Introduction:

$M \subset \mathbb{R}^3$

X, Y - two vector fields on M

$\partial_X Y = \nabla_X Y + \mathrm{II}(X,Y)$ ← second fundamental form.

<u>Exercise 0</u> II is a bilinear form.

$\mathrm{II}(X,X) = |X|^2 \cdot$ {curvature of the curve in the direction of $\vec{X}$}.

Definition $\vec{H} = \operatorname{tr} \mathrm{II} = \sum_{e_i} \mathrm{II}(e_i, e_i)$ - mean curvature vector. $(K = \det \mathrm{II})$

What is the meaning of $\vec{H}$?

<u>1st variation</u> $M_t = M + t\vec{X}$

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(M_t) = -\int \langle \vec{H}, \vec{X} \rangle \, dv_g$$

<u>Exercise 1</u> Prove it using

a) $\frac{d}{dt}\Big|_{t=0} \det(I + tA) = \operatorname{tr}(A)$

b) divergence formula (consequence of the Stokes formula)

$$\int_M \operatorname{div}(X) = \int_{\partial M} \langle \vec{X}, n \rangle$$

n ← outer normal

, where

$$\operatorname{div}(X) = \operatorname{tr}(Y \mapsto \nabla_Y X) = \sum \langle \nabla_{e_i} X, e_i \rangle$$

Examples

1. soap films (Plateau problem)
2. • plane
   • catenoid – surface of revolution for $y = \cosh(x)$ & its scalings.

Exercise 2 plane & catenoid are the only minimal surfaces of revolution

Proposition There are no compact minimal surfaces in $\mathbb{R}^3$.

▶ $M \subset \mathbb{R}^n$ we can introduce Laplace-Beltrami operator (or Laplacian)

$$\Delta f = -\operatorname{div}(\nabla f) \quad \left(\text{In } \mathbb{R}^3,\ \Delta f = \sum \frac{\partial^2}{(\partial x_i)^2}\right)$$

Lemma $\Delta_M \vec{x} = \vec{H}$

▶ **Exercise 3** $\Delta f = -\left(\sum \partial_{e_i} \partial_{e_i} f - \partial_{\nabla^M_{e_i} e_i} f\right)$

$$\partial_{e_i} \vec{x} = e_i$$

$$-\Delta_M \vec{x} = \sum \partial_{e_i} e_i - \nabla_{e_i} e_i = \sum \mathrm{II}(e_i, e_i).$$ ◀

Maximum principle ⇒ proposition.

◀

Min surfaces can be defined for other ambient manifold, e.g. for $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

$$\nabla^{S^n}_X Y = \nabla^M_X Y + \mathrm{II}^{M\subset S^n}(X,Y)$$

<u>Exercise 4</u> $\mathrm{II}^{S^n\subset \mathbb{R}^n}_x (X,Y) = \langle X, Y\rangle \vec{x}$

<u>Corollary</u> $M \subset \mathbb{S}^n$ is minimal iff $\Delta_M \vec{x} = 2\vec{x}$

▶ $H^{M\subset \mathbb{R}^{n+1}} = \sum \left(\mathrm{II}^{M\subset S^n} + \mathrm{II}^{S^n\subset \mathbb{R}^n}\right)(e_i, e_i) =$

$$= H^{M\subset S^n} + \underbrace{\sum \langle e_i, e_i\rangle \vec{x}}_{2\vec{x}}$$ ◀

<u>Exercise 5</u> $M \subset \mathbb{R}^{n+1}$ is such that $\Delta_M \vec{x} = 2\vec{x}$. Then $M \subset \mathbb{S}^n$.

Common application of MS: relations between topology & curvature ($\sim$ geodesics)

Usually uses 2nd variation:

$$Q(\xi,\xi) = \int_M |\nabla^\perp \xi|^2 - \mathrm{tr}\left(R_\Sigma(\cdot, \xi)\cdot, \xi\right) - |\langle \mathrm{II}, X\rangle|^2$$

$$Q(\varphi,\varphi) = \int_M |\nabla\varphi|^2 - \left(Ric_M(\nu,\nu) + |\mathrm{II}|^2\right)\varphi^2$$

for hypersurfaces.

Thm (Σ^n, g) - closed manifold, $Ric_M > 0$
Then $H_{n-1}(\Sigma, g) = 0$.

▶ If $H^{n-1}(\Sigma, g) \neq 0 \Rightarrow$ (hard GMT result) $\exists$ minimal hypersurface (with sing. set of codim ≤ 8) minimizing volume in that homology class. Plug $\varphi = 1$ $\Rightarrow$ contradiction (up to the sing. set). ◀

Geroch thm There is no metric on $\mathbb{T}^3$ with $S_g > 0$

▶

Exercise 6 For $M \subset \Sigma^3$
$$\left(Ric(\nu,\nu) + |\mathrm{II}|^2\right) = \frac{1}{2}\left(S + |\mathrm{II}|^2\right) + K.$$
Claim: There exists a stable $\mathbb{T}^2 \subset \mathbb{T}^3$

$$Q(1,1) = -\int_{\mathbb{T}^2} \frac{1}{2}(\underset{>0}{S} + \underset{\geq 0}{|\mathrm{II}|^2}) + K \quad \& \quad \int K = 0 \quad ◀$$

Equator.	Clifford torus
• A_{cc} - ! embedded annulus in $\mathbb{B}^3$	• $\mathbb{T}_{cl}$ is the unique embedded torus in $\mathbb{S}^3$ ✓
• A_{cc} is the second smallest area.	• $\mathbb{T}_{cl}$ is the second smallest area ✓
• A_{cc} is the second smallest index	• $\mathbb{T}_{cl}$ is the second smallest index ✓
Any surface can be embedded as FBMS in $\mathbb{B}^3$	Any surface can be embedded as MS in $\mathbb{S}^3$ ✓

Exercises

7. $\mathbb{S}^2 \subset \mathbb{S}^n$ has index $n-2$. Any other surface has index $\geq n+1$.
   <u>Hint</u>: Consider variation fields given by the projections of constant vectors fields to the normal bundle.

8. (a) $M \subset \mathbb{S}^3$, $M \approx \mathbb{S}^2$. Show that
   $$\int_M 1+|H|^2 \geq 4\pi. \quad \text{(easy)}$$
   (b) for any $M \subset \mathbb{S}^n$, $\int_M 1+|H|^2 \geq 4\pi$. (hard)

Hints:

(i) $\int_M 1 + |H|^2$ is conformally invariant, i.e. if $\Phi: \mathbb{S}^n \to \mathbb{S}^n$ is a conformal automorphism, then

$$\int_{\Phi(M)} 1 + |\vec{H}_{\Phi(M)}|^2 = \int_M 1 + |\vec{H}_M|^2.$$

(ii) Consider conformal automorphisms given by: stereographic projection onto $\mathbb{R}^n$, then scaling of $\mathbb{R}^n$, then stereographic projection back.

(9) Let $M \subset \mathbb{S}^3$ be minimal. In local isothermic coordinates consider the expressions

$$\omega_2 = (\partial_z \vec{x}, \partial_z \vec{x})\, dz^2$$

$$\omega_4 = (\partial_z^2 \vec{x}, \partial_z^2 \vec{x})\, dz^4,$$

where $(\cdot,\cdot)$ is the standard $\mathbb{C}$-bilinear form on $\mathbb{C}^3$.

(a) Show that $\omega_2 = 0$

(b) Show that ω_4 is a holomorphic quartic differential.

(c) Suppose $M \cong \mathbb{S}^2$, show that $\omega_4 = 0$.

(d) Use this to show that equatorial $\mathbb{S}^2 \subset \mathbb{S}^3$ is the unique minimal sphere in $\mathbb{S}^3$.