation is independent of μ, B. However, to facilitate comparison with previous work, we consider the limit of constant buoyancy frequency, $\mu \ll 1$. Then

$$\begin{aligned} G(R \mid B, 0) &= -(1/2\pi)\textstyle\int_0^\infty \coth B\xi J_0(\xi R)\, d\xi, \\ &\to (4\pi B)^{-1} \ln R, \quad \text{as} \quad B \to 0, \\ &\to -(2\pi R)^{-1}, \quad \text{as} \quad B \to \infty. \end{aligned}$$

An infinite series for this function is given in the appendix to Huppert and Bryan (1976).

As a particular example we consider the case where $\mathscr{A}$ is the unit disc, i.e. the obstacle is a right circular cylinder. The streamlines for the flow when $B \ll 1$ are given in I, for various values of the parameter $HR_0/h_0 = \alpha B$. The solution for strong stratification, $B \gg 1$, behaves with varying α in the same manner as that for $B \ll 1$. For small α the warm eddy is approximately circular and occupies an area to the right (looking downstream) of the obstacle. For larger values the eddy area is closer to the obstacle and is deformed from the circular shape, behaving as a deformable self-gravitating body under the combined influence of a central repulsive field and a uniform background field (see I). As in I, there exists a maximum value of α above which no steady solution could be found. Figure 2 shows the streamlines on $z = 0$ for this value, $\alpha = 0.05$. The dashed circle indicates the edge of the cylinder (and so also of the cold eddy) and the warm eddy is dotted. Calculation of the streamlines at various heights above the obstacle would give a quantitative estimate of the penetration height of the fluid isolated from the free-stream within the closed zero contour. This delineation of the Taylor column has yet to be undertaken. The solutions in both limits, $B \ll 1$ and $B \gg 1$, are very similar. Thus it is to be expected that intermediate values of B would show a smooth transition between these extremes.

5. DISCUSSION

The problem of obtaining the steady, three-dimensional, quasigeostrophic flow field for a stratified rotating fluid with topography has been reduced to maximising, over the class of rearrangements of a given two-dimensional function, a functional related to the total energy of the motion. As in homogeneous flows, there appears to exist a critical value of the free-stream velocity above which a steady solution including all the non-zero potential buoyancy is not possible. Above this critical value at least some of the fluid originally above the obstacle must be swept downstream. The analysis presented above has been for obstacles of everywhere positive height. For obstacles of negative height analogous results may be derived. The vortex above the topography will then be cyclonic and warm, and the cold, anticyclonic vortex will be to the left (looking downstream). The analysis shows that the stratification in the ocean does not preclude the possibility, noted in I, of finding not only trapped, cold, anticyclonic vortices above seamounts but also warm, cyclonic vortices trapped to the right.

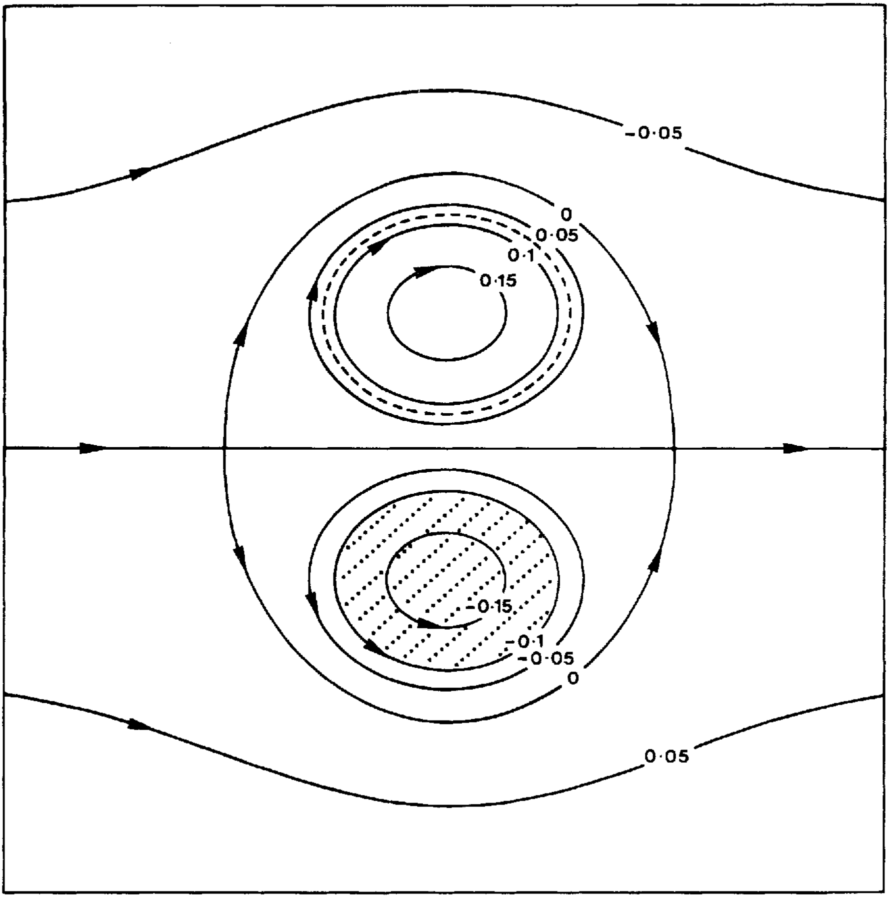

FIGURE 2 Streamlines for steady flow over an infinitesimal cylinder with a trapped eddy, $U/h_0N_0=0.05$. The dashed circle indicates the edge of the cylinder and the warm eddy is dotted. The streamline interval is 0.05.

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