Toric Varieties

References:

Fulton – Introduction to toric varieties

Cox, Little, Schenck – Toric varieties

MOTIVATION

(1) Interesting source of examples

toric varieties = rich class of algebraic varieties that can be described in purely combinatorial terms.

Eg Produce a proper variety that is not projective.

(2) testing ground for conjectures in algebraic geometry, arithmetic geometry, singularity theory

eg a general hypersurface singularity can be resolved in a purely combinatorial way, using toric geometry.

(3) explicit constructions

eg Mumford's uniformization of degenerating abelian varieties; Gross-Siebert programme; Minimal Model Programme.

(4) Use geometric techniques to solve problems in combinatorics.

eg Stanley; Adiprasito-Huh, Katz
"tropical Hodge theory"

Further developments:

- logarithmic geometry
  "sheafification of toric geometry"
- tropical geometry.

Base field: $\mathbb{C}$

§ Abstract definition

Algebraic torus T:
group scheme $/\mathbb{C}$
isomorphic to

$$\mathbb{G}_{m,\mathbb{C}}^n = (\mathbb{C}^*)^n$$

Character lattice

$$M = \mathrm{Hom}(T, \mathbb{C}^*)$$
$$\cong \mathbb{Z}^n$$

explicitly:

$$(\mathbb{C}^*)^n \to \mathbb{C}^* : (z_1, \dots, z_n) \mapsto z_1^{m_1} \cdots z_n^{m_n}$$

with $(m_1, \dots, m_n) \in \mathbb{Z}^n$.

Cocharacter lattice

$$N = \mathrm{Hom}(\mathbb{C}^*, T)$$
$$\cong \mathbb{Z}^n$$

$$\mathbb{C}^* \longrightarrow (\mathbb{C}^*)^n : z \mapsto (z^{m_1}, \dots, z^{m_n})$$

Duality pairing:

$$\langle \cdot, \cdot \rangle : M \otimes N \to \mathbb{Z}$$
$$(\chi, \theta) \mapsto \chi \circ \theta \in \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$$
$$(z \mapsto z^m)$$

$\rightsquigarrow N = M^\vee = \mathrm{Hom}(M, \mathbb{Z})$.

DEF A T-toric variety is a separated normal $\mathbb{C}$-scheme X of finite type together with a dense open embedding

$$T \hookrightarrow X$$

such that the action

$$T \times T \longrightarrow T$$ by

multiplication extends

to an action

$$T \times X \longrightarrow X$$ of T on X.

<u>EXAMPLES</u>

T, $(\mathbb{C}^*)^n \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$,

$(\mathbb{C}^*)^n \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$.

[KEY POINT]: T-toric varieties can be classified combinatorially in terms of <u>fans</u>.

§ <u>FANS</u>

$\sigma^\vee$ n-dimensional rational polyhedral cone $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$

$$\sigma^\vee = \{ \lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_i \in \mathbb{R}_{\geq 0} \}$$

for some $v_1, \ldots, v_r \in M$ that generate $M_{\mathbb{R}}$.

$\rightsquigarrow$ $\sigma^\vee \cap M$ = semigroup of lattice points

$\rightsquigarrow$ $\mathbb{C}[\sigma^\vee \cap M]$ = semigroup algebra

$$\mathbb{C}[\sigma^\vee \cap M] = \left\{ \sum_{m \in \sigma^\vee \cap M} c_m \chi^m \;\middle|\; c_m \in \mathbb{C},\ \text{almost all zero} \right\}$$

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}$$

<u>FACT</u> this is an <u>integrally</u> <u>closed</u> <u>finitely</u> <u>generated</u> $\mathbb{C}$-algebra.

$\rightsquigarrow$ $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ is an affine normal $\mathbb{C}$-scheme of finite type.

$X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$

affine toric variety associated with $\sigma^\vee$.

$$X_\sigma(\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\sigma^\vee \cap M], \mathbb{C})$$

$$= \operatorname{Hom}_{\text{semigr}}(\sigma^\vee \cap M, (\mathbb{C}, \cdot))$$

<u>torus action</u>

$$\begin{array}{ccc} T & \hookrightarrow & X_\sigma \\ \| & & \| \\ \operatorname{Spec} \mathbb{C}[M] & & \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M] \end{array}$$

Torus action

$$T \times X_\sigma \longrightarrow X_\sigma$$

Algebraically

$$\mathbb{C}[\sigma^\vee \cap M] \longrightarrow \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\sigma^\vee \cap M]$$

$$x^m \longmapsto x^m \otimes x^m$$

topologically

$$T(\mathbb{C}) \times X_\sigma(\mathbb{C}) \to X_\sigma(\mathbb{C})$$

$$\mathrm{Hom}(M, \mathbb{C}) \times \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{C}) \longrightarrow \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{C})$$

$$(f, g) \longmapsto f \cdot g$$

EXAMPLES

(1) $\sigma^\vee = M_{\mathbb{R}}$

$$X_\sigma = \mathrm{Spec}\, \mathbb{C}[M] = T$$
$$\cong$$
$$\mathrm{Spec}\, \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

(2) $\sigma^\vee = \mathbb{R}^2_{\geq 0}$ $(n = 2)$

$M = \mathbb{Z}^2$

$$\sigma^\vee \cap M = \mathbb{N}^2$$

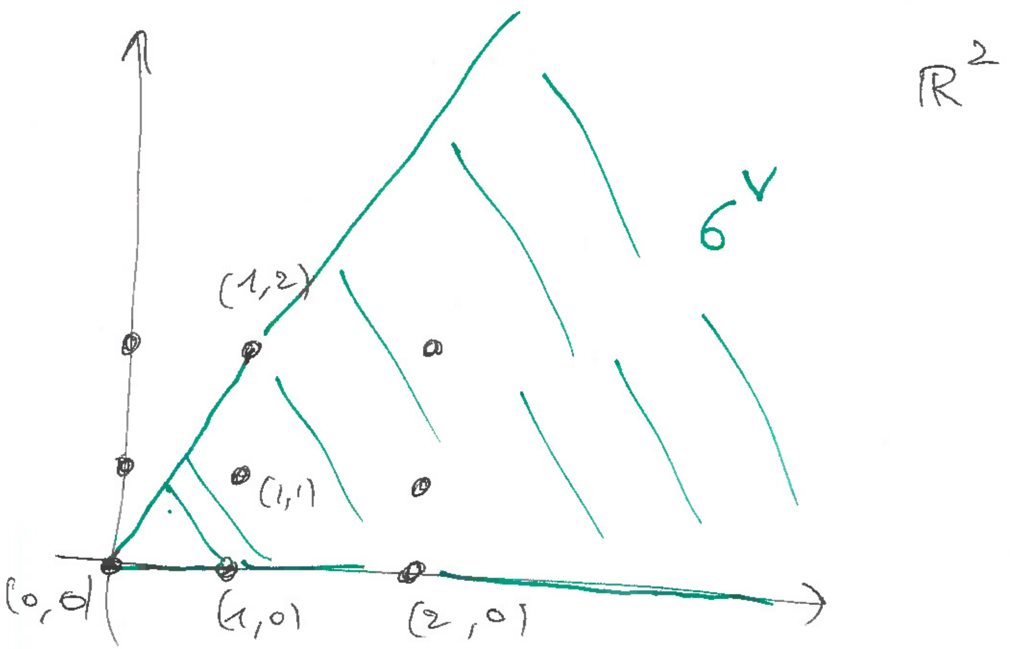
$$X_\sigma = \mathrm{Spec}\, \mathbb{C}[\mathbb{N}^2] \qquad \begin{array}{c} \mathbb{N}^2 \\ \rotatebox{90}{$\in$} \\ (a,b) \\ \downarrow \\ x^a y^b \end{array}$$

$$= \mathrm{Spec}\, \mathbb{C}[x, y]$$

$$= \mathbb{A}^2_{\mathbb{C}}$$

(3) $M = \mathbb{Z}^2$

$\sigma^\vee$ = real cone in $M_{\mathbb{R}} = \mathbb{R}^2$ generated by $(1,0)$ and $(1,2)$



generators of $\sigma^\vee \cap \mathbb{Z}^2$:

$(1,0)$, $(1,2)$, $(1,1)$

Relation:

$(1,0) + (1,2) = 2(1,1)$

$$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2] = \frac{\mathbb{C}[x,y,z]}{(xz - y^2)}$$

$$X_\sigma = \operatorname{Spec} \frac{\mathbb{C}[x,y,z]}{(xz-y^2)}$$

Singular at the origin

[ordinary double point / A_1 singularity]

__Idea__

lattice points in $\sigma^\vee \cap M$ correspond to the characters

$$\chi : T \to \mathbb{C}^*$$

that extend to regular functions

$$\begin{array}{ccc} \bar{\chi} : X_\sigma & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ T & \xrightarrow{\chi} & \mathbb{C}^* \end{array}$$

Now we want to construct general toric varieties by gluing affine ones.

Where do we find gluing data?

$\to$ neat description in terms of __dual__ cones.

$$\sigma = (\sigma^\vee)^\vee = \{ n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq 0 \ \forall m \in \sigma^\vee \}$$

A *fan* in $N_{\mathbb{R}}$ is a finite collection Σ of strictly convex rational polyhedral cones σ such that

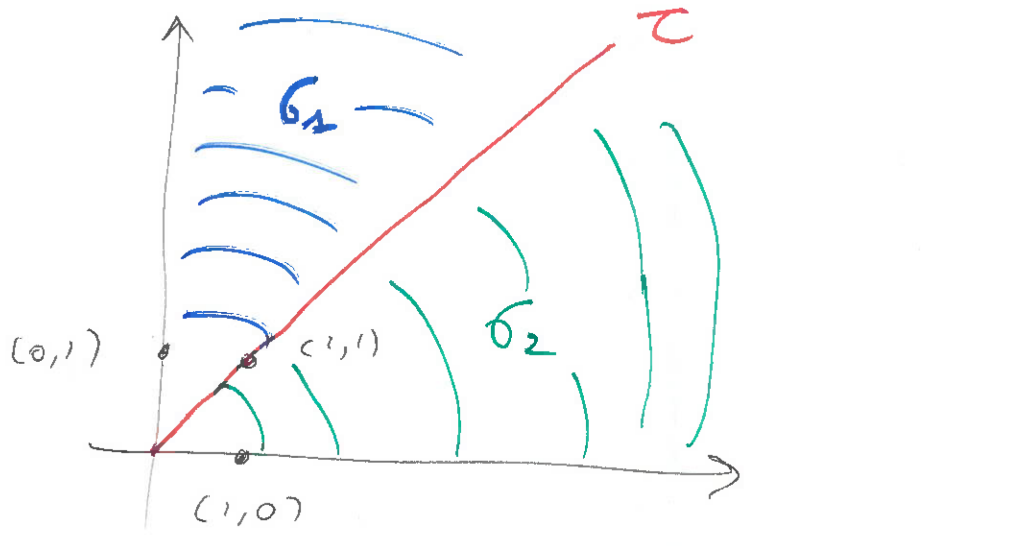
does not contain a line.

(1) if $\sigma \in \Sigma$ and $\tau \leq \sigma$, then $\tau \in \Sigma$.

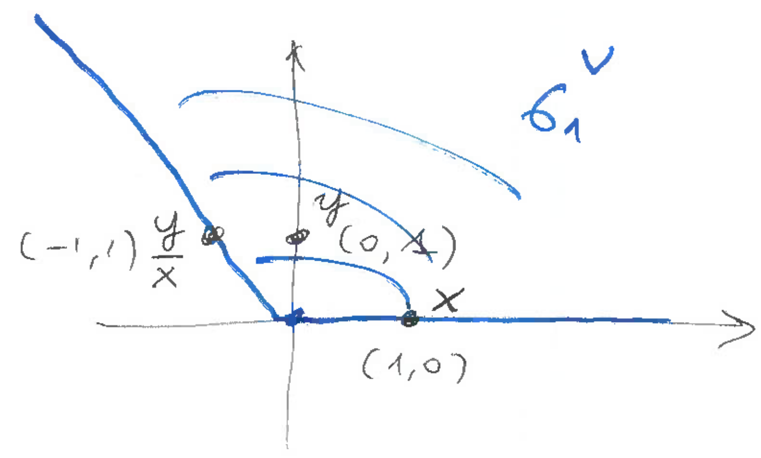
"is a face of"

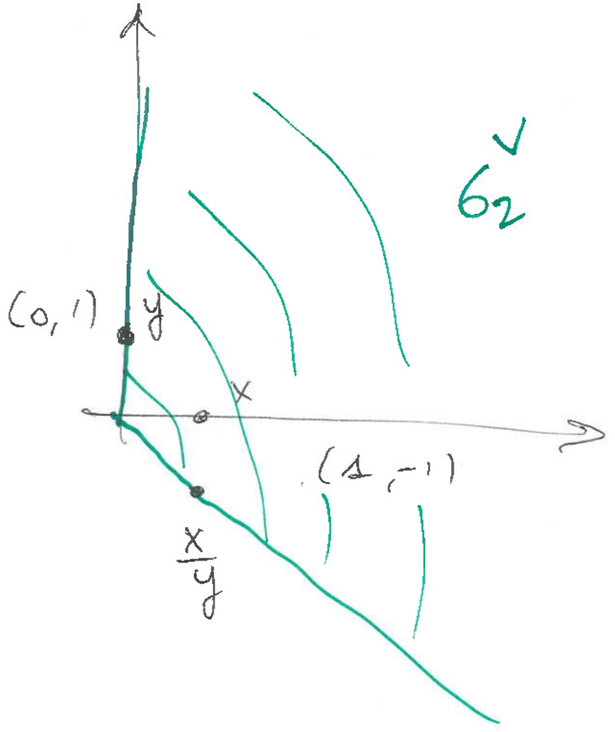
(2) if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a common face of σ and σ'.

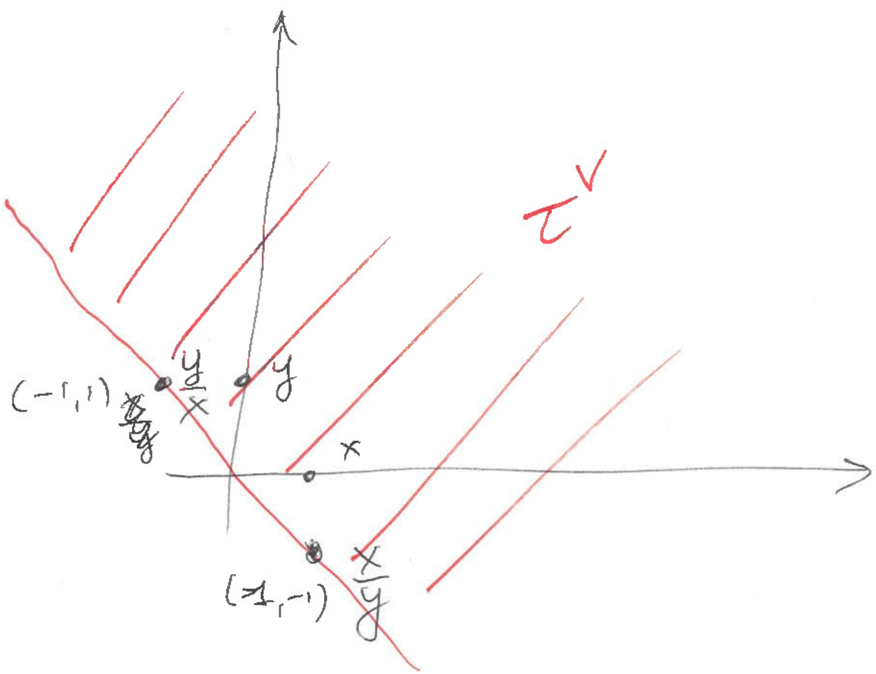
Eg $N = \mathbb{Z}^2$, $N_{\mathbb{R}} = \mathbb{R}^2$



Dual cones in $M_{\mathbb{R}}$







$$\begin{tikzcd}
T \circlearrowright X_\tau = \operatorname{Spec}\mathbb{C}[\tau^\vee\cap M] = \operatorname{Spec}\mathbb{C}[x,y,\tfrac{x}{y},\tfrac{y}{x}] \arrow[r, hook] \arrow[d, hook] & T \circlearrowright X_{\sigma_1} = \operatorname{Spec}\mathbb{C}[\sigma_1^\vee\cap M] = \operatorname{Spec}\mathbb{C}[x,\tfrac{y}{x}] \arrow[d] \\
T \circlearrowright X_{\sigma_2} = \operatorname{Spec}\mathbb{C}[\sigma_2^\vee\cap M] = \operatorname{Spec}\mathbb{C}[\tfrac{x}{y},y] \arrow[r] & X_\Sigma
\end{tikzcd}$$

<u>Gluing</u>

EXERCISE 1 Check that X_Σ is the blowup of $\mathbb{A}^2_{\mathbb{C}}$ at the origin.

THM Every T-toric variety is of the form X_Σ for some fan Σ in $N_{\mathbb{R}}$.

How can we reconstruct Σ from a given T-toric variety X?

To every T-orbit O we can attach a cone σ_O in $N_{\mathbb{R}}$:

$N = \mathrm{Hom}(\mathbb{C}^*, T)$

Consider the cocharacters

$$\theta : \mathbb{C}^* \longrightarrow T$$

that extend to a morphism $\bar{\theta} : \mathbb{A}^1_{\mathbb{C}} \longrightarrow X$ such that $\bar{\theta}(0) \in O$.

__FACTS__:

(1) These cocharacters θ are the lattice points of an __open__ rational polyhedral cone $\mathring{\sigma}_O$ in $N_{\mathbb{R}}$

σ_O = closure of $\mathring{\sigma}_O$.

(2) $\sigma_{O_1} \leq \sigma_{O_2}$ iff

$$O_2 \subseteq \overline{O_1}.$$

(3) These cones form a fan Σ in $N_{\mathbb{R}}$ such that

$$X \cong X_\Sigma$$

EXAMPLE $\quad X = \mathbb{P}^1_{\mathbb{C}} \circlearrowleft \mathbb{C}^*$

$$O_1 = \{0\}, \quad O_2 = \{\infty\},$$

$$O_3 = \mathbb{C}^*.$$

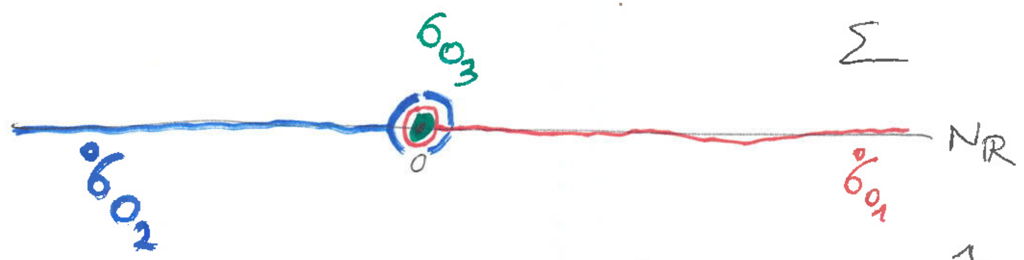
$$\theta : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$z \longmapsto z^n, \quad n \in \mathbb{Z}.$$

cocharacters.

All of these extend to morphisms $\overline{\theta} : \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$

(properness of $\mathbb{P}^1_{\mathbb{C}}$)

$$\overline{\theta}(0) \in \begin{cases} O_1 & \text{if } n > 0 \\ O_3 & \text{if } n = 0 \\ O_2 & \text{if } n < 0. \end{cases}$$



EXERCISE 2. Check that $X_\Sigma \cong \mathbb{P}^1_{\mathbb{C}}$

EXERCISE 3 Find the fans of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{P}^d_{\mathbb{C}}$ for $d \geq 2$.

EXERCISE 4

weighted projective spaces

$\underline{d} = (d_0, \dots, d_n) \in \mathbb{Z}_{>0}^{n+1}$ coprime

$\rightsquigarrow$ weighted projective space

$\mathbb{P}(\underline{d}) = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_n]$

$\deg(x_i) = d_i$

More explicitly:

$$\mathbb{C}^{n+1} \setminus \{\underline{0}\}$$

$$\circlearrowleft$$

$\mathbb{C}^* \ni t$ acts on $(x_0, \dots, x_n)$ by $(t^{d_0} x_0, \dots, t^{d_n} x_n)$.

The quotient of this action is $\mathbb{P}(\underline{d})$.

$$\circlearrowleft$$

~~What is~~ $(\mathbb{C}^*)^{n+1} / \mathbb{C}^* \cong (\mathbb{C}^*)^n$

What is the fan of this toric variety?

<u>EXERCISE 5</u> "Atiyah flop"

$$X = Z(xy - uv) \subset \mathbb{A}^4_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y,u,v]$$

singularity at 0.

smooth $\qquad\qquad$ smooth

$$\begin{tikzcd}
& Z \arrow[dl] \arrow[dr] & \\
Y_1 \arrow[rr, dashed, leftrightarrow, "\text{FLOP}", "\text{birat'l}"'] \arrow[dr] & & Y_2 \arrow[dl] \\
& X &
\end{tikzcd}$$

blow up the Weil divisor $Z(x,u)$ $\qquad$ blow up the Weil divisor $Z(x,v)$

Exceptional locus of $Y_1 \to X$ is a copy of $\mathbb{P}^1_{\mathbb{C}}$.
2

ADVANCED EXERCISE:

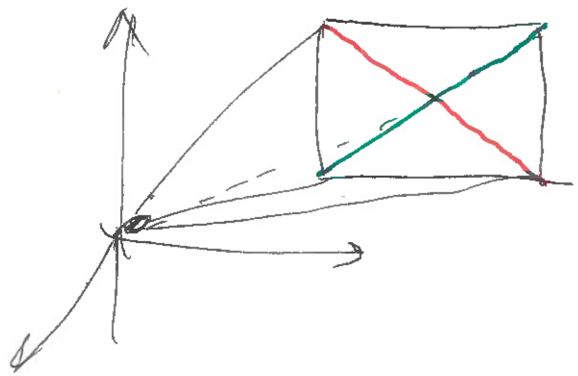
describe $Y_1 \to X$

$Y_2 \to X$

with toric geometry.

$X = X_\sigma$ affine toric

$\mathbb{R}^3$



Many geometric properties of X_Σ can be read off from Σ,

<u>eg</u>

<u>(1)</u> X_Σ is smooth <u>iff</u> every cone in Σ is smooth, i.e., spanned by a subset of a lattice basis for N.

<u>(2)</u> X_Σ is proper / $\mathbb{C}$ <u>iff</u> Σ is complete, <u>ie</u>, the union of the cones in Σ is the whole of $N_{\mathbb{R}}$.

<u>(3)</u> X_Σ is $\mathbb{Q}$-factorial (every Weil divisor is $\mathbb{Q}$-Cartier) <u>iff</u> Σ is simplicial (# ~~gen~~ rays of a cone = dimension of cone)

In this case, X_Σ has only finite quotient singularities (orbifold).

(4) Computation of $\mathrm{Pic}(X_\Sigma)$, ampleness criterion, cohomology, ... in combinatorial way.

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§ Stanley's theorem

Convex polytope K in $\mathbb{R}^n$

= convex hull of finite set of points.

We will assume that $\dim(K) = n$ ($\Leftrightarrow \mathring{K} \neq \emptyset$).

Simplicial convex polytope: all faces in the boundary are simplices.

$\rightsquigarrow$ "f-vector"

$f = (f_0, \dots, f_n)$

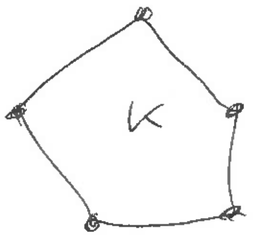
$f_i = \#$ faces of dimension i in ∂K.

eg $f_0 = \#$ vertices,
$f_1 = \#$ edges, ...

PROBLEM: Classify all f-vectors.

EXAMPLE

$n = 2$



$f_0 = f_1$ (Euler's formula)

$f_0 \geq 3$

These are sufficient conditions

$n = 3$

$f_0 - f_1 + f_2 = 2$ (Euler formula)

$3 f_2 = 2 f_1$ (simplicial)

$f_0 \geq 4$

~~nec~~ $\rightarrow$ sufficient conditions

General case?

Given $f_0, \dots, f_{n-1}$, set

$f_{-1} = 1$

and $h_p = \sum_{i=p}^{n} (-1)^{i-p} \binom{i}{p} f_{n-i-1}$

for $p = 0, \dots, n$

Conjecture McMullen 1971

$(f_0, \dots, f_{n-1})$ is an f-vector

iff

(I) "Dehn-Sommerville"
$h_p = h_{n-p}, \ \forall p$

(II) ....

Stanley (1980)

If K has f-vector $(f_0, \dots, f_{n-1})$

then we can construct a complete simplicial fan Σ such that

$\#$ d-dimensional cones in Σ

$= f_{d-1}$, $\forall d$.

Then

$$h_p = \dim H^{2p}(X_\Sigma, \mathbb{Q}), \quad \forall p.$$

$$\boxed{h_p = h_{n-p}}$$

Poincaré duality!