Special holonomy: notes for LSGNT lecture, January 2026

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1 Holonomy groups and the Berger list

An n-dimensonal Riemannian manifold (M, g) has a unique Levi Civita connection on TM which has zero torsion and is compatible with the metric. Fix a base point p in M. The *holonomy group* $G \subset O(TM_p)$ is the group of rotations effected by parallel transport around loops based at p.

Let $G \subset O(n)$. A *torsion-free G-structure* on M is a principle G-bundle $P \to M$, an isomorphism of TM with the bundle associated to the representation of G on $\mathbf{R}^n$ and a G-connection on P which defines a torsion-free connection on TM.

The two notions are essentially equivalent. A torsion-free G-structure defines a metric with holonomy contained in G and a Riemannian manifold with holonomy G has a torsion-free G-structure.

A manifold (M, g) is locally symmmetric if the curvature tensor is covariant constant $\nabla \mathrm{Riem} = 0$. Equivalently the curvature tensor is preserved by parallel transport. Such manifold is locally isometric to a symmetric space G^+/G and has holonomy contained in G. Lie theory gives a complete classification of symmetric spaces.

The holonomy groups G which can arise for simply connected, non-symmetric, irreducible manifolds are severely constrained. The curvature tensor lies in the kernel of

$$s^2\mathbf{g} \subset \Lambda^2 \otimes \Lambda^2 \to \Lambda^4.$$

For most groups $G \subset SO(n)$ this kernel is zero. In the 1950's Marcel Berger produced a list of posibilities which was refined subsequently by other authors. The upshot is the following list:

1. $G = SO(n)$; the generic case.
2. $n = 2m$ and $G = U(m)$, these are *Kähler metrics* on complex manifolds.
3. $n = 2m$ and $G = SU(m)$ these are *Calabi-Yau metrics*, with a covariant constant holomorphic volume form.
4. $n = 4\ell$ and $G = Sp(\ell)$, these are *hyperkähler metrics* with three compatible Kähler structures.

5. $n = 4\ell$ and $G = Sp(\ell)Sp(1)/\pm 1$, these are *quaternion Kähler metrics* with a triple of complex structures at each point but not necessarily any global complex structures.

6. the exceptional case $G_2 \subset SO(7)$.

7. the exceptional case $\mathrm{Spin}(7) \subset SO(8)$.

James Simons showed this list almost coincides with the list of groups $G \subset SO(n)$ which act transitively on the sphere S^{n-1}.

The cases (2)-(5) can be studied via complex geometry.

Yau's Theorem states that a Calabi-Yau structure on a compact manifold is equivalent to a complex structure with non-vanishing holomorphic m-form and a Kahler class.

For the more special case $Sp(\ell) \subset SU(2\ell)$ we need a holomorphic symplectic structure.

A quaterion Kähler manifold M has a twistor space $Z \to M$ which is a complex manifold with a (non-holomorphic) fibration over M having holomorphic S^2-fibres. (Example: $\mathbf{CP}^{2\ell+1} \to \mathbf{HP}^{\ell}$.)

For the rest of these notes we concentrate on the exceptional cases (6),(7). However there are reasons to think of $SU(3)$ also as an exceptional case.

2 Description of the exceptional cases

Different approaches are possible using

- exteror algebra.
- spinors
- octonions, triality. ...

We will emphasise the first route.

Begin with $\mathbf{R}^6 = \mathbf{C}^3$. We have an $SU(3)$-invariant symplectic form ω and complex 3-form $\Theta = dz_1 dz_2 dz_3 = \tilde{\rho} - i\rho$.

Now take $\mathbf{R}^7 = \mathbf{C}^3 \times \mathbf{R}$ with co-ordinate t on $\mathbf{R}$. We have a 3-form

$$\phi = \rho + \omega dt,$$

which is preserved by $SU(3) \subset SO(6) \subset SO(7)$.

Define G_2 to be the stabiliser in $SO(7)$ of ϕ.

Next consider $\mathbf{R}^4$ with orientation and Euclidean structure. There is a space Λ^+ of self-dual 2-forms with standard basis

$$\omega_i = dx_0 dx_i + dx_j dx_k$$

(ijk) cyclic. Take $\mathbf{R}^7 = \mathbf{R}^4 \times \mathbf{R}^3$ with coordinates y_i on $\mathbf{R}^3$ and write

$$\phi' = dy_1 dy_2 dy_3 - \sum \omega_i dy_i.$$

If we have identify $\mathbf{R}^3$ with Λ^+ we see that this form ϕ' is preserved by $SO(4)$.

It is easy to write down an isomorphism between $\mathbf{C}^3 \times \mathbf{R}$ and $\mathbf{R}^4 \times \mathbf{R}^3$ which takes ϕ to ϕ'. (Exercise).

Then we see that $SO(4) \subset G_2$ and it follows easily that

- G_2 acts transitively on the unit sphere $S^6 \subset \mathbf{R}^7$;
- the stabiliser of a point in S^6 is $SU(3)$, so $S^6 = G_2/SU(3)$.

Hence $\dim G_2 = 14$. It is an exceptional Lie group of rank 2. A maximal torus in G_2 can be obtained either from that in $SU(3)$ or in $SO(4)$.

On $\mathbf{R}^7$ we have a 4-form $*\phi = \tilde{\rho} dt + \omega^2$.

Take $\mathbf{R}^8 = \mathbf{R}^7 \times \mathbf{R}$ with co-ordinate s on $\mathbf{R}$. We have a 4-form

$$\Omega = \phi ds + *\phi,$$

which is preserved by $G_2 \subset SO(7) \subset SO(8)$. Define a group G to be the stabiliser of Ω in $SO(8)$.

Next take $\mathbf{R}^8 = \mathbf{C}^4$. Write $\tilde{\omega}$ for the standard symplectic form and $\tilde{\Theta}$ for the complex 4-form $dz_1 dz_2 dz_3 dz_4$. Define and $SU(4)$-invariant 4-form

$$\Omega' = \tilde{\omega}^2 + \mathrm{Re}(\Theta).$$

It is easy to write down an isomorphism between $\mathbf{R}^7 \times \mathbf{R}$ and $\mathbf{C}^4$ wich takes Ω to Ω'. (Exercise)

Then we see that $SU(4) \subset G$ and it follows easily that

- G acts transitively on the unit sphere $S^7 \subset \mathbf{R}^8$;
- the stabiliser of a point in S^7 is G_2, so $S^7 = G/G_2$.

Hence dim $G = 21$. In fact this subgroup $G \subset SO(8)$ is isomorphic to Spin(7), the double cover of $SO(7)$.

One way to see this is to consider the G-invariant decomposition

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21},$$

where Λ^2_{21} is the Lie algebra of G. The action on the other factor gives a homomorphism

$$G/\pm 1 \rightarrow SO(7)$$

which one checks is an isomorphism.

Another way is to consider the spin representation S of Spin(7). This can be considered as an 8-dimensional real vector space and we get a subgroup Spin(7) $\subset SO(8)$.

The conclusion of the above is that

- a Riemannian 6-manifold has holonomy (contained in) $SU(3)$ if it has covariant constant forms ω, ρ, algebraically equivalent to the model above at each point;
- a Riemannian 7-manifold M has holonomy (contained in) G_2 if it has a covariant constant form ϕ, algebraically equivalent to the model above at each point;
- a Riemannian 8-manifold M has holonomy (contained in) Spin(7) if it has a covariant constant form Ω, algebraically equivalent to the model above at each point.

An alternative spinorial approach is

- In dimension 6, the positive spin representation is on $\mathbf{C}^4$ and Spin(6) act transitively on the unit sphere S^7. The group $SU(3) \subset SO(6)$ is the stabiliser of a unit spinor.
- In dimension 7, the spin representation is on $\mathbf{R}^8$ and Spin(7) act transitively on the unit sphere S^7. The group $G_2 \subset SO(7)$ is the stabiliser of a unit spinor.
- In dimension 8, the positive spin representation is on $\mathbf{R}^8$ and Spin(8) act transitively on the unit sphere S^7. The group Spin(7) $\subset SO(8)$ is the stabiliser of a unit spinor.

It follows that manifolds with holonomy (contained in) $SU(3), G_2,$ Spin(7) are exactly those with a covariant constant unit spinor field.

Additional facts

1. Define a cross product $\mathbf{R}^7 \times \mathbf{R}^7 \to \mathbf{R}^7$ by
$$\phi(u, v, w) = \langle u \times v, w \rangle.$$
This defines the product on the imaginary octonions.

2. Part of the symmetry of the 4-form $\Omega \in \Lambda^4 \mathbf{R}^8$ can be seen as follows. Let $\mathbf{A}$ be 3-dimensional affine space over the field of two elements. This contains 8 points and 14 planes. Take a real vector space V with basis elements e_i corresponding to the points p_i of $\mathbf{A}$. Then a monomial $\pm e_i \wedge e_j \wedge e_k \wedge e_l$ occurs in Ω if and only if p_i, p_j, p_k, p_l lie on a plane.

   Similarly for the 3-form ϕ and the projective plane over this field (the Fano plane).

3. The 3-form ϕ defines the Euclidean structure on $\mathbf{R}^7$ via the quadratic form $v \mapsto i_v(\phi)^2 \wedge \phi$. Thus it defines the 4-form $*\phi$. Similarly the 4-form Ω determines the Euclidean structure on $\mathbf{R}^8$.

4. The $GL(7)$ orbit of ϕ in $\Lambda^3(\mathbf{R}^7)$ is *open*. The $GL(8)$ orbit of Ω in $\Lambda^4(\mathbf{R}^8)$ is a submanifold of dimension 64-21=43.

5. A closed 4-form on M^8 which is algebraically equivalent to Ω at each point defines a torsion-free Spin(7) structure. Similarly for a 3-form on M^7 algebraically equivalent to the model at each point and such that $\phi, *\phi$ are both closed. (Theorem of Fernandez and Gray)

6. Metrics with holonomy G_2, Spin(7) have zero Ricci curvature.

3 Geometry in special holonomy manifolds

Submanifolds

A subset $X \subset \mathrm{Gr}_p(\mathbf{R}^n)$ is *calibrated* by a p-form $a \in \Lambda^p$ if for any p-dimensional subspace $\Pi \subset \mathbf{R}^7$ the restriction of a is $\leq$ the volume form with equality if $\alpha \in X$. If M is an n-manifold and α is a closed p-form on M which is algebraically equivalent to a at each point we get the notion of a calibrated submanifold. These are minimal submanifolds (in the compact case, absolutely area minimising in their homology class).

In $(\mathbf{C}^3, \omega, \Theta)$ the set of complex lines is calibrated by ω, similarly for planes and ω^2. Complex curves and surfaces in Kähler manifolds are calibrated.

Special Lagrangian 3-planes $\Pi \subset \mathbf{C}^3$ are defined by the conditions that ω and $\tilde{\rho}$ restrict to zero. They are calibrated by ρ. We get the notion of Special Lagrangian submanifolds in Calabi-Yau threefold. More generally we can rotate the holomorphic form by a phase $e^{i\theta}$ so for each θ we get a class of $SLag_\theta$ submanifolds..

In $\mathbf{R}^7, \phi$ a 4-dimensional subspace Π is *coassociative* if $\phi|_\Pi = 0$. They are calibrated by $*\phi$. A 3-dimensional subspace is *associative* if its orthogonal complement is co-associative. The group G_2 acts transitively on the set of associative/co-associative subspaces. We get the notions of associative and co-associative submanifolds in a manifold (M, ϕ) of holonomy G_2.

In $CY^3 \times \mathbf{R}$, $SLag_0 \times$ point and complex curve $\times \mathbf{R}$ are associative while $SLag_{\pi/2} \times \mathbf{R}$ and Complex surface $\times$ point are co-associative.

In $(\mathbf{R}^8, \Omega)$ a 4-plane Π is *Cayley* if for any normal vector v the contraction $i_v \Omega$ vanishes on Π. These are calibrated by Ω and we get the notion of Cayley submanifolds.

All of these submanifolds have elliptic deformation theories.

Gauge Theory

On a manifold M^8 with torsion free G_2 structure Ω we have $\Lambda^2 = \Lambda^2_{21} + \Lambda^2_7$. A connection on a bundle $E \to M$ is called a Spin(7)-instanton if the component F_7 of its curvature is zero. Tihis is an elliptic equation moduo gauge equivalence. The quadratic form

$$\alpha \mapsto \alpha^2 \wedge \Omega$$

has opposite signs on Λ^2_7 and $Lambda^2_{21}$. It follows from Chern-Weil theory that such a connection minimises the L^2 norm of the curvature over all connections on E. In the case when M is a Calabi-Yau 4-fold, Hermitian Yang-Mills connections on stable holomorphic bundles are Spin(7)-instantons.

Similarly, on a manifold M^7 with torsion free G_2 structure ϕ we have $\Lambda^2 = \Lambda^2_{14} + \Lambda^2_7$ and we define G_2-instantons in the same way. This is also an elliptic equation, when suitably formulated. On $CY^3 \times \mathbf{R}$ the pull-back of Hermitian-Yang Mills connections on CY^3 are G_2-instantons.

4 Moduli theory of exceptional holonomy structures

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The decomposition of Λ^4 as a Spin(7) representation is

$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}.$$

Here Λ^4_1 is spanned by Ω and $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$ is the tangent space to the $GL(8, \mathbf{R})$ orbit of Ω.

The map $v \mapsto i_v(\Omega)$ defines an irreducible component $\Lambda^3_8 \subset \Lambda^3$.

It follows that an infinitesimal deformation of a torsion free Spin(7) structure Ω on a manifold M^8 is given by a closed form σ in $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$.

The Lie derivative of Ω along a vector field v is $di_v\Omega$. The map $v \mapsto i_v(\Omega$ defines an irreducible component $\Lambda^3_8 \subset \Lambda^3$.

So we expect that the tangent space to the moduli space $\mathcal{M}$ of torsion-free Spin(7)-structures to be

$$T\mathcal{M} = \frac{\text{closed forms } \sigma \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{35}}{\text{Im } d : \Omega^3_8 \to \Omega^4}.$$

The component $d : \Omega^3_8 \to \Omega^4_1 \oplus \Omega^4_7$ can be identified with the Dirac operator and this implies that its cokernel is $\mathbf{R}\Omega$. The space Λ^4_{35} is the set of anti-self dual forms Λ^4_-. It follows that

$$T\mathcal{M} = \mathbf{R} \oplus H^4_-$$

where H^4_- is the space of closed anti-self dual 4-forms, which forms a maximal negative subspace for the cup product form on H^4.

It can be shown that the map $\Omega \to [\Omega] \in H^4(M)$ embeds $\mathcal{M}$ locally as a submanifold of H^4 of dimension $1 + b^4_-$.

Similarly in the G_2- case the cohomology class of ϕ gives a local equivalence between the moduli space and H^3.

5 Constructions

All known constructions of compact manifolds of exceptional holonomy involve singular perturbation techniques.

Resolving orbifolds(Joyce, Joyce-Karigiannis)

A prototype is the Kummer construction for $K3$ surfaces (real dimension 4). In a complex geometry description we start with a complex torus X and consider the involution $x \mapsto -x$. This has 16 fixed points. The quotient M_0 of the torus by the involution has 16 singular points, with neighbourhoods modelled on $\mathbf{C}^2/\pm 1$. We get a smooth complex surface M by blowing up the singular points, so M contains 16 2-spheres of self-intersection -2.

The Calabi-Yau metrics on M depend on a choice of Kahler class, which specifies the area of each -2 sphere. Consider the case when these are all equal to a small number ϵ. The Calabi-Yau metric can be contained by a gluing construction using

- The flat metric on M_0;
- The Eguchi-Hanson metric on $Y = T^*S^2$, which is hyperkahler and asymptotic to $\mathbf{C}^2/\pm 1$.

In the simplest case of the higher dimensional situation, Joyce consider flat 7-dimension orbifold M_0 with singular set locally equivalent to $T^3 \times \mathbf{C}^2/\pm 1$, then glues in $T^3 \times Y$.

Twisted connected sums (Kovalev, Corti *et al*)

There is another prototype with $K3$ surfaces. Let S_1, S_2 be rational elliptic surfaces containing isomorphic smooth elliptic curves D_1, D_2. Form a singular surface M_0 by identifying D_1 and D_2. It can be shown that M_0 has a smoothing M_t.

For small t the Calabi-Yau metric on M_t can be described as follows. The complement $S_1 \setminus D_1$ has a Tian-Yau hyperkahler metric which is complete and asymptotic to the cylinder $T^3 \times \mathbf{R}$. Similarly for $S_2 \setminus D_2$. The metric on M_t is obtained by gluing these two asymptotically cylindrical metrics with a very long neck (length $\sim -\log |t|$).

In higher dimensions, we take a pair of complex 3-folds Z_1, Z_2 and anti-canonical divisors D_1, D_2 with trivial normal bundles. Then $Z_i \setminus D_i$ have Tian-Yau metrics, asymptotic to $D_i \times S^1 \times \mathbf{R}$.

Now take the products $S^1 \times (Z_i \setminus D_i)$ and glue the cylindrical ends but *interchanging* the S^1 factors. This requires a more subtle matching condition on the divisors D_i.