Symplectic Manifolds, Moment Maps and Hamiltonian Reduction

Yankı Lekili

LSGNT
4 Nov 2022

Symplectic Manifolds

-Classical Mechanics

Newton's second law is the second order ODE

$$m\ddot{q} = F(q)$$

In celestial mechanics, one is primarily concerned with *conservative* force fields , i.e. we assume that work done in moving a particle from one position to other is independent of the path. Mathematically, this means that the vector valued function $F(q)$ is given as a gradient

$$F(q) = -\nabla_q V$$

of a scalar function V.

Ex. Consider $F(q_1, q_2) = q_2\partial_{q_1} - q_1\partial_{q_2}$ on $\mathbb{R}^2$. Is it conservative?

In the Hamiltonian formulation, we can write equations of motion as

$$\dot{q} = \partial H / \partial p \tag{1}$$

$$\dot{p} = -\partial H / \partial q \tag{2}$$

where $p = m\dot{q}$ is the momentum and $H(q, p)$ is the Hamiltonian function given by the sum of kinetic and potential energy:

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

Note that if $V(q) = const.$, $\dot{q} = const.$. This is the law of inertia. In particular, the trajectories are “geodesics” traversed so as to keep the energy, H, constant.

Hamilton (1835) $H : \mathbb{R}^{2n} \to \mathbb{R}$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$.

$$\dot{q} = \partial H / \partial p$$
$$\dot{p} = -\partial H / \partial q$$

The phase flow ϕ_t^H of H is the one-parameter group of transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$

$$(p(0), q(0)) \to (p(t), q(t))$$

For any function $f : \mathbb{R}^{2n} \to \mathbb{R}$, classical observable, we determine the evolution $f_t = f(p(t), q(t))$ of such f,

$$\frac{df_t}{dt} = \{H, f_t\}$$

where

$$\{H, f\} = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}$$

Energy conservation law: $\{H, H\} = 0$.

The Poisson bracket is antisymmetric, and satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

We can express an anti-symmetric bracket via an anti-symmetric matrix η^{ij}

$$\{f, g\}(x) = \eta^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

To understand what Jacobi identity means, let $A = \mathcal{O}(\mathbb{R}^{2n})$ be algebra of classical observables. Define one-parameter family of operators

$$U_t : A \to A$$
$$f \to f_t$$

Ex. Jacobi identity holds if and only if U_t is automorphism of A that respects the Poisson bracket (for any H).

Besides the Jacobi, we also have the Leibniz rule:

$$\{H, FG\} = \{H, F\}G + F\{H, G\}$$

In terms of evolution with respect to H, this simply says

$$d(FG)/dt = (dF/dt)G + F(dG/dt)$$

Definition: A mechanical system is a Poisson bracket $\{\ ,\ \}$ on the algebra of differentiable functions $\mathcal{O}(M)$ on some space M called the phase space and a Hamiltonian function H.

Noether's theorem: If F is a symmetry, i.e $\{F, H\} = 0$, then its conserved $\{H, F\} = 0$.
Poisson's theorem: If F, G are conservation laws (i.e $\{H, F\} = \{H, G\} = 0$), then $\{F, G\}$ is also a conservation law ($\{H, \{F, G\}\} = 0$).

Ex. Describe the motion for the harmonic oscillator $H(p, q) = \frac{p^2}{2m} + k\frac{q^2}{2}$, where k is the rigidity coefficient of the string.

Symplectic Manifolds

If $\det(\eta^{ij}) \neq 0$, then define

$$\omega = \eta_{ij} dx^i \wedge dx^j$$

Then, Jacobi identity holds if and only if $d\omega = 0$.

Def. A manifold M equipped with a non-degenerate closed 2-form ω is called a symplectic manifold.

Ex. Show that a 2-form ω on M^{2n} is non-degenerate if and only if ω^n is a volume form on M.

Ex. Show that on a compact symplectic manifold (M, ω), the class $[\omega] \in H^2(M)$ is non-zero.

On $\mathbb{R}^{2n}$ with co-ordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ the standard symplectic structure is defined by

$$\omega_{std} = \sum_i dq_i \wedge dp_i$$

In terms of the associated Poissson bracket,

$$\{q_j, p_k\} = \delta_{jk}, \{q_j, q_k\} = 0, \{p_j, p_k\} = 0$$

Thm. (Darboux) Any symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{std})$.

On a general symplectic manifold M, given a (Hamiltonian) function $H : M \to \mathbb{R}$, we can define a vector field

$$\omega(X_H, \cdot\,) = dH$$

X_H is called the Hamiltonian vector field, and its flow (motion) is the Hamiltonian flow associated with H.

Ex. Show that

$$(\mathcal{O}(M), \{\,,\,\}) \to (Vect(M), [\,,\,])$$
$$H \to X_H$$

is a Lie algebra homomorphism. The image is denoted by $ham(M, \omega)$.

Examples

- Let Q be any smooth manifold, T^*Q is a symplectic manifold. Given $(q_1, \ldots, q_n)$ local co-ordinates for Q, and co-ordinates $(p_1, \ldots, p_n)$ for the fiber, one can construct a 1-form
$$\alpha = \sum_i p_i dq_i$$
Ex. Check that α is intrinsically defined. We define $\omega = -d\alpha$ as the canonical symplectic form on T^*Q.
- Suppose (M, g, J) is Kähler manifold, define $\omega(X, Y) = g(JX, Y)$, then (M, ω) is symplectic.
- G be a real Lie group with Lie algebra $\mathfrak{g}$. G acts on $\mathfrak{g}$ via adjoint action, and thus also on the dual space $\mathfrak{g}^*$. Then every G-orbit $M \subset \mathfrak{g}^*$ has a canonical symplectic form: the corresponding Poisson bracket is given by $\{x, y\} = [x, y]$ where $x, y \in \mathfrak{g}$ are considered as functions on $\mathfrak{g}^*$.

Symplectomorphism group

A vector field X is a *Hamiltonian* vector field if $\iota_X \omega$ is exact. More generally, a vector field X on M is called *symplectic* if $\iota_X \omega$ is closed.
If ϕ_t is a flow of a symplectic vector field, by Cartan's magic formula, we have:

$$\frac{d}{dt}\phi_t^* \omega = \mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega = 0$$

Cor. (Liouville's theorem) Hamiltonian flow $\phi_t^H : M \to M$ preserves volume (i.e. Hamiltonian flow is incompressible).

Def. $\mathrm{Symp}(M,\omega) = \{\phi : M \to M : \phi^*\omega = \omega\}$

$$\{X : \iota_X\omega \in \mathrm{Ker} d\} = \mathrm{Lie}(\mathrm{Symp}(M,\omega))$$

$Ham(M,\omega) \subset \mathrm{Symp}(M,\omega)$ is the unique subgroup of $\mathrm{Symp}(M,\omega)$ with Lie algebra $ham(M,\omega) = \{X : \iota_X\omega \in \mathrm{Im} d\}$

Ex. Let M be the 2-torus $\mathbb{R}^2/\mathbb{Z}^2 = \{(x,y)\ mod 1\ \}$ equipped with the symplectic form $dx \wedge dy$. Consider the flow ϕ_t given by

$$\dot{x} = 1, \quad \dot{y} = \alpha, \quad \text{for } \alpha \in \mathbb{R}$$

Show that the flow on M is symplectic but not Hamiltonian.

Ex. M as in the previous exercise. Consider the Hamiltonian $H : M \to \mathbb{R}$ given by $H(x,y) = \cos(2\pi x)$. Describe the Hamiltonian flow.

Integrable systems

By Noether's theorem, the integral curves of a Hamiltonian flow are contained in the level sets of the conservations laws, i.e F such that $\{H, F\} = 0$. Such F are also called *integrals of motion*.
If there are many such independent conservation laws, one can characterise the motion by intersecting the level sets.

Def. A Hamiltonian (mechanical) system (M^{2n}, ω, H) is completely integrable if it possesses n integrals of motion $H_1 = H, H_2, \ldots, H_n$ such that

i) $\{H_i, H_j\} = 0$ for all i, j,

ii) $\mu : M \to \mathbb{R}^n, x \to (H_1(x), \ldots, H_n(x))$ has no critical points.

Arnold-Liouville theorem

Thm. Let $H_1, \ldots, H_n$ define a completely integrable system on (M, ω). For $b \in \mathbb{R}^n$, let $M_b = \mu^{-1}(b)$ where $\mu = (H_1, \ldots, H_n)$.

- If M_b is non-empty and compact, then each of its connected components is a diffeomorphic to a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$
- One can choose (angle) co-ordinates $(\phi_1, \ldots, \phi_n)$ identifying connected component of M_b with $\mathbb{R}^n/\mathbb{Z}^n$ such that the flows of the vector field $X_{H_1}, \ldots, X_{H_n}$ are linear.
- The restriction of the symplectic form to M_b is trivial.
- In a neighborhood of M_b, one can choose (action) co-ordinates $(a_1, \ldots, a_n)$ complementary to $(\phi_1, \ldots, \phi_n)$ such that the a_j are integrals of motion and the symplectic form in these co-ordinates is given by

$$\omega = \sum da_j \wedge d\phi_j$$

The connected components of the level sets M_b are *Lagrangian* tori.

The equations of motion (Hamiltonian flow with respect to $H = H_1$) in these co-ordinates are given by

$$\dot{a}_j = 0, \quad \dot{\phi}_j = \omega_j(a),\, j = 1, \ldots, n$$

for some $\omega_j(a) = dH/da_j \in \mathbb{R}$. Thus, the motion on the invariant torus is conditionally periodic, that is

$$a_j = a_j(0) \quad , \quad \phi_j(t) = \omega_j t + \phi_j(0)$$

for some constants $\omega_j = \omega_j(a(0))$.

Def. A submanifold $L \subset M^{2n}$ is called isotropic if $\omega|L = 0$ and is called Lagrangian if in addition $\dim L = n$.

Ex. Show that an isotropic submanifold of M^{2n} has $\dim L \leq n$.

Moment map

Let G be a connected Lie group acting on a symplectic manifold (M, ω) by symplectomorphims. We call such an action a symplectic G-action.

- $H : M \to \mathbb{R}$ any function such that X_H is complete, then its flow defines a symplectic $\mathbb{R}$-action.
- More generally, if $H_1, \ldots, H_k$ are k commuting functions ($\{H_i, H_j\} = 0$ for all i, j), their flows define a symplectic $\mathbb{R}^k$-action.

One way to interpret Arnold-Liouville theorem is that we can find co-ordinates such that a neighborhood of the fiber $\mathbb{T}^n$, can be identified (symplectically) with $\mathbb{T}^n \times B$ with $B \subset \mathbb{R}^n$ and the orbits of the original system gets identified with $\mathbb{T}^n \times \{b\}$, for $b \in B$. We shall next study the case there is a global action of a torus.

Hamiltonian group actions

Given a G-action on M, by linearizing the action, every element $a \in \mathfrak{g}$ defines a vector field ξ_a on M. The action is symplectic if $d(\iota_{\xi_a}\omega) = 0$ for all a. The correct definition of a global Hamiltonian action is given as follows:

Def. A symplectic action of G on M is called *Hamiltonian* if there exists a map, called the *moment map*,

$$\mu : M \to \mathfrak{g}^*$$

such that

- For any $a \in \mathfrak{g}$, the function $H_a(x) = \langle \mu(x), a \rangle$ is the Hamiltonian for the vector field ξ_a.
- For any $a, b \in \mathfrak{g}$, $\{H_a, H_b\} = H_{[a,b]}$
- μ is G-equivariant.

Ex. Show that the last condition is automatic if G is connected.

Suppose a torus $\mathbb{T}$ acts on M^{2n}. Choose a basis a_i of the Lie algebra $\mathfrak{t} = \mathrm{Lie}\mathbb{T}$. If we can find H_i such that $\iota_{a_i}\omega = dH_i$, then for $a = \sum_i \lambda_i a_i$, we can define $\mu(x)$ by

$$\langle \mu(x), a \rangle = \sum_i \lambda_i H_i(x)$$

Ex. Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n | |t_i| = 1\}$ act on $\mathbb{C}^n$ by $(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$. Then check that

$$\mu(z_1, \ldots, z_n) = -\frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$$

is a moment map.

Convexity

Thm. (Atiyah, Guillemin-Sternberg)
The image of a compact connected symplectic manifold under the moment map of an (quasi) effective Hamiltonian torus action is a Delzant polytope. In fact,

- $\mu(M)$ it is the convex hull of $\mu(M^{\mathbb{T}})$,
- $\mu^{-1}(b)$ is connected for all $b \in \mathbb{R}^n$.

Thm. (Delzant) When the torus acting is half the dimension of the manifold and the action is effective, the polytope determines the manifold. More precisely, there is a 1-1 correspondence

$$\{ \text{ toric manifolds } \} \leftrightarrow \{ \text{ Delzant polytopes } \}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \leftrightarrow \mu(M)$$

Polytopes

Def. A rational convex polytope P is a subset of $\mathbb{R}^n$ defined as the intersection of halfplanes

$$S_{a,b} = \{x \in \mathbb{R}^n : a_1x_1 + \ldots + a_nx_n \leq b\}, a_i \in \mathbb{Q}, b \in \mathbb{R}^n$$

We say that P is a *Delzant (unimodular) polytope* if it is a convex rational polytope such that each point on a k-dimensional facet has a neighborhood isomorphic via an integral affine transformation to a neighborhood of the origin in $[0, \infty)^{n-k} \times \mathbb{R}^k$.

Ex. Let M be compact symplectic manifold with a Hamiltonian S^1-action, prove that there exists a fixed point of the S^1-action. In this case, it is enough to check that Delzant condition at each vertex for the polytope.

Examples

Ex. S^1 acts on $(S^2, \omega_{st} = d\theta \wedge dh)$ by rotations with moment map $\mu = h$ equal to the height function and moment polytope $[-1, 1]$.

Ex. Consider the $\mathbb{T}^2$-action on $\mathbb{CP}^2$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] \to [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2]$$

This has moment map

$$\mu[z_0, z_1, z_2] = \frac{-1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right)$$

The moment map image is a triangle with vertices $(0, 0), (-1/2, 0), (0, -1/2)$.

Examples

Ex. The Lie algebra $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ with the Lie bracket $[\vec{a}, \vec{b}] = \vec{a} \times \vec{b}$ (exterior product).
Consider the diagonal action of $G = SO(3)$ on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ by

$$\Phi(x, y) \to (\Phi x, \Phi y)$$

Show that the moment map $\mu : \mathbb{R}^6 \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is given by

$$\mu(\vec{x}, \vec{y}) = \vec{x} \times \vec{y}$$

This is called the *angular momentum*.

Examples

Ex. Consider the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, which can be described explicitly as

$$\{(x, y, [a : b]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : ay = bx\}$$

Construct a Hamiltonian $\mathbb{T}^2$ action on this. Compute the moment map and decribe its image. (Generalize to $\mathcal{O}(n)$.)

Ex. Consider the natural action of $U(k)$ on $(\mathbb{C}^{k\times n}, \omega_{std})$ Identify the Lie algebra $\mathfrak{u}(k)$ with its dual via the inner product $(A, B) = trace(A^*B)$. Prove that the moment map for this action is given by

$$\mu(A) = \frac{i}{2}AA^*, \text{ for } A \in \mathbb{C}^{k\times n}$$

Hamiltonian Reduction

Thm. (Marsden-Weinstein-Meyer) Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $p \in \mathfrak{g}^*$ be a regular value of μ and $\iota : \mu^{-1}(p) \to M$ be the inclusion map. Assume the stabilizer $G_p \subset G$ of p acts freely on $\mu^{-1}(p)$, then

- orbit space $M_{red} = \mu^{-1}(p)/G_p$ is a manifold,
- $\pi : \mu^{-1}(p) \to M_{red}$ is a principal G_p-bundle,
- there exists a symplectic form ω_{red} on M_{red} satisfying $\iota^*\omega = \pi^*\omega_{red}$.

Examples

Consider S^1 action on $\mathbb{C}^{n+1}$ diagonally by

$$e^{i\theta}(z_0, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n)$$

The moment map is

$$\mu(z_0, z_1, \ldots, z_n) = -\frac{1}{2} \sum_i |z_i|^2$$

For every $r > 0$, we get a symplectic structure on

$$\mu^{-1}(r^2/2)/S^1 = S^{2n+1}/S^1 = \mathbb{CP}^n$$

The associated symplectic form is the unique form ω_{FS} such that

$$\pi^* \omega_{FS} = \omega_{\mathbb{C}^{n+1}}|_{S^{2n+1}}$$

where $\pi : S^{2n+1} \to \mathbb{CP}^n$ is the quotient map.

Ex. For the action of $U(k)$ on $\mathbb{C}^{k\times n}$ with moment map computed above show that

$$\mu^{-1}(\frac{\mathrm{Id}}{2i})/U(k) = \mathbb{G}(k,n)$$

is the Grassmannian of k-panes in $\mathbb{C}^n$.

Ex. Suppose Q is a smooth manifold with a free proper action of a Lie group G. Then the corresponding action of G on T^*X is Hamiltonian. Compute the moment map. Show that there is a symplectomorphism

$$T^*(X/G) \simeq \mu^{-1}(0)/G$$

Some other follow-up topics

- Morse theory for Hamiltonians
- Equivariant cohomology and localization
- Relation with GIT (Kempf-Ness)
- Infinite-dimensional Hamiltonian reduction (moduli of flat connections)
- Quantization

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