POINCARÉ DUALITY EXERCISES

MICHAEL SINGER

Exercise 1. Verify that the homology and cohomology groups of the n-sphere S^n satisfy Poincaré duality. Do the same for complex projective space $\mathbb{C}P^n$.

Exercise 2. What are the homology groups (with real coefficients) of the real projective plane $\mathbb{R}P^2$? Do they satisfy Poincaré duality? Explain why or why not.

Exercise 3. Describe the subvariety $M := \{X_1 X_2 = 0\}$ inside $\mathbb{C}P^2$, with homogeneous coordinates $(X_0 : X_1 : X_2)$. Compute its cohomology groups. Do they satisfy Poincaré duality? Discuss why or why not. [Hint: Mayer–Vietoris (see below) can be used to compute the cohomology of M.]

Exercise 4. Let Σ be a compact oriented surface (2-manifold) of genus g. Recall that $H_1(\Sigma, \mathbb{R})$ is of dimension $2g$. Describe a basis $(e_1, \ldots, e_g, f_1, \ldots, f_g)$ of this vector space which satisfies

$$e_j \cdot f_k = \delta_{jk}$$

(the intersection pairing in homology). (A description with a good picture is fine.)

Exercise 5. Compute the intersection form on $H_2(S^2 \times S^2)$ and on $H_2(\mathbb{C}P^2)$. What is the intersection form on $H_2(M)$ if M is the blow-up of $\mathbb{C}P^2$ in a point?

Exercise 6. Prove the Poincaré Lemma, that is compute the de Rham cohomology groups of $\mathbb{R}^n$, by induction on n.

[Hint: having done the case $n = 1$, write a closed k-form on $\mathbb{R}^{n+1}$ in the form

$$A = \alpha + dy \wedge \beta$$

where the variables in $\mathbb{R}^{n+1}$ are $(x, y) = (x_1, \ldots, x_n, y)$. Here neither α or β involve dy. (That is, think of α as a family of k-forms on $\mathbb{R}^n$, depending parametrically on the extra variable y and similarly for β.)

For the induction, we want to get rid of the variable y as far as we can. Do this by solving $\partial\lambda/\partial y = \beta$. Now show that $A - d\lambda$ doesn't feature y or dy and is a closed form k-form on $\mathbb{R}^n$. This is the inductive step.]

Remark: There are many other ways of proving the Poincaré Lemma...

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Exercise 7. The definition of compact cohomology $H^*_c(M)$ is given by taking closed forms with compact support modulo the forms that are d of a form with compact support. Prove the Poincaré Lemma

$$H^k_c(\mathbb{R}^n) = \begin{cases} 0 \text{ if } k < n; \\ \mathbb{R} \text{ if } k = n. \end{cases}$$

In thinking about this, first guess or decide what the map $H^n_c(\mathbb{R}^n) \to \mathbb{R}$ might be and then try to prove it's an isomorphism. It is worth thinking about small values of n before diving in to the general case.

Exercise 8. Let $M = U \cup V$ where U and V are open subsets. For each k, we have a short exact sequence of complexes[1]

$$0 \to \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \to 0$$

where the first (non-zero) map is direct sum of restrictions and the next is difference of restrictions. Show that this is indeed exact. You will probably need a partition of unity subordinate to the cover, namely ρ_U, ρ_V smooth functions on M such that $\rho_U + \rho_V = 1$, ρ_U with support in U, ρ_V with support in V, to prove that the map to $\Omega^k(U \cap V)$ is surjective, for example.

Hence obtain the Mayer–Vietoris sequence in de Rham cohomology

$$\ldots \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(M) \to \ldots.$$

Give an explicit description of the connecting homomorphism $H^k(U \cap V) \to H^{k+1}(M)$ using the partition of unity. [This is a standard 'diagram chase' but if you've not come across these things before, it is likely to be challenging.] M need not be compact in this exercise.

Exercise 9. In the situation of Ex 8, show that for compactly supported forms show that the correct analogue of the short exact sequence of complexes is

$$0 \to \Omega^k_c(U \cap V) \to \Omega^k_c(U) \oplus \Omega^k_c(V) \to \Omega^k_c(M) \to 0$$

and obtain the corresponding long exact sequence for compactly supported cohomology.

Exercise 10. According to Poincaré duality, if Y^k is an compact oriented submanifold of M^n then the linear function $H^k(M, \mathbb{R}) \to \mathbb{R}$ defined by

$$\alpha \mapsto \int_Y \alpha$$

must also be given by

$$\alpha \mapsto \int_M \alpha \wedge \eta_Y$$

for η_Y some closed $(n-k)$-form on M.

For the case that Y is the origin in $\mathbb{R}^n$, show that we can take $\eta = \beta(x)dx$ where $\beta(x)$ is a compactly supported smooth function on $\mathbb{R}^n$, $dx =$

[1]This means that the given horizontal maps commute with d.

$dx^1 \dots dx^n$ is the standard volume element and the total mass of η is equal to 1:

$$\int \eta = \int_{\mathbb{R}^n} \beta(x) dx = 1.$$

Remark: by replacing η by $\delta^{-n}\beta(x/\delta)dx$, we may take the support of η to be as close as we like to 0. This is true in general: one can choose η_Y to be supported as close as we like to Y.

Exercise 11. Suppose that M is oriented and has a finite good cover,

$$M = \bigcup_{j=1}^{N} U_j$$

where the U_j and all finite non-empty intersections of the U_j are diffeomorphic to $\mathbb{R}^n$. (We do not assume that M is compact.)[2]

Prove that the pairing between $H^k(M)$ and $H_c^{n-k}(M)$ induced by integration of wedge product of forms identifies each space with the dual of the other (Poincaré duality).

Of course this is a substantial exercise. The ingredients are the Poincaré Lemmas for ordinary and compactly supported cohomology of $\mathbb{R}^n$, the Mayer–Vietoris sequences, the five lemma from homological algebra and induction on N. It is worth looking at the case $N = 2$ to see how the argument would go.

[2]It is a fact that locally finite good covers exist for any M (one can use a metric and consider geodesic balls for this). So if M is compact, there will be a finite good cover. But non-compact spaces like $\mathbb{R}^n$ that are 'simple at infinity' may also admit finite good covers.