Exercise: Read Milnor's book on Morse theory. If you want a more detailed and time consuming version, see Audin-Damian's book.

(Some of the below exercises are adapted from Milnor's book, so if you can't solve them, you can look them up.)

Exercise: Consider the complex projective space $\mathbb{CP}^n$ with projective coordinates $[z_0 : z_1 : \ldots : z_n]$ and define the function:

$$f : \mathbb{CP}^n \to \mathbb{R},\ f([z_0 : \ldots : z_n]) = \frac{\sum_{i=1}^n j|z_j|^2}{|z_0|^2 + \ldots + |z_n|^2}$$

Prove that f is Morse and compute homology of $\mathbb{CP}^n$ this way.

Exercise: Compute the homology of based loop space $\Omega(S^n)$ for $n > 2$ using the energy functional and the standard metric. Conclude that any two non-conjugate points on S^n are joined by infinitely many geodesics for a generic metric.

Exercise: Prove the following theorem of Lefschetz using Morse theory (this proof is due to Andreotti and Frankel). Let $M \subset \mathbb{C}^n$ be a non-singular affine algebraic variety in complex n-space with real dimension $2k$, then

$$H_i(M; \mathbb{Z}) = 0 \text{ for } \ i > k$$

(Hint: Choose a generic point $p \in \mathbb{C}^n$ and consider the squared-distance function $L_p : M \to \mathbb{R}$. Show that this is a Morse function if p is generic, and compute the index of its critical points.).

Exercise: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with a critical point at p. Assume that the Hessian of f at p has nullity 1, and not all derivatives of order ≥ 3. vanish at p. Show that there are local coordinates x, y around p in which f is given by the expression:

$$f(x, y) = f(p) \pm x^2 \pm y^m$$

for some integer $m \geq 3$.

Exercise: Given an example of a function on the 2-torus T^2 with exactly 3 critical points.

Exercise: Consider $\mathbb{R}P^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}/(x, y, z) \sim (-x, -y, -z)$. Let $f : \mathbb{R}P^2 \to \mathbb{R}$ be the function sending $(x, y, z) \mapsto y^2 + 2z^2$. Show that this is a Morse function. Compute its critical points and the gradient flow lines between them.

Exercise: Repeat the above exercise for the function $f : T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto \cos(2\pi x) + \cos(2\pi y)$.