Symplectic Manifolds, Moment Maps and Hamiltonian Reduction

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Symplectic Manifolds
-Classical Mechanics

Newton's second law is the second order ODE

\[ m\ddot{q} = F(q) \]

In celestial mechanics, one is primarily concerned with conservative force fields, i.e. we assume that work done in moving a particle from one position to other is independent of the path. Mathematically, this means that the vector valued function \( F(q) \) is given as a gradient

\[ F(q) = -\nabla_q V \]

of a scalar function \( V \).

**Ex.** Consider \( F(q_1, q_2) = q_2 \partial_{q_1} - q_1 \partial_{q_2} \) on \( \mathbb{R}^2 \). Is it conservative?
In the Hamiltonian formulation, we can write equations of motion as

\[
\dot{q} = \frac{\partial H}{\partial p} \tag{1}
\]

\[
\dot{p} = -\frac{\partial H}{\partial q} \tag{2}
\]

where \( p = m \dot{q} \) is the momentum and \( H(q, p) \) is the Hamiltonian function given by the sum of kinetic and potential energy:

\[
H(q, p) = \frac{p^2}{2m} + V(q)
\]

Note that if \( V(q) = \text{const.} \), \( \dot{q} = \text{const.} \). This is the law of inertia. In particular, the trajectories are “geodesics” traversed so as to keep the energy, \( H \), constant.
Hamilton (1835) $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$.

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

The phase flow $\phi_t^H$ of $H$ is the one-parameter group of transformations $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$

$$(p(0), q(0)) \rightarrow (p(t), q(t))$$

For any function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, classical observable, we determine the evolution $f_t = f(p(t), q(t))$ of such $f$,

$$\frac{df_t}{dt} = \{H, f_t\}$$

where

$$\{H, f\} = \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p}$$

Energy conservation law: $\{H, H\} = 0$. 
The Poisson bracket is antisymmetric, and satisfies the Jacobi identity

\[ \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \]

We can express an anti-symmetric bracket via an anti-symmetric matrix \( \eta^{ij} \)

\[ \{f, g\}(x) = \eta^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \]

To understand what Jacobi identity means, let \( A = \mathcal{O}(\mathbb{R}^{2n}) \) be algebra of classical observables. Define one-parameter family of operators

\[ U_t : A \rightarrow A \]

\[ f \rightarrow f_t \]

**Ex.** Jacobi identity holds if and only if \( U_t \) is automorphism of \( A \) that respects the Poisson bracket (for any \( H \)).
Besides the Jacobi, we also have the Leibniz rule:
\[ \{ H, FG \} = \{ H, F \} G + F \{ H, G \} \]

In terms of evolution with respect to \( H \), this simply says
\[ \frac{d(FG)}{dt} = \frac{dF}{dt} G + F \frac{dG}{dt} \]

**Definition:** A mechanical system is a Poisson bracket \( \{ , \} \) on the algebra of differentiable functions \( \mathcal{O}(M) \) on some space \( M \) called the phase space and a Hamiltonian function \( H \).

**Noether’s theorem:** If \( F \) is a symmetry, i.e \( \{ F, H \} = 0 \), then its conserved \( \{ H, F \} = 0 \).

**Poisson’s theorem:** If \( F, G \) are conservation laws (i.e \( \{ H, F \} = \{ H, G \} = 0 \)), then \( \{ F, G \} \) is also a conservation law (\( \{ H, \{ F, G \} \} = 0 \)).

**Ex.** Describe the motion for the harmonic oscillator
\[ H(p, q) = \frac{p^2}{2m} + k \frac{q^2}{2} \], where \( k \) is the rigidity coefficient of the string.
Symplectic Manifolds

If $\det(\eta^{ij}) \neq 0$, then define

$$\omega = \eta_{ij} dx^i \wedge dx^j$$

Then, Jacobi identity holds if and only if $d\omega = 0$.

**Def.** A manifold $M$ equipped with a non-degenerate closed 2-form $\omega$ is called a symplectic manifold.

**Ex.** Show that a 2-form $\omega$ on $M^{2n}$ is non-degenerate if and only if $\omega^n$ is a volume form on $M$.

**Ex.** Show that on a compact symplectic manifold $(M, \omega)$, the class $[\omega] \in H^2(M)$ is non-zero.
On \( \mathbb{R}^{2n} \) with co-ordinates \( (q_1, \ldots, q_n, p_1, \ldots, p_n) \) the standard symplectic structure is defined by

\[
\omega_{\text{std}} = \sum_i dq_i \wedge dp_i
\]

In terms of the associated Poisson bracket,

\[
\{q_j, p_k\} = \delta_{jk}, \quad \{q_j, q_k\} = 0, \quad \{p_j, p_k\} = 0
\]

**Thm.** (Darboux) Any symplectic manifold is locally symplectomorphic to \( (\mathbb{R}^{2n}, \omega_{\text{std}}) \).
On a general symplectic manifold $M$, given a (Hamiltonian) function $H : M \to \mathbb{R}$, we can define a vector field

$$\omega(X_H, \cdot) = dH$$

$X_H$ is called the Hamiltonian vector field, and its flow (motion) is the Hamiltonian flow associated with $H$.

**Ex.** Show that

$$(\mathcal{O}(M), \{\ ,\ \}) \to (\text{Vect}(M), [\ ,\ ] )$$

$H \to X_H$

is a Lie algebra homomorphism. The image is denoted by $\text{ham}(M, \omega)$. 
Examples

- Let $Q$ be any smooth manifold, $T^* Q$ is a symplectic manifold. Given $(q_1, \ldots, q_n)$ local co-ordinates for $Q$, and co-ordinates $(p_1, \ldots, p_n)$ for the fiber, one can construct a 1-form

$$\alpha = \sum_i p_i dq_i$$

**Ex.** Check that $\alpha$ is intrinsically defined. We define $\omega = -d\alpha$ as the canonical symplectic form on $T^* Q$.

- Suppose $(M, g, J)$ is Kähler manifold, define $\omega(X, Y) = g(JX, Y)$, then $(M, \omega)$ is symplectic.

- $G$ be a real Lie group with Lie algebra $\mathfrak{g}$. $G$ acts on $\mathfrak{g}$ via adjoint action, and thus also on the dual space $\mathfrak{g}^*$. Then every $G$-orbit $M \subset \mathfrak{g}^*$ has a canonical symplectic form: the corresponding Poisson bracket is given by $\{x, y\} = [x, y]$ where $x, y \in \mathfrak{g}$ are considered as functions on $\mathfrak{g}^*$. 
A vector field $X$ is a Hamiltonian vector field if $\iota_X \omega$ is exact. More generally, a vector field $X$ on $M$ is called symplectic if $\iota_X \omega$ is closed.

If $\phi_t$ is a flow of a symplectic vector field, by Cartan’s magic formula, we have:

$$\frac{d}{dt} \phi_t^* \omega = \mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega = 0$$

**Cor.** (Liouville’s theorem) Hamiltonian flow $\phi^H_t : M \to M$ preserves volume (i.e. Hamiltonian flow is incompressible).
Def. \( \text{Symp}(M, \omega) = \{ \phi : M \to M : \phi^*\omega = \omega \} \)

\[ \{ X : \iota_X \omega \in \text{Ker}d \} = \text{Lie}(\text{Symp}(M, \omega)) \]

\( \text{Ham}(M, \omega) \subset \text{Symp}(M, \omega) \) is the unique subgroup of \( \text{Symp}(M, \omega) \) with Lie algebra \( \text{ham}(M, \omega) = \{ X : \iota_X \omega \in \text{Im}d \} \)

Ex. Let \( M \) be the 2-torus \( \mathbb{R}^2/\mathbb{Z}^2 = \{(x, y) \mod 1 \} \) equipped with the symplectic form \( dx \wedge dy \). Consider the flow \( \phi_t \) given by

\[ \dot{x} = 1, \quad \dot{y} = \alpha, \quad \text{for } \alpha \in \mathbb{R} \]

Show that the flow on \( M \) is symplectic but not Hamiltonian.

Ex. \( M \) as in the previous exercise. Consider the Hamiltonian \( H : M \to \mathbb{R} \) given by \( H(x, y) = \cos(2\pi x) \). Describe the Hamiltonian flow.
Integrable systems

By Noether’s theorem, the integral curves of a Hamiltonian flow are contained in the level sets of the conservations laws, i.e $F$ such that $\{H, F\} = 0$. Such $F$ are also called integrals of motion. If there are many such independent conservation laws, one can characterise the motion by intersecting the level sets.

**Def.** A Hamiltonian (mechanical) system $(M^{2n}, \omega, H)$ is completely integrable if it possesses $n$ integrals of motion $H_1 = H, H_2, \ldots, H_n$ such that

1) $\{H_i, H_j\} = 0$ for all $i, j$,

2) $\mu : M \to \mathbb{R}^n, x \mapsto (H_1(x), \ldots, H_n(x))$ has no critical points.
Arnold-Liouville theorem

**Thm.** Let $H_1, \ldots, H_n$ define a completely integrable system on $(M, \omega)$. For $b \in \mathbb{R}^n$, let $M_b = \mu^{-1}(b)$ where $\mu = (H_1, \ldots, H_n)$.

- If $M_b$ is non-empty and compact, then each of its connected components is diffeomorphic to a torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.
- One can choose (angle) co-ordinates $(\phi_1, \ldots, \phi_n)$ identifying connected component of $M_b$ with $\mathbb{R}^n/\mathbb{Z}^n$ such that the flows of the vector field $X_{H_1}, \ldots, X_{H_n}$ are linear.
- The restriction of the symplectic form to $M_b$ is trivial.
- In a neighborhood of $M_b$, one can choose (action) co-ordinates $(a_1, \ldots, a_n)$ complementary to $(\phi_1, \ldots, \phi_n)$ such that the $a_j$ are integrals of motion and the symplectic form in these co-ordinates is given by

$$\omega = \sum da_j \wedge d\phi_j$$
The connected components of the level sets $M_b$ are *Lagrangian* tori.

The equations of motion (Hamiltonian flow with respect to $H = H_1$) in these co-ordinates are given by

$$\dot{a}_j = 0, \quad \dot{\phi}_j = \omega_j(a), \; j = 1, \ldots, n$$

for some $\omega_j(a) = dH/da_j \in \mathbb{R}$. Thus, the motion on the invariant torus is conditionally periodic, that is

$$a_j = a_j(0), \; \phi_j(t) = \omega_j t + \phi_j(0)$$

for some constants $\omega_j = \omega_j(a(0))$.

**Def.** A submanifold $L \subset M^{2n}$ is called isotropic if $\omega|L = 0$ and is called Lagrangian if in addition $\dim L = n$.

**Ex.** Show that an isotropic submanifold of $M^{2n}$ has $\dim L \leq n$. 
Moment map

Let $G$ be a connected Lie group acting on a symplectic manifold $(M, \omega)$ by symplectomorphisms. We call such an action a symplectic $G$-action.

- $H: M \to \mathbb{R}$ any function such that $X_H$ is complete, then its flow defines a symplectic $\mathbb{R}$-action.
- More generally, if $H_1, \ldots, H_k$ are $k$ commuting functions ($\{H_i, H_j\} = 0$ for all $i, j$), their flows define a symplectic $\mathbb{R}^k$-action.

One way to interpret Arnold-Liouville theorem is that we can find co-ordinates such that a neighborhood of the fiber $\mathbb{T}^n$, can be identified (symplectically) with $\mathbb{T}^n \times B$ with $B \subset \mathbb{R}^n$ and the orbits of the original system gets identified with $\mathbb{T}^n \times \{b\}$, for $b \in B$. We shall next study the case there is a global action of a torus.
Hamiltonian group actions

Given a $G$-action on $M$, by linearizing the action, every element $a \in \mathfrak{g}$ defines a vector field $\xi_a$ on $M$. The action is symplectic if $d(\iota_{\xi_a} \omega) = 0$ for all $a$. The correct definition of a global Hamiltonian action is given as follows:

**Def.** A symplectic action of $G$ on $M$ is called *Hamiltonian* if there exists a map, called the *moment map*,

$$\mu : M \rightarrow \mathfrak{g}^*$$

such that

- For any $a \in \mathfrak{g}$, the function $H_a(x) = \langle \mu(x), a \rangle$ is the Hamiltonian for the vector field $\xi_a$.
- For any $a, b \in \mathfrak{g}$, $\{H_a, H_b\} = H_{[a,b]}$
- $\mu$ is $G$-equivariant.

**Ex.** Show that the last condition is automatic if $G$ is connected.
Suppose a torus $\mathbb{T}$ acts on $M^{2n}$. Choose a basis $a_i$ of the Lie algebra $\mathfrak{t} = \text{Lie}\mathbb{T}$. If we can find $H_i$ such that $\iota_{a_i}\omega = dH_i$, then for $a = \sum_i \lambda_i a_i$, we can define $\mu(x)$ by

$$\langle \mu(x), a \rangle = \sum_i \lambda_i H_i(x)$$

**Ex.** Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n | |t_i| = 1\}$ act on $\mathbb{C}^n$ by $(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1z_1, \ldots, t_nz_n)$. Then check that

$$\mu(z_1, \ldots, z_n) = -\frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$$

is a moment map.
Convexity

**Thm. (Atiyah, Guillemin-Sternberg)**
The image of a compact connected symplectic manifold under the moment map of an (quasi) effective Hamiltonian torus action is a Delzant polytope. In fact,

- $\mu(M)$ it is the convex hull of $\mu(M^\mathbb{T})$,
- $\mu^{-1}(b)$ is connected for all $b \in \mathbb{R}^n$.

**Thm. (Delzant)** When the torus acting is half the dimension of the manifold and the action is effective, the polytope determines the manifold. More precisely, there is a 1-1 correspondence

\[
\{ \text{toric manifolds} \} \leftrightarrow \{ \text{Delzant polytopes} \} \\
(M^{2n}, \omega, \mathbb{T}^n, \mu) \leftrightarrow \mu(M)
\]
Polytopes

**Def.** A rational convex polytope $P$ is a subset of $\mathbb{R}^n$ defined as the intersection of halfplanes

$$S_{a,b} = \{x \in \mathbb{R}^n : a_1x_1 + \ldots + a_nx_n \leq b\}, \ a_i \in \mathbb{Q}, \ b \in \mathbb{R}^n$$

We say that $P$ is a *Delzant (unimodular) polytope* if it is a convex rational polytope such that each point on a $k$-dimensional facet has a neighborhood isomorphic via an integral affine transformation to a neighborhood of the origin in $[0, \infty)^{n-k} \times \mathbb{R}^k$.

**Ex.** Let $M$ be compact symplectic manifold with a Hamiltonian $S^1$-action, prove that there exists a fixed point of the $S^1$-action. In this case, it is enough to check that Delzant condition at each vertex for the polytope.
Examples

**Ex.** $S^1$ acts on $(S^2, \omega_{st} = d\theta \wedge dh)$ by rotations with moment map $\mu = h$ equal to the height function and moment polytope $[-1, 1]$.

**Ex.** Consider the $\mathbb{T}^2$-action on $\mathbb{C}P^2$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] \rightarrow [z_0, e^{i\theta_1}z_1, e^{i\theta_2}z_2]$$

This has moment map

$$\mu[z_0, z_1, z_2] = \frac{-1}{2} \left( \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right)$$

The moment map image is a triangle with vertices $(0, 0), (-1/2, 0), (0, -1/2)$. 
Examples

Ex. The Lie algebra $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ with the Lie bracket $[\vec{a}, \vec{b}] = \vec{a} \times \vec{b}$ (exterior product).
Consider the diagonal action of $G = SO(3)$ on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ by

$$\Phi(x, y) \rightarrow (\Phi x, \Phi y)$$

Show that the moment map $\mu : \mathbb{R}^6 \rightarrow \mathfrak{so}(3)^* \cong \mathbb{R}^3$ is given by

$$\mu(\vec{x}, \vec{y}) = \vec{x} \times \vec{y}$$

This is called the angular momentum.
Examples

Ex. Consider the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, which can be described explicitly as

$$\{(x, y, [a : b]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : ay = bx\}$$

Construct a Hamiltonian $\mathbb{T}^2$ action on this. Compute the moment map and describe its image. (Generalize to $\mathcal{O}(n)$.)

Ex. Consider the natural action of $U(k)$ on $(\mathbb{C}^{k\times n}, \omega_{std})$ Identify the Lie algebra $u(k)$ with its dual via the inner product $(A, B) = \text{trace}(A^*B)$. Prove that the moment map for this action is given by

$$\mu(A) = \frac{i}{2}AA^*, \text{ for } A \in \mathbb{C}^{k\times n}$$
**Thm.** (Marsden-Weinstein-Meyer) Let \((M, \omega, G, \mu)\) be a Hamiltonian \(G\)-space for a compact Lie group \(G\). Let \(p \in g^*\) be a regular value of \(\mu\) and \(\iota: \mu^{-1}(p) \to M\) be the inclusion map. Assume the stabilizer \(G_p \subset G\) of \(p\) acts freely on \(\mu^{-1}(p)\), then

- orbit space \(M_{\text{red}} = \mu^{-1}(p)/G_p\) is a manifold,
- \(\pi: \mu^{-1}(p) \to M_{\text{red}}\) is a principal \(G_p\)-bundle,
- there exists a symplectic form \(\omega_{\text{red}}\) on \(M_{\text{red}}\) satisfying \(\iota^*\omega = \pi^*\omega_{\text{red}}\).
Examples

Consider $S^1$ action on $\mathbb{C}^{n+1}$ diagonally by

$$e^{i\theta}(z_0, \ldots, z_n) = (e^{i\theta}z_0, \ldots, e^{i\theta}z_n)$$

The moment map is

$$\mu(z_0, z_1, \ldots, z_n) = -\frac{1}{2} \sum_i |z_i|^2$$

For every $r > 0$, we get a symplectic structure on

$$\mu^{-1}(r^2/2)/S^1 = S^{2n+1}/S^1 = \mathbb{C}\mathbb{P}^n$$

The associated symplectic form is the unique form $\omega_{FS}$ such that

$$\pi^*\omega_{FS} = \omega_{\mathbb{C}^{n+1}|S^{2n+1}}$$

where $\pi : S^{2n+1} \to \mathbb{C}\mathbb{P}^n$ is the quotient map.
Ex. For the action of $U(k)$ on $\mathbb{C}^{k \times n}$ with moment map computed above show that

$$\mu^{-1}(\frac{\text{Id}}{2i})/U(k) = G(k, n)$$

is the Grassmannian of $k$-panes in $\mathbb{C}^n$.

Ex. Suppose $Q$ is a smooth manifold with a free proper action of a Lie group $G$. Then the corresponding action of $G$ on $T^*X$ is Hamiltonian. Compute the moment map. Show that there is a symplectomorphism

$$T^*(X/G) \simeq \mu^{-1}(0)/G$$
Some other follow-up topics

- Morse theory for Hamiltonians
- Equivariant cohomology and localization
- Relation with GIT (Kempf-Ness)
- Infinite-dimensional Hamiltonian reduction (moduli of flat connections)
- Quantization
References

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