The theory of minimal surfaces:
Plateau Problem

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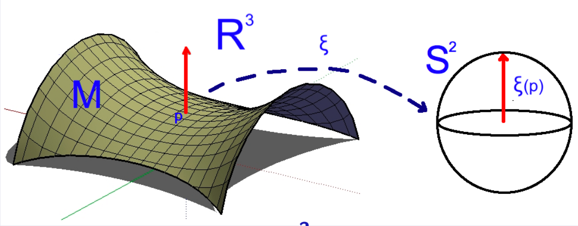
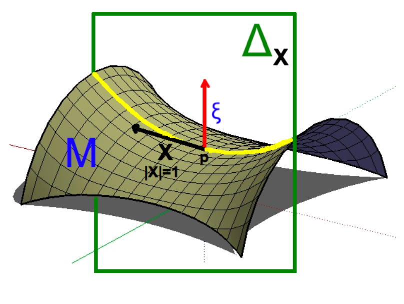


Let $\mathbf{M}$ be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to $\mathbf{M}$. Since $\langle \xi, \xi \rangle = 1$

$$d\xi_p \colon T_p\mathbf{M} \to T_{\xi(p)}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

Definition

The map $\mathbf{A} = -d\xi$ is the **shape operator** of $\mathbf{M}$.



$|\langle \mathbf{A}X, X\rangle|$ = curvature at p of the planar curve $\mathbf{\Delta}_X \cap \mathbf{M}$.
(The yellow curve in the picture)

- $\mathbf{\Delta}_X$ the plane normal to $\mathbf{M}$ at p and containing X.

Definition

$$\mathbf{A}_p \colon T_p\mathbf{M} \to T_p\mathbf{M}$$

- $\mathbf{A}_p$ is symmetric.
- The eigenvalues k_1, k_2 of $\mathbf{A}_p$ are the **principal curvatures** of $\mathbf{M}$ at p.
- $\mathbf{K_G} = \det(\mathbf{A}) = k_1 k_2$ is the **Gauss curvature**.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature**.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the shape operator** or **norm of the second fundamental form**.

Gauss equation

$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K_G}$$

Question

What does the mean curvature say about $\mathbf{M}$?

First Variation Formula

Let $\mathbf{M}_t^{\phi}$ be a smooth normal variation of $\mathbf{M}$ fixing the boundary, i.e. let $\phi \in C_0^{\infty}(\mathbf{M})$ and

$$\mathbf{M}_t^{\phi} = \{p + t\phi(p)\xi(p) \mid p \in \mathbf{M}\}, \text{ then}$$

$$\frac{d}{dt}\mathrm{Area}(\mathbf{M}_t^{\phi})\Big|_{t=0} = -2\int_{\mathbf{M}} \mathbf{H}\phi$$

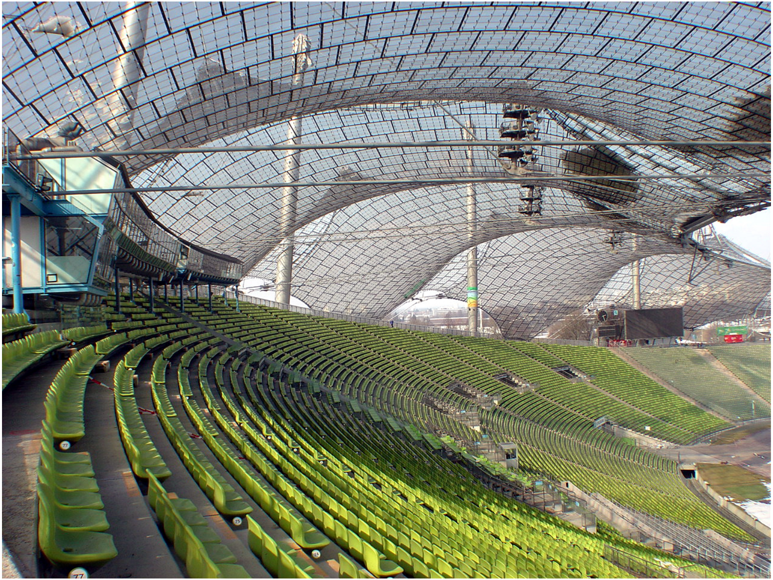
First Variation Formula (HW)

$$\frac{d}{dt}\text{Area}(\mathbf{M}_t^{\phi})\Big|_{t=0} = -2\int_{\mathbf{M}} \mathbf{H}\phi, \quad \phi \in C_0^{\infty}(\mathbf{M})$$

Definition

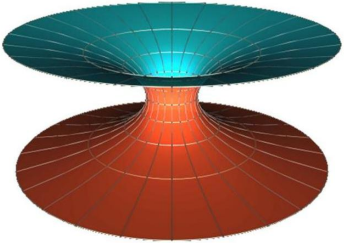
$\mathbf{M}$ is a **minimal surface** $\iff$ $\mathbf{M}$ is a critical point for the area functional $\iff$ $\mathbf{H} \equiv 0$.

Munich Olympic Stadium (Frei Otto)



Catenoid

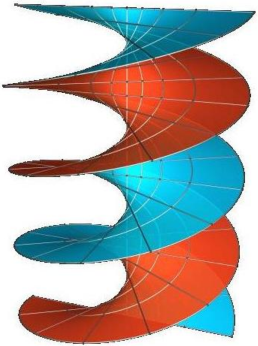
Image by Matthias Weber

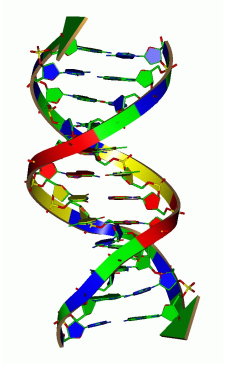


- In 1741, **Euler** discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.
- In 1776, **Meusnier** verified that the catenoid is locally a solution of Lagrange's equation.
- Together with the plane, the catenoid is the only minimal surface of revolution (1860 **Bonnet**). (HW)

Helicoid

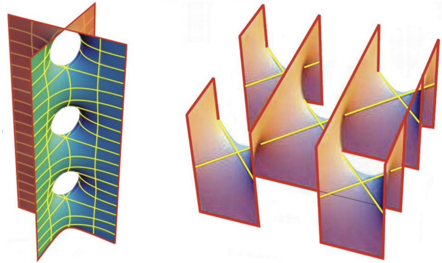
Images by Matthias Weber





- Shape used by **Archimedes** to pump water in 250 BC.
- Proved to be minimal by **Meusnier** in 1776.
- Together with the plane, the helicoid is the only ruled minimal surface (proved by **Catalan** in 1842). (HW)
- The shape of a string of **DNA** resembles that of a helicoid.

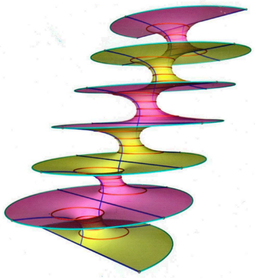
Scherk Surfaces



- Discovered in 1834 by **Scherk**.
- Infinite genus.

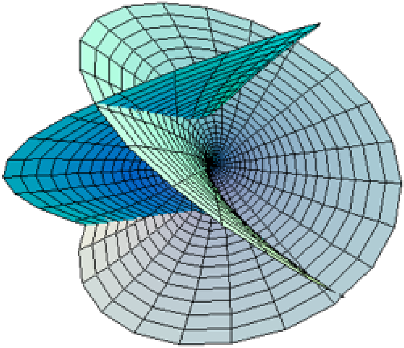
Riemann minimal examples.

Image by Matthias Weber



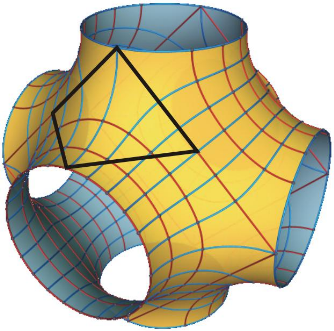
- Discovered in 1860 by **Riemann**, these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ.
- After appropriate scalings, they converge to catenoids as $t \to 0$ or to helicoids as $t \to \infty$.
- The Riemann minimal examples have the property that every horizontal plane intersects the surface in a circle or in a line.
- These surfaces have genus 0.

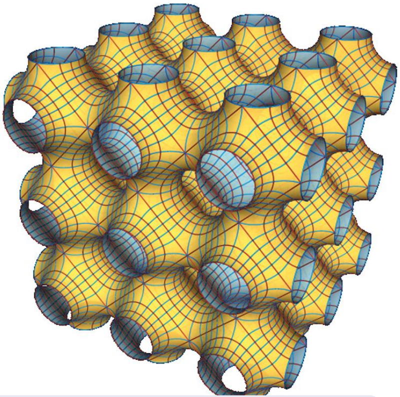
Enneper Surface



- Discovered in 1864 by **Enneper**.
- Together with the Catenoid, it is the only minimal surface whose Gauss map covers the sphere exactly once (1989 **Osserman**).

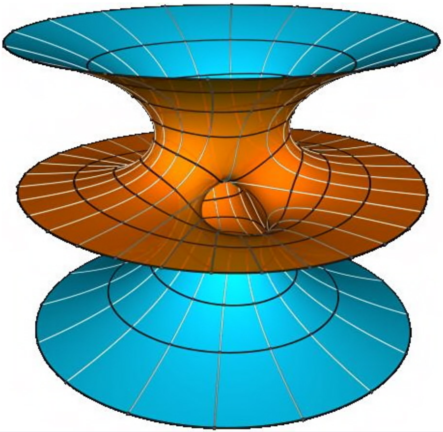
Schwarz Primitive triply-periodic surface. Images by Weber





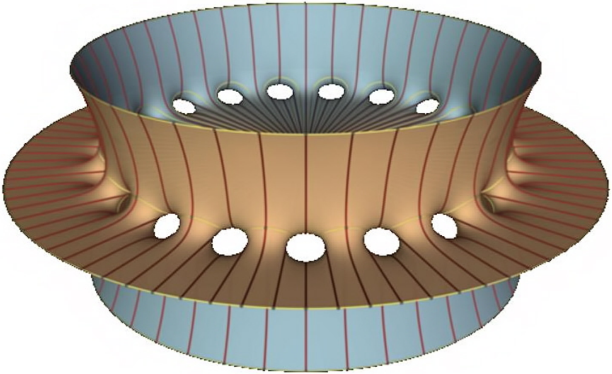
- Discovered by **Schwarz** in the 1880's, it is also called the **P**-surface.
- Such a structure, common to any triply-periodic minimal surface (**TPMS**), is also known as a **crystallographic cell** or **space tiling**.

Costa Torus



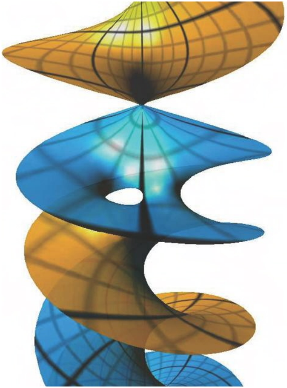
- Discovered in 1982 by **Costa**.
- Finally an interesting topology!

Costa-Hoffman-Meeks surfaces. Image by M. Weber



- Discovered by **Hoffman** and **Meeks** in 1983.
- Adding handles to a minimal surfaces that has more than one end.

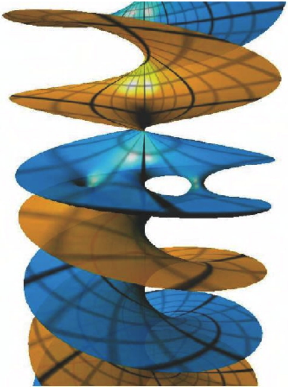
Helicoid with a handle



- Discovered in 1993 by **Hoffman**, **Karcher** and **Wei**.

Helicoid with 2 handles

Images by Matthias Weber



- Discovered in 2013 by **Hoffman**, **Trazet** and **White**

First Variation Formula

$$\frac{d}{dt}\text{Area}(\mathbf{M}_t^{\phi})\Big|_{t=0} = -2\int_{\mathbf{M}} \mathbf{H}\phi, \quad \phi \in C_0^{\infty}(\mathbf{M})$$

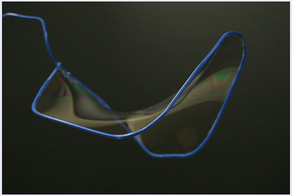
Definition

$\mathbf{M}$ is a **minimal surface** $\iff$ $\mathbf{M}$ is a critical point for the area functional $\iff$ $\mathbf{H} \equiv 0$.

Proposition (HW)

(Locally) Area minimizing surfaces are minimal surfaces.

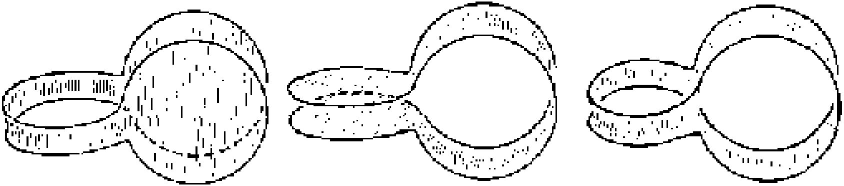
Soap films are
minimal surfaces



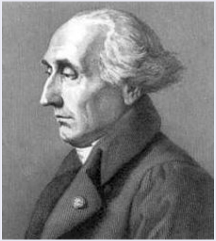
Joseph Plateau (1801 - 1883)

Plateau proved that a soap film minimises area **among nearby surfaces**. (Surface tension is at work.)

Here is a wire that bounds more than one soap film.



Joseph Lagrange
(1736 - 1813)



- **Given a boundary, does there exist an area minimizing minimal surface spanning it?**
- This became known as the **Plateau Problem**.

Remark

- Dipping a closed wire in and out of soapy water is **NOT** a good way to solve Plateau problem!
- In his 1902 thesis **Lebesgue** constructed the now called Lebesgue integral in an effort to solve Plateau problem.

It was solved independently by

Jesse Douglas
(1897 - 1965)
Fields Medal in 1936

Tibor Rado
(1895 - 1965)

Idea for a proof of Plateau Problem

Let Γ be a curve in $\mathbb{R}^3$ and let $\mathcal{F}$ be the set of surfaces spanning Γ. Let

$$\mathcal{A}_{\boldsymbol{\Gamma}} := \inf_{\mathbf{M} \in \mathcal{F}} \mathrm{Area}(\mathbf{M}) \in (0, \infty).$$

Then, **if there exists** a surface $\mathcal{M}$ in $\mathcal{F}$ such that $\mathrm{Area}(\mathcal{M}) = \mathcal{A}_\Gamma$, then this surface solves Plateau problem.

The infimum needs not be realized by a surface!

Digression: Isoperimetric Problem

Given a Jordan (plane simple closed) curve, let L and A denote its length and exclosed area. Then:

$$\mathbf{L^2 \geq 4\pi A},$$

with equality if and only if the curve is a circle.

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- If the enclosed area is fixed, the solution **minimizes** the perimeter.
- If the perimeter is fixed, the solution **maximizes** the area.

Potted history of the Isometric Problem

The circle encloses a greater area than a regular polygon of the same perimeter. (**Zenodorus** 200–120 BC)

Contribution by **Bernoulli** in 1691 and **Euler** in 1744.

It was finally proved by **Steiner** in 1838. But was it really?

Isoperimetric Problem

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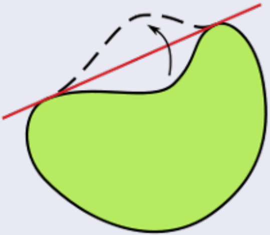
$$\mathbf{L^2 \geq 4\pi A},$$

with equality if and only if the curve is a circle.

Proof (Steiner)

For any curve that IS NOT a circle, there is a **method (given by Steiner)** by which one finds a curve that encloses greater area (with the same perimeter). **Therefore** the circle has the greatest area.



Therefore?

Proof of the Isoperimetric Problem (Steiner)

For any curve that is not a circle, **there is a method** (given by Steiner) by which one finds a curve that encloses greater area (with the same perimeter). **Therefore** the circle has the greatest area.

Prove that among all positive integers, the integer 1 is the largest (Perron)

For any integer that is not 1, **there is a method** (to take the square) by which one finds a larger positive integer. **Therefore** 1 is the largest integer.

Remark

- Steiner's proof works as long as the infimum of the ratio is achieved.
- The proof of the Isoperimetric Problem was completed by **Weierstrass** in 1879. (**Steiner** — 1838)

DIRICHLET'S PRINCIPLE, CONFORMAL MAPPING, AND MINIMAL SURFACES

(1950)

R. COURANT

INSTITUTE FOR MATHEMATICS AND MECHANICS
NEW YORK UNIVERSITY, NEW YORK

Thus a minimizing sequence cannot in general be expected to yield the solution of the problem by a mere passage to the limit. The essential point in the "direct variational methods" is to introduce appropriate additional constructions that produce convergence.

Plateau Problem

Given a curve in $\mathbb{R}^3$, does there exist an **area minimizing minimal** surface spanning it?

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- Before Douglas and Radó, it was solved for certain families of curves.
- There are different proofs of “Plateau problem” depending on the way it is framed.
- And even within the same “framing” there are different proofs.

Douglas and Radó's Plateau Problem:

Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve and let $D \subset \mathbb{R}^2$ be the open unit disk. Let $\mathcal{F}_\Gamma$ be the set of maps $u\colon \bar{D} \to \mathbb{R}^3$ such that the following holds:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$. $\quad (\int\int_D \|\nabla u\|^2 dxdy < \infty)$
- $u\colon \partial D \to \Gamma$ is monotone and onto.

Let $\mathcal{A}_\Gamma := \inf_{u \in \mathcal{F}_\Gamma} \mathrm{Area}(u(D)) \in (0, \infty)$.
Then there exists a map $w \in \mathcal{F}_\Gamma$ such that

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Remark

- $w(D)$ minimizes area among disks.
- The map w is not necessarily a parametrization of a surface. Indeed ∇w "could" be zero (branch points) somewhere. (1970 **Osserman** and 1973 **Gulliver**)

Let $\mathcal{F}_\Gamma$ be the set of maps $u\colon \bar{D} \to \mathbb{R}^3$ such that the following holds:

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Idea for a proof of Plateau Problem

Using the direct method in the calculus of variations introduced by Hilbert and Zaremba around 1900:

- Let u_k be a sequence of maps in $\mathcal{F}_\Gamma$ with

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We are going to follow **Courant**'s proof.

Two red flags!

- The area depends on the image but the diffeomorphism group of the disk is not compact.
- Bounding the area of the image of the map does not give much control on the map itself.

The area depends on the image of the map but the diffeomorphism group of the disk in NOT compact.

Fix $u\colon D \to \mathbb{R}^3$ and let $\phi_k\colon D \to D$ be a non converging sequence of diffeomorhisms. The maps

$$u_k\colon D \to \mathbb{R}^3, \quad u_k = u \cdot \phi_k,$$

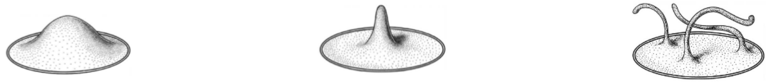
are all parametrizations of the SAME surface, but they do not converge.

Example

$$\phi_k(z) = \frac{z}{k(1 - \|z\|) + 1}, \qquad k \in \mathbb{N}, \quad z \in D \subset \mathbb{C}.$$

Bounding the area of the image of the map does not give much control on the map itself.

Frank Morgan, Geometric Measure Theory: A Beginners Guide



Reminder

Let $u(x, y)\colon D \to \mathbb{R}^3$.

- The Area of (the image of) u is

$$A(u) := A(u(D)) = \iint_D \sqrt{\|u_x\|^2 \|u_y\|^2 - \langle u_x, u_y \rangle^2} dxdy$$

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- Note that $A(u) \leq E(u)$.
- If $\|u_x\|^2 = \|u_y\|^2$ and $\langle u_x, u_y\rangle = 0$ then $A(u) = E(u)$.

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- If $\|u_x\|^2 = \|u_y\|^2$ and $\langle u_x, u_y\rangle = 0$ then $A(u) = E(u)$.

Definition/Proposition

- When $\|u_x\|^2 = \|u_y\|^2$ and $\langle u_x, u_y\rangle = 0$, the u is called an **isothermal** (conformal) parametrization.
- A parametrization is **isothermal** (conformal) if and only if it preserves angles. (HW)

Proposition (Lichtenstein, 1916)

Global isothermal reparametrizations exist.

$$\Downarrow$$

$$\mathcal{A}_\Gamma := \inf_{u\in\mathcal{F}_\Gamma} \text{Area}(u) = \mathcal{A}_\Gamma^{iso} := \inf_{u\in\mathcal{F}_\Gamma^{iso}} \text{Area}(u),$$

where $\mathcal{F}_\Gamma$ the set of maps $u\colon \bar{D} \to \mathbb{R}^3$ such that:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$.
- $u\colon \partial D \to \Gamma$ is monotone and onto.

And $u \in \mathcal{F}_\Gamma^{iso}$ if $u \in \mathcal{F}_\Gamma$ and it is an isothermal parametrization.

Proof

- $\mathcal{A}_\Gamma \leq \mathcal{A}_\Gamma^{iso}$ because $F_\Gamma \supset F_\Gamma^{iso}$.
- $\mathcal{A}_\Gamma \geq \mathcal{A}_\Gamma^{iso}$ because given $u \in F_\Gamma$ there exists $u^* \in F_\Gamma^{iso}$ with $u^*(D) = u(D) \implies A(u^*) = A(u)$.

Proposition

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Let

$$\mathcal{A}_\Gamma := \inf_{u\in\mathcal{F}_\Gamma} A(u), \quad \mathcal{A}_\Gamma^{iso} := \inf_{u\in\mathcal{F}_\Gamma^{iso}} A(u)(= E(u)), \quad \mathcal{E}_\Gamma := \inf_{u\in\mathcal{F}_\Gamma} E(u)$$

Then

$$\mathcal{A}_\Gamma = \mathcal{A}_\Gamma^{\mathbf{iso}} = \mathcal{E}_\Gamma \quad (\text{Proof: } \mathcal{A}_\Gamma = \mathcal{A}_\Gamma^{\mathbf{iso}} \geq \mathcal{E}_\Gamma \geq \mathcal{A}_\Gamma).$$

- $A(u) = Area(u) = \iint_D \sqrt{\|u_x\|^2\|u_y\|^2 - \langle u_x, u_y\rangle^2}dxdy$.
- $E(u) = Energy(u) = \frac{1}{2}\iint_D \|u_x\|^2 + \|u_y\|^2 dxdy$.
- Recall that $A(u) \leq E(u)$.

Since

$$\mathcal{A}_\Gamma = \mathcal{E}_\Gamma,$$

to solve Plateau Problem, it suffices to solve a new problem.

New problem: Plateau/Dirichlet Problem

Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve and let $D \subset \mathbb{R}^2$ be the open unit disk. Let $\mathcal{F}_\Gamma$ be the set of maps $u \colon \bar{D} \to \mathbb{R}^3$ such that the following holds:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$.
- $u \colon \partial D \to \Gamma$ is monotone and onto.

Let

$$\mathcal{E}_\Gamma := \inf_{u \in \mathcal{F}_\Gamma} \mathrm{E}(u) \quad \left(E(u) = \frac{1}{2} \iint_D \|u_x\|^2 + \|u_y\|^2 dxdy \right).$$

Prove that there exists a map $w \in \mathcal{F}_\Gamma$ such that

$$\mathrm{E}(w) = \mathcal{E}_\Gamma.$$

Plateau/Dirichlet Problem

Let

$$\mathcal{E}_\Gamma := \inf_{u \in \mathcal{F}_\Gamma} \mathrm{E}(u) \quad \left(E(u) = \frac{1}{2} \iint_D \|u_x\|^2 + \|u_y\|^2 dxdy \right).$$

Prove that there exists a map $w \in \mathcal{F}_\Gamma$ such that

$$\mathrm{E}(w) = \mathcal{E}_\Gamma.$$

Bounding the area of the image of the map does not give much control on the map itself.

This is NO LONGER a red flag because we are not dealing with area anymore.

The group of conformal (angle preserving) maps from the disk to the disk is also NOT compact and the Dirichlet energy is INVARIANT under conformal maps.

Fix $u\colon D \to \mathbb{R}^3$ be isothermal and let $\phi_k\colon D \to D$ be a non converging sequence of conformal maps. The maps

$$u_k\colon D \to \mathbb{R}^3, \quad u_k = u \cdot \phi_k,$$

have the SAME Dirichlet energy, but they do not converge.

Example

$$\phi_k(z) = e^{i\theta} \frac{z_k - z}{1 - \bar{z}_k z} = e^{0} \frac{\frac{k-1}{k} - z}{1 - \frac{k-1}{k} z}, \qquad k \in \mathbb{N}, \quad z \in D \subset \mathbb{C}.$$

Let $\mathcal{F}_\Gamma$ be the set of maps $u\colon \bar{D} \to \mathbb{R}^3$ such that:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$.
- $u\colon \partial D \to \Gamma$ is monotone and onto.

And let $u \in \mathcal{F}_\Gamma^*$ if

- $u \in \mathcal{F}_\Gamma$.
- $u(1,0) = p_1$, $u(-1,0) = p_2$ and $u(0,1) = p_3$, where p_1, p_2 and p_3 are three distinct points in Γ.

Since the group of conformal maps from D to D acts triply-transitively on ∂D then the following holds.

$$\mathcal{E}_\Gamma := \inf_{u \in \mathcal{F}_\Gamma} \mathrm{E}(u) = \mathcal{E}_\Gamma^* := \inf_{u \in \mathcal{F}_\Gamma^*} \mathrm{E}(u),$$

$$\mathcal{E}_\Gamma := \inf_{u \in \mathcal{F}_\Gamma} \mathrm{E}(u) = \mathcal{E}^*_\Gamma := \inf_{u \in \mathcal{F}^*_\Gamma} \mathrm{E}(u),$$

Proposition

Let $u \in \mathcal{F}^*_\Gamma$ and let $\phi\colon D \to D$ be a conformal map.

$$u \cdot \phi \in \mathcal{F}^*_\Gamma \iff \phi = id.$$

No more red flags!

Plateau/Dirichlet Problem v2

Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve and let $D \subset \mathbb{R}^2$ be the open unit disk. Let $\mathcal{F}_\Gamma^*$ be the set of maps $u\colon \bar{D} \to \mathbb{R}^3$ such that the following holds:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$.
- $u\colon \partial D \to \Gamma$ is monotone and onto.
- $u(1,0) = p_1$, $u(-1,0) = p_2$ **and** $u(0,1) = p_3$**, where** p_1, p_2 **and** p_3 **are three distinct points in** Γ**.**

Let

$$\mathcal{E}_\Gamma^* := \inf_{u \in \mathcal{F}_\Gamma^*} \mathrm{E}(u) \quad \left(E(u) = \frac{1}{2} \iint_D \|u_x\|^2 + \|u_y\|^2 dxdy\right).$$

Prove that there exists a map $w \in \mathcal{F}_\Gamma^*$ such that

$$\mathrm{E}(w) = \mathcal{E}_\Gamma^*.$$

Plateau/Dirichlet Problem v2

Let

$$\mathcal{E}_\Gamma^* := \inf_{u \in \mathcal{F}_\Gamma^*} \mathrm{E}(u) \quad \left(E(u) = \frac{1}{2} \iint_D \|u_x\|^2 + \|u_y\|^2 dxdy \right).$$

Prove that there exists a map $w \in \mathcal{F}_\Gamma^*$ such that

$$\mathrm{E}(w) = \mathcal{E}_\Gamma^*.$$

Idea for a proof of the Plateau/Dirichlet Problem

- Let u_k be a sequence of maps in $\mathcal{F}_\Gamma^*$ with

$$\lim_{k \to \infty} E(u_k) = \mathcal{E}_\Gamma^*$$

- With extreme care, prove that u_k (or possibly a subsequence) converges to the desired map.

The proof of Plateau/Dirichlet Problem v2

Ingredient # 1 (Courant)

Let $\mathcal{F}^*_\Gamma(K) \subset \mathcal{F}^*_\Gamma$ be the set of maps $u\colon \bar{D} \to \mathbb{R}^3$ such that the following holds:

- $u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$ and $E(u) \leq K$.
- $u\colon \partial D \to \Gamma$ is monotone and onto.
- $u(1,0) = p_1$, $u(-1,0) = p_2$ and $u(0,1) = p_3$, where p_1, p_2 and p_3 are three distinct points in Γ.

Then $\mathcal{F}^*_\Gamma(K)$ is equicontinuous on ∂D and by the Arzelá-Ascoli Theorem, $\mathcal{F}^*_\Gamma(K)$ is compact in the topology of uniform convergence on ∂D.

Read

The parametrizations of Γ converge. But we still don't know what happens in the interior.

The proof of Plateau/Dirichlet Problem v2

Ingredient # 2 (Dirichlet Problem)

Let $w \in C^0(\bar{D}) \cap W^{1,2}(D)$, then there is a unique solution

$$u \in C^0(\bar{D}) \cap W^{1,2}(D) \cap C^\infty(D)$$

to the problem

$$\Delta u = 0, \quad u|_{\partial D} = w|_{\partial D}.$$

In fact u minimizes the Dirichlet energy in the space of maps that are equal to w on ∂D, in particular $E(u) \leq E(w)$.

Read

We can work with harmonic ($\Delta u = 0$) maps.

$$\mathcal{E}^*_\Gamma := \inf_{u \in \mathcal{F}^*_\Gamma} \mathrm{E}(u) \quad \left(E(u) = \frac{1}{2} \iint_D \|u_x\|^2 + \|u_y\|^2 dxdy \right).$$

Prove that there exists a map $w \in \mathcal{F}^*_\Gamma$ such that $\mathrm{E}(w) = \mathcal{E}^*_\Gamma$.

Proof of the Plateau/Dirichlet Problem

- Let w_k be a sequence of maps in $\mathcal{F}^*_\Gamma$ with $\lim_{k\to\infty} E(w_k) = \mathcal{E}^*_\Gamma \quad (\implies E(w_k) \leq 2\mathcal{E}^*_\Gamma)$
- Using Ingredient # 2, let $u_k \in \mathcal{F}^*_\Gamma$ be the harmonic map with $u_k|_{\partial D} = w_k|_{\partial D}$. In particular, $E(u_k) \leq E(w_k) \leq 2\mathcal{E}^*_\Gamma$.
- Using Ingredient # 1, the family of maps $u_k|_{\partial D} \colon \partial D \to \Gamma$ is equicontinuous, that is, up to a subsequence, $u_k|_{\partial D}$ converges uniformly to $\gamma \colon \partial D \to \Gamma$ and γ is monotone and onto.

- In sum, $u_k \in \mathcal{F}_\Gamma^*$ is a sequence of harmonic maps with

$$\lim_{k\to\infty} E(u_k) = \mathcal{E}_\Gamma^* := \inf_{u\in\mathcal{F}_\Gamma^*} \mathrm{E}(u)$$

and $u_k|_{\partial D}$ converges uniformly to $\gamma\colon \partial D \to \Gamma$, with γ monotone and onto.

- In sum, $u_k \in \mathcal{F}_\Gamma^*$ is a sequence of harmonic maps with

$$\lim_{k\to\infty} E(u_k) = \mathcal{E}_\Gamma^* := \inf_{u\in\mathcal{F}_\Gamma^*} \mathrm{E}(u)$$

  and $u_k|_{\partial D}$ converges uniformly to $\gamma\colon \partial D \to \Gamma$, with γ monotone and onto.
- Because u_k is harmonic, so is $(u_j - u_k)$ and the maximum principle gives that

$$\sup_D |u_j - u_k| = \max_{\partial D} |u_j - u_k| (\to 0).$$

- In sum, $u_k \in \mathcal{F}_\Gamma^*$ is a sequence of harmonic maps with

$$\lim_{k\to\infty} E(u_k) = \mathcal{E}_\Gamma^* := \inf_{u\in\mathcal{F}_\Gamma^*} \mathrm{E}(u)$$

  and $u_k|_{\partial D}$ converges uniformly to $\gamma\colon \partial D \to \Gamma$, with γ monotone and onto.
- Because u_k is harmonic, so is $(u_j - u_k)$ and the maximum principle gives that

$$\sup_D |u_j - u_k| = \max_{\partial D} |u_j - u_k| (\to 0).$$

- Therefore u_k converges uniformly to $u_\infty\colon D \to \mathbb{R}^3$ with $u_\infty|_{\partial D} = \gamma$.

- In sum, $u_k \in \mathcal{F}_\Gamma^*$ is a sequence of harmonic maps with

$$\lim_{k\to\infty} E(u_k) = \mathcal{E}_\Gamma^* := \inf_{u\in\mathcal{F}_\Gamma^*} \mathrm{E}(u)$$

and $u_k|_{\partial D}$ converges uniformly to $\gamma\colon \partial D \to \Gamma$, with γ monotone and onto.

- Because u_k is harmonic, so is $(u_j - u_k)$ and the maximum principle gives that

$$\sup_D |u_j - u_k| = \max_{\partial D} |u_j - u_k| (\to 0).$$

- Therefore u_k converges uniformly to $u_\infty\colon D \to \mathbb{R}^3$ with $u_\infty|_{\partial D} = \gamma$.
- Since $(u_j - u_k)$ is harmonic, similar estimates hold for its derivatives and thus u_k converges uniformly to the desired map!

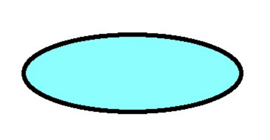
Q.E.D.

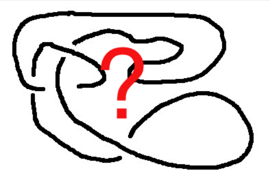
Open problem:

Given a closed wire, how many minimal surfaces spanning such wire are there?

Open problem:

What is the shape of a minimal surface?





Open Problem:

Consider two curves on parallel planes.

- How many **minimal** surfaces are there spanning such curves?
- What is the **shape** of such a **minimal** surface?