POINCARÉ DUALITY NOTES

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1. Transverse intersection

Let M be an oriented manifold of dimension n without boundary. Let Y and Z be closed submanifolds. If $p \in Y \cap Z$, we say that Y and Z meet transversely at p if

$$T_pY + T_pZ = T_pM \text{ or equivalently } N_p^*Y \cap N_p^*Z = 0 \tag{1.1}$$

Here T_p is the tangent space and N_p^*Y is the conormal space, that is the annihilator of T_pY inside T_p^*M. If Y and Z meet transversely at p then in a neighbourhood of p, $Y \cap Z$ is a submanifold of codimension equal to the sum of the codimensions of Y and Z (implicit function theorem).

Say that Y and Z intersect transversely if the intersect transversely for all points p of $Y \cap Z$. Then $Y \cap Z$ is a (possibly not connected) submanifold of M.

If Y and Z are of complementary dimensions k and $n-k$, and at least one of them is compact, then transverse intersection implies that the intersection is a finite set of points.

Transversality is a generic condition, analogous to intersection of hyperplanes in $\mathbb{R}^n$. If A and B are affine subspaces of $\mathbb{R}^n$ of codimensions a and b respectively then $A \cap B$ is 'almost always' an affine subspace of codimension $a+b$. Similarly, if Y and Z are two submanifolds of M then there are arbitrarily small perturbations Y', Z' of Y and Z so that Y' an Z' intersect transversely.

2. Intersection number

Let M^n be oriented and Y^k and Z^{n-k} be closed oriented submanifolds. The intersection number $Y \cdot Z$ of Y and Z is defined to be

$$Y \cdot Z \sum_{p \in Y' \cap Z'} (\pm 1) \tag{2.1}$$

where: Y' and Z' are small transversely intersecting perturbations of Y and Z and p is counted positively if the orientation of $T_pY \oplus T_pZ$ agrees with that of T_pM and negatively otherwise.

Example When two oriented circles in $\mathbb{R}^2$ intersect in two points, one will be counted positively and the other negatively, so the intersection number will be zero.

Remark One can make this construction even if we don't have orientations, provided we work mod 2.

Our submanifolds Y and Z represent homology classes in $H_k(M)$ and $H_{n-k}(M)$ respectively and $(Y, Z) \mapsto Y \cdot Z$ can be extended to define a $\mathbb{Z}$-bilinear map

$$H_(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z} \tag{2.2}$$

3. Poincaré duality (in homology)

Our first statement of Poincaré duality is that *if M is compact* this is a *perfect pairing*: for fixed Z, the functional $Y \mapsto Y \cdot Z$ is 0 if and only if Z is a torsion

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class. Passing to real (or rational) coefficients gets rid of the torsion and we have a *nondegenerate pairing*

$$H_k(M, \mathbb{R}) \times H_{n-k}(M, \mathbb{R}) \to \mathbb{R} \tag{3.1}$$

so that each of H_k and H_{n-k} is identified with the dual of the other.
Simple consequences Recall the Betti numbers $b_k(M) = \dim H_k(M, \mathbb{R})$. Then for a compact oriented manifold, $b_k(M) = b_{n-k}(M)$ by Poincaré duality. It follows immediately that if $n = \dim M$ is odd then the Euler characteristic

$$\chi(M) = \sum_{k=1}^{n} (-1)^k b_k(M) \tag{3.2}$$

equals zero.

Suppose n is even, $n = 2m$. Then on the middle-dimensional homology we have a perfect pairing

$$H_m(M, \mathbb{Z}) \times H_m(\mathbb{Z}) \to \mathbb{Z} \tag{3.3}$$

which will be skew if m is odd and symmetric if m is even. If m is odd, it follows that $b_m(M)$ is *even* because a skew form on a real vector space can only be non-degenerate if the dimension of that vector space is even. (This checks with the fact that $b_1(\Sigma) = 2g$ for a compact Riemann surface of genus g.)

If m is even then we have a *symmetric* $\mathbb{Z}$-bilinear form and this is an important and subtle invariant of M. Over $\mathbb{R}$, the only invariant of such a form is its signature (number of positive and negative eigenvalues). Over $\mathbb{Z}$ the situation is much more subtle.

Hasse–Minkowski gives that if Q is a non-degenerate $\mathbb{Z}$-bilinear form then

- If Q is *indefinite* then
$$Q = \mathrm{diag}(\pm 1, \ldots, \pm 1)$$
or
$$Q = rH \oplus sE_8$$
where
$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
and
$$E_8 = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$
- If Q is *definite* there are only finitely many possibilities (including the standard from ± 1) for each fixed rank.

Note that H arises as the intersection form on $H_2(S^2 \times S^2)$ and the combination $3H \oplus 2E_8$ arises as the intersection form on $H_2(K3)$.

In the first case, $n = 4$, Donaldson proved in the early 1980s, using instanton moduli spaces, that if M is smooth, compact, oriented and simply connected, then if the intersection form is definite, it must be standard. On the other hand, at about the same time, Freedman proved that *all* intersection forms are realised by *topological* (compact, oriented, simply connected) 4-manifolds. This discrepancy led to the discovery (or perhaps invention) of fake $\mathbb{R}^4$'s: smooth manifolds which are homemorphic but not diffeomorphic to $\mathbb{R}^4$.

4. De Rham cohomology and Poincaré duality

On our oriented n-manifold M, consider the de Rham complex $\Omega^*(M)$. This is a *differential graded complex*. Ω^k is the space of smooth differential forms of degree k on M. Wedge product gives the algebra structure:

$$(\alpha,\beta)\mapsto \alpha\wedge\beta,\ \beta\wedge\alpha = (-1)^{jk}\alpha\wedge\beta$$

if α is of degree j and β is of degree k. Exterior derivative d increases form degree by 1 and satisfied $\mathrm{d}^2 = 0$. The 'graded Leibniz rule' is

$$\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^j\alpha\wedge\mathrm{d}\beta$$

if α is of degree j.

Because $\mathrm{d}^2 = 0$, define

$$H^k(M) = \ker(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))/\mathrm{d}\Omega^{k-1}(M) \tag{4.1}$$

sometimes called the de Rham cohomology to distinguish it from other favourite definitions you may know.

Differential forms are designed to be integrated over oriented submanifolds. That is, for oriented Y of dimension k, we have a linear functional

$$\Omega^k(M)\to\mathbb{R},\ \alpha\mapsto\int_Y\alpha.$$

This works even if Y has a boundary and then

$$\int_Y \mathrm{d}\gamma = \int_{bY}\gamma$$

where bY denotes the (oriented) boundary of Y (Stokes Theorem). If Y has no boundary then the above formula shows that integration over Y descends to give a well-defined functional on $H^k(M)$.

The wedge product combines with integration to give the bilinear map

$$(\alpha,\beta)\mapsto\int_M\alpha\wedge\beta$$

which is identically zero unless α and β are of complementary degree. This is defined initially as a pairing on $\Omega^*(M)$ but also descends to de Rham cohomology, giving a pairing

$$H^k(M,\mathbb{R})\times H^{n-k}(M,\mathbb{R})\to\mathbb{R}. \tag{4.2}$$

Our second statement of Poincaré duality is that this pairing is non-degenerate, identifying H^k and H^{n-k} naturally as duals of each other.

5. The Poincaré dual of a submanifold

Our non-degenerate pairing (4.2) has the consequence that if $f : H^k(M,\mathbb{R})\to\mathbb{R}$ is any linear map, then it is representable by a unique class $\eta_f\in H^{n-k}(M,\mathbb{R})$, so

$$f(\alpha) = \int_M\alpha\wedge\eta_f. \tag{5.1}$$

In particular, integration over a closed oriented submanifold Y^k (or more generally smooth k-cycle) defines uniquely a class $\eta_y\in H^{n-k}(M)$ often called the Poincaré dual of Y:

$$\int_Y\alpha = \int_M\alpha\wedge\eta_Y. \tag{5.2}$$

The two versions of Poincaré duality are 'the same' in the sense that, with the above notation,

$$Y\cdot Z = \pm\int\eta_Y\wedge\eta_Z; \tag{5.3}$$

(transvers) intersection of submanifolds (or cycles) corresponds to wedge product of the Poincaré dual classes in cohomology.

6. Currents—a unifying idea

In $\mathbb{R}^n$, a distribution u is a continuous linear functional on the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions of compact support. I shall not give the definition of the topology on $C_c^\infty(\mathbb{R}^n)$, you can find this in standard books such as Hörmander, volume 1 or de Rham's 'Variétés differentiables'. Denote the space of distributions by $C^{-\infty}(\mathbb{R}^n)$.

If f is continuous (or even just L^1) then it defines a distribution through the rule

$$\phi \mapsto \int f(x)\phi(x)\,\mathrm{d}x$$

for any $\phi \in C_c^\infty(\mathbb{R}^n)$. The other classic example is the Dirac δ, which is just evaluation at the origin,

$$\delta_0 : \phi \mapsto \phi(0).$$

Similarly, a current on an oriented manifold M is a continuous linear functional on the space of compactly supported form $\Omega_c^*(M)$. We say that a current T is homogeneous of *dimension* k if

$$T[\alpha] = 0$$

unless $\alpha \in \Omega_c^k(M)$. The space of such currents will be denoted $\Omega_{-\infty,k}(M)$.

Closed oriented submanifolds (and indeed smooth chains) of dimension k as well as smooth (or not so smooth) differential forms of *degree* $n-k$ both give examples of currents of dimension k. Indeed if Y is our usual closed submanifold of dimension k,

$$\alpha \mapsto \int_Y \alpha \tag{6.1}$$

and is continuous. (If this functional were not continuous in α, we might conclude we had the wrong definition of continuity!) Similarly, if $\phi \in \Omega_c^{n-k}(M)$, then

$$\alpha \mapsto \int \phi \wedge \alpha \tag{6.2}$$

is a continuous linear functional. Note that the coefficients of ϕ do not need to be smooth for this integral to be well-defined. Indeed, a more-or-less equivalent way to think of currents is as differential forms whose coefficients are distributions rather than smooth functions.

We chose to grade currents by dimension. In de Rham's book he also uses degree, with the formula degree $= n - \dim$. On

Distributions in $\mathbb{R}^n$ can be differentiated and also multiplied by smooth functions. (You cannot in general multiply distributions, though you can in some circumstances if you know about the so-called wave-front set, which measures the singularities of distributions. For example, in $\mathbb{R}$ you can't multiply $\delta(x)$ by itself, but in $\mathbb{R}^2$ with coordinates (x, y) you can multiply $\delta(x)$ (integration over the y-axis) by $\delta(y)$ (integration over the x-axis). The result is δ_0, evaluation at 0.) These operations are extended by 'testing' against a smooth function of compact support:

$$\partial_j u[\phi] = -u[\partial_j\phi], \quad fu[\phi] = u[f\phi] \tag{6.3}$$

where $\partial_j = \partial/\partial x_j$, u is a distribution, f is a smooth function and $\phi \in C_c^\infty(\mathbb{R}^n)$ is a test function. The point is that if u is smooth, in the sense that

$$u[\phi] = \int u(x)\phi(x)\,\mathrm{d}x$$

then (6.3) are evidently correct formulae, but they continue to make sense even if u is a more general distribution like the Dirac δ.

7. Homology of currents

The sign is annoying in (6.3) and becomes even more confusing when we replace a partial derivative by exterior derivative! For this reason, define the boundary bT of a current T using the same principle,

$$\mathrm{b}T[\alpha] = T[\mathrm{d}\alpha] \tag{7.1}$$

This *decreases dimension* by one (and accordingly increases degree by one). If T is a current of dimension given by integration over a k-chain, then $\mathrm{b}T$ is the oriented boundary of this chain and (7.1) is Stokes theorem again. If T is given by integration against a smooth form β of degree $n-k$, then $\mathrm{b}\beta = \pm d\beta$.

It follows from (7.1) that $\mathrm{b}^2 = 0$ and so we have a complex $(\Omega_{-\infty,*}, \mathrm{b})$ of which we can take the homology. Define these groups to be

$$H_{-\infty,k}(M) = \ker(\mathrm{b} : \Omega_{-\infty,k} \to \Omega_{-\infty,k-1})/\mathrm{b}\Omega_{-\infty,k+1} \tag{7.2}$$

We have duality with the usual de Rham complex built in from the ground here in the sense that (if M is compact and oriented) we have the picture

$$\begin{array}{ccccc} \cdots & \Omega^k(M) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{k+1}(M) & \cdots \\ & \updownarrow & & \updownarrow & \\ \cdots & \Omega_{-\infty,k}(M) & \xleftarrow[\ \mathrm{b}\ ]{} & \Omega_{-\infty,k+1}(M) & \cdots \end{array} \tag{7.3}$$

where the vertical arrows denote (topologically) dual spaces. You have to be careful[1] about deducing that the cohomology of the top row is dual to that of the second row, but this can be done, giving

$$H^k(M) \text{ is dual to } H_{-\infty,k}(M). \tag{7.4}$$

This would give Poincaré duality if we knew that $H_{-\infty,k}(M)$ is isomorphic to $H^{n-k}(M)$. This is not obvious a priori because currents are so much more general than smooth forms. However, it is in fact true, as follows by regularization of currents.

8. Regularization of distributions in $\mathbb{R}^n$

Let $\tau(x)$ be a smooth function with compact support in the unit ball, symmetric ($\tau(-x) = \tau(x)$) and total mass equal to 1:

$$\int \tau(x)\,\mathrm{d}x = 1.$$

Let $\tau_\varepsilon = \varepsilon^{-n}\tau(x/\varepsilon)$ so τ_ε is symmetric, has total mass one and is supported in $|x| < \varepsilon$. If f is smooth it is easy to see that

$$\int \tau_\varepsilon(x) f(x)\,\mathrm{d}x \to f(0)$$

as $\varepsilon \to 0$, so $\tau_\varepsilon(x)$ should be thought of as a smooth approximation to the Dirac distribution δ_0.

If f is continuous or perhaps only L^1 on $\mathbb{R}^n$, then the convolution

$$f_\varepsilon = \tau_\varepsilon * f \tag{8.1}$$

that is,

$$f_\varepsilon(x) = \int \tau_\varepsilon(x-y) f(y)\,\mathrm{d}y \tag{8.2}$$

[1] This is to do whether the ranges of these operators are closed or not. If the cohomology groups are finite dimensional there is no problem. This issue also arises when you try to prove Serre duality by the same method (paper of Serre, 1955)

is C^∞ and $f_\varepsilon \to f$ in the topology of convergence with all derivatives on compact sets as $\varepsilon \to 0$. The smoothness follows essentially by differentiation under the integral sign: derivatives with respect to x fall on τ_ε, which is smooth.

If ϕ is any test function (smooth function with compact support) then

$$\int u_\varepsilon(x)\phi(x)\,\mathrm{d}x = \int \tau_\varepsilon(x-y)f(y)\phi(x)\,\mathrm{d}x = \int u(x)\phi_\varepsilon(x) \tag{8.3}$$

since τ is symmetric. In this form we can define u_ε for any distribution by the formula

$$u_\varepsilon[\phi] = u[\phi_\varepsilon]. \tag{8.4}$$

The remarkable thing about this is that u_ε is smooth: since $\phi_\varepsilon = \int \tau_\varepsilon(x-y)\phi(y)\,\mathrm{d}y$, the distribution u_ε is given by integration of ϕ against the function

$$y \mapsto u[\tau_\varepsilon(\cdot - y)]$$

which will be smooth in y. It can also be shown that $u_\varepsilon \to u$ as $\varepsilon \to 0$ (in the topology of distributions, which we haven't discussed) and that the support of u_ε is contained in the ε-neighbourhood of the support of u.

We shall now generalize this to regularization of currents. The idea is the same, but for the purposes of topology, we want to know that if a current is closed then its regularization is in the same homology class.

9. Homotopy formulae

Suppose that f_t is a 1-parameter family of diffeomorphisms of M ($t \in [0,1]$), arising as the flow of some vector field X. By Cartan's formula,

$$L_X\alpha = \mathrm{d}(\iota_X\alpha) + \iota_X \mathrm{d}\alpha \tag{9.1}$$

for any form α on M. On the left,

$$L_X\alpha = \frac{\mathrm{d}}{\mathrm{d}t} f_t^*(\alpha)\bigg|_{t=0}$$

is the Lie derivative of α with respect to X. Observe that if $\mathrm{d}\alpha = 0$ this gives

$$L_X\alpha = \mathrm{d}(\iota_X\alpha)$$

so that the infinitesimal variation of α is exact. By integration we obtain a homotopy formula relating $f_1^*(\alpha)$ to $f_0^*(\alpha) = \alpha$.

By the fundamental theorem of calculus

$$f^*\alpha - \alpha = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f_t^*\alpha\,\mathrm{d}t, \tag{9.2}$$

where we've written $f = f_1$ for the time-1 flow of X. Using Cartan's formula with α replaced by $f_t^*\alpha$

$$f^*\alpha - \alpha = \mathrm{d}F*\alpha + F*\mathrm{d}\alpha \tag{9.3}$$

where the operator F is defined by

$$F^*\beta = \int_0^1 \iota_X f_t^*(\beta)\,\mathrm{d}t \tag{9.4}$$

for any form β on M. The notation F^* is perhaps a bit fanciful: it is supposed to convey that it is some integrated version of pull-back by f_t, roughly speaking.

In particular if $\mathrm{d}\alpha = 0$ then $[\alpha] = [f^*(\alpha)]$ in de Rham cohomology. It is reasonable that a cohomology class should not change under a diffeomorphism that is connected to the identity.

To relate this to regularization, let $s_y(x) = x + y$ for fixed $y \in \mathbb{R}^n$ and write $\alpha_y = s_y^*\alpha$. Let S_y^* be the corresponding homotopy operator so

$$s_y^*\alpha - \alpha = \mathrm{d}S_y^*\alpha + S_y^*\mathrm{d}\alpha \tag{9.5}$$

for every fixed y. Finally multiply both sides by $\tau_\varepsilon(y)$ and integrate with respect to y. Set

$$R_\varepsilon^* \alpha = \int s_y^* \alpha \tau_\varepsilon(y)\, \mathrm{d}y \tag{9.6}$$

and

$$A_\varepsilon^* \gamma = \int S_y^* \gamma\, \tau_\varepsilon(y)\, \mathrm{d}y. \tag{9.7}$$

Since the total mass of τ_ε is 1 we obtain

$$R_\varepsilon^* \alpha - \alpha = \mathrm{d}A_\varepsilon^* \alpha + A_\varepsilon^* \mathrm{d}\alpha. \tag{9.8}$$

A key point about this formula is that R_ε^* commutes with d (since this is true of s_y^* for each fixed y, and we are then taking a linear combination of these, admittedly an infinite one, by integration against $\tau_\varepsilon(y)$). And also the operators only increase support by a ε, so if α has compact support, the same will be true of $R_\varepsilon^*\alpha$ and $A_\varepsilon^*\alpha$. Because of this, we have the corresponding formulae for currents by defining

$$R_\varepsilon T[\alpha] = T[R_\varepsilon^*\alpha],\ A_\varepsilon T[\alpha] = T[A_\varepsilon^*\alpha] \tag{9.9}$$

we automatically have

$$R_\varepsilon T - T = \mathrm{b}A_\varepsilon T + A_\varepsilon \mathrm{b}T \tag{9.10}$$

(Note that we need to know the stated fact about R_ε^* and A_ε^* mapping forms with compact support to forms with compact support in order that (9.9) be well-defined.)

It is somewhat tricky, but possible, to transfer and patch this $\mathbb{R}^n$ formula to an arbitrary manifold. Once one has (9.10) on a manifold M, it is immediate that $H_{-\infty,k}(M)$ is isomorphic to $H^{n-k}(M)$. Indeed, if T is a current of dimension k with $\mathrm{b}T = 0$ then

$$T = R_\varepsilon T + \mathrm{b}A_\varepsilon T, \tag{9.11}$$

showing that the smooth current $R_\varepsilon T$ represents the same homology class as T. Since $R_\varepsilon T$ is a smooth form of degree $n-k$, the result follows. Subject to all the other unproved statements, this would complete the approach to Poincaré duality via the (co)homology of currents.

10. Other approaches to Poincaré duality

In the book of Bott and Tu, 'Differential forms in algebraic topology' they use a (finite) good cover of M, the Mayer–Vietoris sequence and the Poincaré lemma for both cohomology and cohomology with compact supports.

- A *good cover* of M is cover by open sets U_j with the property that all non-empty intersections are diffeomorphic to $\mathbb{R}^n$. A compact manifold always has a finite good cover, opening up the possibility of an induction on the number of sets in a good cover.
- If $M = U \cup V$, then the Mayer–Vietoris sequence is a long exact sequence relating $H^*(M)$ to $H^*(U)$, $H^*(V)$ and $H^*(U \cap V)$. There are two versions one for H^* and one for H_c^*, cohomology with compact supports. Both are needed.

Poincaré Lemma

- $H^0(\mathbb{R}^n) = \mathbb{R}$, $H^j(\mathbb{R}^n) = 0$ if $j > 0$;
- $H_c^n(\mathbb{R}^n) = \mathbb{R}$, $H^j(\mathbb{R}^n) = 0$ if $j < n$.

The Poincaré is very important! It says there is no 'local' cohomology: so the de Rham cohomology groups are capturing global topological properties of a space.

You are invited to consider this proof in more detail on the Exercise sheet.

Hodge Theory If you choose a riemannian metric g on M then there is the Hodge $*$-operator

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M).$$

This is an algebraic operator and $*^2 = (-1)^{k(n-k)}$ on Ω^k. Moreover $\alpha \wedge *\alpha = |\alpha|^2(*1)$ where the pointwise length-squared of α is defined by the riemannian metric g on M and $*1$ is a fancy way of writing the volume-form on M determined by the metric.

Let $\delta = *\mathrm{d}*$. This is a first-order differential operator $\Omega^k(M) \to \Omega^{k-1}(M)$. Hodge theory says that if M is compact, then in each de Rham class, there is a unique *harmonic representative*, that is a form which satisfies

$$\mathrm{d}\alpha = 0, \quad \delta\alpha = 0.$$

(One should think of this as the representative in the cohomology class having smallest L^2 norm. But that is another topic!)

Using our harmonic representatives, we can prove that the pairing between $H^k(M)$ and $H^{n-k}(M)$ is non-degenerate trivially, as follows: given a non-zero class in $H^k(M)$, let α be its harmonic representative. We need to know that there is $\beta \in H^{n-k}(M)$ which pairs non-trivially with α. But $*\alpha$ fits the bill. Indeed, $\delta\alpha = 0$ implies that $\mathrm{d}(*\alpha) = 0$, so $*\alpha$ does represent a class in $H^{n-k}(M)$. And moreover

$$\int \alpha \wedge *\alpha = \int |\alpha|^2(*1) > 0$$

since $\alpha \neq 0$. Thus the pairing is non-degenerate.