

JACOBIANS AND A LITTLE MORE: NOTES FOR THE LSGNT

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Abstract

These are notes for a talk given to the LSGNT. Given the existence of Hilbert schemes, we construct the Picard scheme Pic_X of a projective variety X as the Hilbert scheme that parametrizes the projective spaces of the appropriate dimension inside the Hilbert scheme of sufficiently positive effective divisors on X . **After constructing it we show that Pic_X represents the sheafification of the naive Picard functor.** We go on to prove the Riemann–Roch theorem for curves, following [MM]¹. Material in **blue** is there either for background or as commentary while material in **purple** lies, perhaps, slightly to one side of the main development.

1 Grassmannians and Hilbert schemes

This is just a statement of the results proved in the first half of [LC].

Suppose that V is an n -dimensional vector space over a field k . Then, for every r , there is a smooth projective k -variety $Gr(V, r)$ that classifies the r -dimensional quotients of V . Mumford gives one of the standard constructions of this in [LC], pp. 32–33, by explicit co-ordinates and equations. He then uses the Grassmannian to construct Hilbert schemes:

Theorem 1.1 *If k is a field and X is a projective k -scheme with a fixed ample line bundle $\mathcal{O}(1)$ and if $p : \mathbb{Z} \rightarrow \mathbb{Z}$ is a fixed polynomial, then there is a projective k -scheme $\text{Hilb}_{X,p}$ that classifies closed projective subschemes Z of X whose Hilbert polynomial is p . That is, for any k -scheme T , the morphisms $T \rightarrow \text{Hilb}_{X,p}$ are exactly the closed subschemes \mathcal{Z} of $X \times T$ that are flat and projective over T and whose geometric fibres have Hilbert polynomial p .*

PROOF: The idea is that for every sufficiently large integer N , depending on X and p , every Z with Hilbert polynomial p is an intersection of a certain number M of hypersurfaces of degree N , and then we can construct $\text{Hilb}_{X,p}$ inside the Grassmannian $Gr(H^0(X, \mathcal{O}_X(N)), r)$ where $r = h^0(X, \mathcal{O}_X(N)) - M$. \square

¹They write: “The classical construction of the Jacobian, as well as the abstract construction by A. Weil, both used the Riemann–Roch theorem in an essential way. On the other hand, Chow subsequently gave a projective construction of the Jacobian which used only the Riemann theorem [Lemma 6.1 below]. This exhibited the basic character of the Jacobian in a new way. It was taken up by Matsusaka and later by Grothendieck in their work on the Picard variety, and by now it appears that in the study of linear systems of divisors on a variety, the theory develops most naturally and with the fewest artificialities if the construction of the Picard variety comes first.”

In fact, these ideas give the Hilbert scheme $\text{Hilb}_{X/S,p}$ for any quasi-projective morphism $X \rightarrow S$: the T -points are the closed subschemes of $X \times_S T$ which are **flat and projective** over T and whose Hilbert polynomial is p .

For example, Hilbert schemes can make it very easy to construct quotients. Recall that, given a morphism $X \rightarrow S$, an equivalence relation on X is a closed subscheme R of $X \times_S X$ satisfying the usual set-theoretical requirements of an equivalence relation.

Corollary 1.2 *Suppose that $X \rightarrow S$ is quasi-projective and that $R \subset X \times_S X$ is a closed subscheme that defines an equivalence relation on X and is flat and projective over X . Then the quotient X/R exists as a quasi-projective S -scheme and the quotient morphism $X \rightarrow X/R$ is flat and projective.*

PROOF: The diagram

$$\begin{array}{ccc} R & \hookrightarrow & X \times_S X \\ & \searrow^{pr_2} & \downarrow^{pr_2} \\ & & X \end{array}$$

exhibits the morphism $pr_2 : R \rightarrow X$ as an X -valued point of $\text{Hilb}_{X/S}$. Take X/R to be the image of the corresponding morphism $X \rightarrow \text{Hilb}_{X/S}$. \square

Corollary 1.3 *Suppose that $G \rightarrow S$ is a flat projective group scheme that acts freely on a quasi-projective S -scheme X . Then the quotient X/G exists as a quasi-projective S -scheme and the quotient morphism $X \rightarrow X/G$ is a principal G -bundle.*

PROOF: The definition of “acts freely” is the assumption that the morphism $G \times_S X \rightarrow X \times_S X : (g, x) \mapsto (g(x), x)$ is a closed embedding. Then $G \times_S X$ is exhibited as a flat and projective equivalence relation on X . \square

Corollary 1.4 *If X and Y are S -schemes and $X \rightarrow S$ is flat and projective then the functor $\text{Mor}(X, Y)$, whose set of T -points is the set of T -morphisms $X_T \rightarrow Y_T$, is representable by a disjoint countable union $\text{Mor}(X, Y)$ of quasi-projective S -schemes.*

PROOF: A morphism can be identified with its graph, so $\text{Mor}(X, Y)$ is represented by a subscheme of $\text{Hilb}_{X \times_S Y/S}$. \square

A *Brauer–Severi* scheme is a flat projective morphism all of whose geometric fibres are isomorphic to projective space. Note that if $g : Y \rightarrow G$ is flat and projective and s is a geometric point of G such that $Y_s \cong \mathbb{P}_{k(s)}^n$ then there is an étale neighbourhood U of s such that $X_U \rightarrow U$ is a Brauer–Severi scheme. This is because \mathbb{P}^n is *rigid*: $H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) = 0$, where \mathcal{T} is the tangent sheaf. This is equivalent to the statement that there is a Zariski neighbourhood V of s such that $X_V \rightarrow V$ is a Brauer–Severi scheme.

Corollary 1.5 *A Brauer–Severi scheme $g : Y \rightarrow G$ has a dual Brauer–Severi scheme $g^\vee : Y^\vee \rightarrow G$ such that for each geometric point $s \in G$ the fibre Y_s^\vee is naturally isomorphic to the projective space dual to the projective space Y_s .*

PROOF: Construct $g^\vee : Y^\vee \rightarrow G$ as the component of $\text{Hilb}_{Y/G}$ that classifies the hyperplanes in the fibres of $g : Y \rightarrow G$. \square

The Brauer–Severi scheme $g : Y \rightarrow G$ is a *Zariski bundle* if there is a vector bundle \mathcal{E} on G such that $Y \cong \mathbb{P}(\mathcal{E})$.

Proposition 1.6 *The following are equivalent:*

- (1) g is a Zariski bundle.
- (2) g has sections everywhere Zariski locally on G .
- (3) X has a line bundle that restricts to $\mathcal{O}(1)$ on each geometric fibre of g .
- (4) g^\vee is a Zariski bundle.

PROOF: Exercise. \square

2 Picard schemes as Hilbert schemes

Suppose that $f : X \rightarrow S$ is a flat projective morphism whose geometric fibres are reduced and irreducible and that S is connected. We call such an object *a flat family of projective varieties*. We construct the Picard scheme as the Hilbert scheme that classifies the projective spaces (the complete linear systems) that lie in the Hilbert scheme that classifies the effective divisors on X . In turn, this is a re-writing of Chow’s construction [C] (whose review by Segre [S] makes Chow’s idea very clear) of the Jacobian of a curve in terms of Hilbert schemes and the sheafification of functors rather than Chow varieties and universal mapping properties.

Fix a divisor class H on X that is ample relative to S . Fix also a numerical polynomial $p : \mathbb{Z} \rightarrow \mathbb{Z}$ and consider divisor classes E on X such that $\chi(X_s, \mathcal{O}(-E + nH)) = p(n)$ for all n (equivalently, for all $n \gg 0$). We combine these into a group-valued functor

$$\text{PicFunc}_p = \text{PicFunc}_{X/S,p} : T \mapsto \text{Pic}_p(X \times_S T) / \text{Pic}(T)$$

of S -schemes, where $\text{Pic}_p(X \times_S T)$ is the set of line bundles E on $X \times_S T$ such that

$$\chi(X_t, \mathcal{O}((-E + nH))) = p(n)$$

for all geometric points $t \in T$ and for all $n \gg 0$ (equivalently, for all n).

Then there is an integer m_0 such that for all $m \geq m_0$ and for all divisors E with $\chi(X_s, \mathcal{O}(-E + nH)) = p(n)$ the linear systems $|E + mH|$ are very ample and are of the same dimension. So, if we define $q(n) = p(n + m_0)$, we have an isomorphism $\text{PicFunc}_p \cong \text{PicFunc}_q$ of functors given by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}(m_0H)$ and the representability (or otherwise) of PicFunc_p is equivalent to that of PicFunc_q .

In fact, we shall construct a certain quasi-projective S -scheme $\widetilde{\text{Pic}}_p = \text{Pic}_{X/S,p}$ and then observe that it represents the *sheafification* $\widetilde{\text{PicFunc}}_p$, in the big étale site $Et(S)$, of PicFunc_p . (This notion will be explained but *after* we have constructed Pic_p .) That is, there is not always a scheme that represents $\text{PicFunc}_{X/S,p}$ but Pic_p is the best possible substitute.

Theorem 2.1 *There is a quasi-projective S -scheme Pic_p and a morphism of functors $\gamma : \text{PicFunc}_p \rightarrow \text{Pic}_p$ which is an isomorphism étale locally. That is, there is an étale cover $T \rightarrow S$ such that γ_T is an isomorphism. If f has sections Zariski locally then γ is an isomorphism.*

PROOF: Write $\text{Hilb}_{X/S,p}^0 = V$. This classifies effective Cartier divisors on X all of which are very ample and which move in complete linear systems of the same dimension. There is a morphism of functors $\beta : V \rightarrow \text{PicFunc}_p$ given by mapping the effective Cartier divisor E to the class of $\mathcal{O}(E)$. So, for every T and every T -valued point x of PicFunc_p , the fibre $\beta^{-1}(x)$ is a Zariski $\mathbb{P}^{p(n)-1}$ -bundle over T .

Since $V \rightarrow S$ is quasi-projective there is an open subscheme G of $\text{Hilb}_{V/S}$ (maybe with a countable infinity of connected components) that classifies the subschemes of V that are geometrically isomorphic to $\mathbb{P}^{p(n)-1}$. So the universal subscheme $\epsilon : \mathcal{U} \rightarrow G$ of $V \times_S G$ is a Brauer–Severi scheme.

There is a morphism $\gamma : \text{PicFunc}_p \rightarrow G$ of functors given by $\mathcal{O}(E) \mapsto |E|$. Consider the commutative diagram

$$\begin{array}{ccc} V & & \\ \beta \downarrow & \searrow \delta & \\ \text{PicFunc}_p & \xrightarrow{\gamma} & G \end{array}$$

where $\delta = \gamma \circ \beta : V \rightarrow G$. Observe that δ is, *a priori*, merely a morphism of functors; however, it is a morphism of representable functors, and so, by Yoneda's lemma, is a morphism of the objects that represent those functors. That is, δ is a morphism of S -schemes. The fibres of δ are those of ϵ , so we can identify $\mathcal{U} = V$ and $\delta = \epsilon : V \rightarrow G$ is a Brauer–Severi scheme. So V is a Hilbert scheme classifying Cartier divisors on X and δ collapses V onto its own Hilbert scheme G , the geometric fibres being the complete linear systems of divisors on X .

Observe that the dual Brauer–Severi scheme $\delta^\vee : V^\vee \rightarrow G$ contains a copy of $X \times_S G$ consisting of those members of a fibre of δ that contain a given point of X . Therefore, if $f : X \rightarrow S$ has a section everywhere Zariski locally on S , so does δ^\vee . Therefore δ is a Zariski bundle too. So δ has sections Zariski locally, and then so does γ . That is, γ is Zariski locally an isomorphism, so an isomorphism.

In any case f has sections everywhere locally in the étale topology on S , since $f : X \rightarrow S$ is generically smooth. Therefore γ is an isomorphism locally in the étale topology on S and we define $\text{Pic}_p = G$. \square

We have constructed Pic_p by reducing to certain very ample divisors. Now

that we have constructed it, however, we can consider the Hilbert scheme V that classifies arbitrary effective divisors with $\chi(X, \mathcal{O}(-E + nH)) = p(n)$.

Proposition 2.2 *For every component V of the Hilbert scheme of divisors on X the geometric fibres of $\delta : V \rightarrow \text{Pic}_p$ are projective spaces (as schemes).*

PROOF: There is still a commutative diagram

$$\begin{array}{ccc} V & & \\ \beta \downarrow & \searrow \delta & \\ \text{PicFunc}_p & \xrightarrow{\gamma} & \text{Pic}_p \end{array}$$

and the geometric fibres of β are projective spaces. Since γ is an isomorphism locally in the étale topology on S the result is proved. \square

Proposition 2.3 *If $f : X \rightarrow S$ is also smooth then $\text{Pic}_p \rightarrow S$ is projective.*

PROOF: We prove that $G \rightarrow S$ is proper via the valuative criterion.

Suppose that R is a DVR with $\text{Frac}(R) = K$ and that x is a K -point of G lying over an R -point of S . Then x is a Severi–Brauer scheme over K , so becomes isomorphic to \mathbb{P}^N , say, over a finite extension L of K . The smoothness of $X \rightarrow S$ implies that every specialization of a Cartier divisor is itself a Cartier divisor, so that complete linear systems of Cartier divisors specialize to complete linear systems, so this \mathbb{P}_L^N extends to \mathbb{P}_B^N where B denote the integral closure of R in L . That is, the K -point x extends to a B -point of G , and this is enough. \square

As mentioned previously, (2) of the theorem can be stated as: Pic_p represents the sheafification of PicFunc_p , as follows.

The sheafification \tilde{F} of a contravariant set- or group-valued functor F is modelled on the sheafification \tilde{F} of a presheaf F of sets or groups on a topological space X , except that the phrase “open cover” is replaced by “étale cover” and instead of writing $X = \cup U_i$ we take $U = \sqcup U_i$ and consider the morphism $U \rightarrow X$. Then $U \times_X U = \sqcup(U_i \cap U_j)$.

If F were a representable functor on the category of S -schemes for some fixed base S then it would be a sheaf. (This is a theorem.) That is, for any S -scheme T , any étale cover $T' \rightarrow T$ and any $\xi \in F(T')$ such that the the images ξ_1 and ξ_2 of ξ in $F(T' \times_T T')$ under the two maps $F(T') \rightarrow F(T' \times_T T')$ are equal, there exists a unique $\eta \in F(T)$ that maps to ξ under the map $F(T) \rightarrow F(T')$. We sheafify an arbitrary functor (a pre-sheaf) as follows.

First, given an S -morphism $T' \rightarrow T$, define $\overline{H}^0(T' \rightarrow T, F)$ to consist of those elements ξ of $F(T')$ such that there is an étale cover $v : T'' \rightarrow T' \times_T T'$ for which $v^*\xi_1 = v^*\xi_2$. Then we define \tilde{F} by $\tilde{F}(T) = \lim_{T' \rightarrow T} \overline{H}^0(T' \rightarrow T, F)$. Then \tilde{F} is a sheaf and there is an obvious map of functors (presheaves) $F \rightarrow \tilde{F}$.

Let hcf denote “highest common factor”. Then set $r(p) = \text{hcf}_m p(m) = \text{hcf}_{m \gg 0} p(m)$.

Corollary 2.4 (of the proof) *The obstruction to the representability of PicFunc_p lies in the $r(p)$ -torsion of the image of the natural map $\text{Br}(S) \rightarrow \text{Br}(\text{Pic}_p)$ of Brauer groups.*

PROOF: γ is an isomorphism if and only if δ is a Zariski bundle, so the obstruction Ob_p to the representability of PicFunc_p lies in $\text{Br}(\text{Pic}_p)[p(n)]$. On the other hand, since δ is a Zariski bundle étale locally on S , $\text{Ob}_p \in \text{im}(\text{Br}(S) \rightarrow \text{Br}(\text{Pic}_p))$. Tensoring by $\mathcal{O}(H)$ shows that G is independent of n for $n \gg 0$, so $\text{Ob}_p \in \text{im}(\text{Br}(S) \rightarrow \text{Br}(\text{Pic}_p))[r(p)]$. \square

Corollary 2.5 *Suppose that $f : X \rightarrow S$ is a flat family of smooth curves of genus $g \geq 2$ and define $r(n, g) = \text{lcf}(n + 1 - g, 2g - 2)$. Then the obstruction to the representability of PicFunc^n is of order dividing $r(n, g)$. In particular, PicFunc^g and PicFunc^{g-2} are representable. Moreover, if Pic^a has an S -point and PicFunc^b is representable then PicFunc^{b+ma} is representable for all $m \in \mathbb{Z}$.*

Corollary 2.6 *Suppose that $X \rightarrow S$ is a torsor under the abelian scheme $A = \text{Aut}_{X/S}^0$ and that Θ is a divisor on X which defines a principal polarization on A . Then $\text{PicFunc}_{X/S}^0$ is representable.*

PROOF: Say $\dim A = g$. Then $\chi(X, n\Theta) = n^g \Theta^g / g!$, by Hirzebruch–Riemann–Roch (\mathcal{T}_X is trivial, so its Chern classes are trivial), and $\Theta^g = g!$, so that we can take $p(n) = n^g$ and then $r(p) = 1$. \square

Lemma 2.7 *If X is a projective k -scheme the tangent space at the identity to the Picard functor is naturally isomorphic to $H^1(X, \mathcal{O}_X)$.*

PROOF: We must classify those line bundles \mathcal{L} on $\tilde{X} = X \otimes_k k[\epsilon]$ whose restriction to X is trivial. Pick an open cover of \tilde{X} which locally trivializes \mathcal{L} . Every open cover of \tilde{X} is of the form $\{U_i \otimes k[\epsilon]\}$ where $\{U_i\}$ is an open cover of X so \mathcal{L} is locally generated by s_i where $s_i = \tilde{f}_{ij} s_j$, $(\tilde{f}_{ij}) \in \Gamma(U_{ij} \otimes k[\epsilon], \mathcal{O}^*)$. Then $\tilde{f}_{ij} = f_{ij} + \epsilon g_{ij}$ where $f_{ij} \in \Gamma(U_{ij}, \mathcal{O}^*)$ and $g_{ij} \in \Gamma(U_{ij}, \mathcal{O})$. Moreover, $f_{ij} = h_i / h_j$ with $h_i \in \Gamma(U_i, \mathcal{O}^*)$; after replacing s_i by $h_i^{-1} s_i$ we can assume that each $f_{ij} = 1$, and then (g_{ij}) defines the class in $H^1(X, \mathcal{O})$ that we sought. The converse argument is similar. \square

Proposition 2.8 *Suppose that $X \rightarrow S$ and $Y \rightarrow S$ are flat families of projective varieties with projections $p : X \times Y \rightarrow X$ and $q : X \times Y \rightarrow Y$. Then the morphism*

$$\phi : \text{PicFunc}_{X/S} \times \text{PicFunc}_{Y/S} \rightarrow \text{PicFunc}_{X \times_S Y/S} : (\mathcal{L}, \mathcal{M}) \mapsto p^* \mathcal{L} \otimes q^* \mathcal{M}$$

of functors induces an isomorphism $\text{Pic}_{X/S}^0 \times_S \text{Pic}_{Y/S}^0 \xrightarrow{\cong} \text{Pic}_{X \times_S Y/S}^0$ of commutative group schemes over S .

PROOF: Assume that X and Y have S -points x_0 and y_0 , respectively. Then we have homomorphisms

$$\text{Pic}_X^0 \times \text{Pic}_Y^0 \xrightarrow{\phi} \text{Pic}_{X \times Y}^0 \xrightarrow{\psi} \text{Pic}_X^0 \times \text{Pic}_Y^0,$$

where $\psi(\mathcal{N}) = p^*(\mathcal{N}|_{X \times \{y_0\}}) \otimes q^*(\mathcal{N}|_{\{x_0\} \times Y})$, and $\psi \circ \phi$ is the identity. Since ϕ induces an isomorphism on tangent spaces, by Lemma 2.7, the result is proved in this case.

Since points exist locally in the étale topology, it follows that we get an isomorphism after sheafifying, which is enough. \square

3 The theorem of the cube and its corollaries

The results here are from [AV]. The only difference is that, since we have already constructed the Picard scheme of a flat family of projective varieties, the theorem of the cube becomes very easy. S denotes an arbitrary base scheme.

Theorem 3.1 (*the theorem of the cube*) *Suppose that $X \rightarrow S$ and $Y \rightarrow S$ are flat families of projective varieties, that Z is a connected S -scheme and that $x_0 \in X(S), y_0 \in Y(S)$ and $z_0 \in Z(S)$ are given. Suppose that \mathcal{L} is a line bundle on the cube $X \times_S Y \times_S Z$ and that \mathcal{L} is trivial on each of the three faces $\{x_0\} \times Y \times Z$ etc. of the cube. Then \mathcal{L} is trivial.*

PROOF: \mathcal{L} defines a morphism $r : Z \rightarrow \text{Pic}_{X \times_S Y}^0$. We know that $\text{Pic}_{X \times_S Y}^0$ is naturally isomorphic to $\text{Pic}_X^0 \times_S \text{Pic}_Y^0$ and the composite morphisms $Z \rightarrow \text{Pic}_X^0$ and $Z \rightarrow \text{Pic}_Y^0$ take Z to 0, since \mathcal{L} is trivial on the faces $X \times_S \{y_0\} \times_S Z$ and $\{x_0\} \times_S Y \times_S Z$, so that r takes Z to 0 also. Since $\text{Pic}_{X \times_S Y}^0$ represents the Picard functor there is then a line bundle \mathcal{M} on Z that pulls back to \mathcal{L} . But \mathcal{L} is trivial on the copy $\{x_0\} \times_S \{y_0\} \times_S Z$ of Z , so \mathcal{M} , and then \mathcal{L} itself, is trivial. \square

Fix an abelian scheme $A \rightarrow S$.

Corollary 3.2 *Suppose that Y is a scheme and $f, g, h : Y \rightarrow A$ are morphisms. Then for all line bundles \mathcal{L} on A the line bundles $(f + g + h)^*\mathcal{L} \otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes h^*\mathcal{L}$ and $(f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L}$ on Y are isomorphic.*

PROOF: Let $p_i : A^3 \rightarrow A$ be the projections, $m_{ij} = p_i + p_j : A^3 \rightarrow A$ and $m = \sum p_i : A^3 \rightarrow A$. Define

$$\mathcal{M} = m^*\mathcal{L} \otimes \bigotimes m_{ij}^*\mathcal{L}^{-1} \otimes \bigotimes m_i^*\mathcal{L},$$

a line bundle on A^3 , and denote by $q : A^2 \rightarrow A^3$ the face map $(a, a') \mapsto (0, a, a')$. Then $q^*\mathcal{M}$ is trivial. By symmetry, \mathcal{M} is trivial on each of the other two faces of A^3 , so \mathcal{M} is trivial on the cube A^3 . Pulling back \mathcal{M} under the morphism $(f, g, h) : Y \rightarrow A^3$ gives the result. \square

For any $n \in \mathbb{Z}$ we let $[n] : A \rightarrow A$ denote multiplication by n .

Corollary 3.3 *For any $\mathcal{L} \in \text{Pic}(A)$ we have $[n]^*\mathcal{L} \cong \mathcal{L}^{n(n+1)/2} \otimes [-1]^*\mathcal{L}^{n(n-1)/2}$.*

PROOF: Take $Y = A$, $f = [n], g = [1]$ and $h = [-1]$. Write $\mathcal{L} = \mathcal{O}(L)$ and use additive notation. Then

$$[n+1]^*L - 2[n]^*L + [n-1]^*L = L + [-1]^*L,$$

by Corollary 3.2, and we conclude by induction. \square

Corollary 3.4 *If $S = \mathbf{Spec} k$ then $[n]$ is finite and $\deg[n] = n^{2\dim A}$.*

PROOF: Take any ample divisor class M on A and define $L = M + [-1]^*M$. Then L is ample² and $[n]^*L = n^2L$. So $[n]^*(\text{something ample})$ is ample, which shows that $[n]$ is finite. Taking the top self-intersections then gives $\deg[n](L)^{\dim A} = n^{2\dim A}(L)^{\dim A}$; since $(L)^{\dim A} \neq 0$ we are done. \square

Fix also a torsor $X \rightarrow S$ under $A \rightarrow S$ and a line bundle $\mathcal{L} = \mathcal{O}(L)$ on X .

Corollary 3.5 *(the theorem of the square) For any S -scheme T , any $a, b \in A(T)$ and any line bundle $\mathcal{L} = \mathcal{O}(L)$ on X the line bundle $\mathcal{O}_{X_T}(L_a + L_b - L_{a+b} - L)$ on X_T is trivial in $\text{Pic}_{X_T/T}$. That is, it pulls back from a line bundle on T , locally in the étale topology. (Here L_a denotes the translate t_a^*L .)*

PROOF: It is enough to prove this étale locally on S so we can suppose that $X = A$.

Then apply Corollary 3.2, taking $X = A$, f and g the constant maps to a and b , respectively, and h to be the identity. \square

Corollary 3.6 *For any line bundle $\mathcal{O}(L)$ on X there is a homomorphism $\phi_L : A \rightarrow \text{Pic}_{X/S}^0$ of abelian schemes over S given at the level of points by $\phi_L(a) = t_a^*L - L$.*

PROOF: For any S -scheme T we have a group homomorphism $\phi_L : A(T) \rightarrow \text{Pic}_X^0(T)$ defined by $\phi_L(a) = t_a^*L - L$. This is functorial in T and so defines a homomorphism of group schemes, as stated. \square

Define $K(L) = \ker \phi_L$.

Corollary 3.7 *If L is ample then $K(L)$ is finite over S .*

PROOF: Since $A \rightarrow S$ is proper we can assume that $S = \mathbf{Spec} k$ for an algebraically closed field k , and then we can also assume that $X = A$.

If L is ample and $K(L)$ is not finite then there is a non-trivial abelian subscheme B of A contained in $K(L)$. Put $M = L|_B$. Then M is ample and $\phi_M = 0$. That is, $m^*M - p_1^*M - p_2^*M$ is trivial on $B \times B$, where m is the addition map. Pull this back under $j : B \rightarrow B \times B : b \mapsto (b, -b)$; then $M + [-1_B]^*M$ is trivial. But $M + [-1_B]^*M$ is ample, contradiction. \square

Remark: We shall prove later that $K(L)$ is flat over S .

Proposition 3.8 *If X is smooth and projective over a field k then $(\text{Pic}_X^0)_{red}$ is an abelian variety over k . (This is not completely obvious if k is not perfect.)*

PROOF: Put $P = \text{Pic}_X^0$ and $Q = P \otimes \bar{k}$, where \bar{k} is an algebraic closure of k . Then Q/Q_{red} is a finite commutative group scheme over \bar{k} , so that $[m]$ annihilates it if m is a multiple of its order. That is, Q_{red} contains the image of Q under $[m]$.

² $[-1]^*M$ is not the same thing as $-M$!

But Q_{red} is certainly an abelian variety, and so $[m]$ is finite on it, by Corollary 3.4, so that the image of $[m]$ is all of Q_{red} . Then P_{red} is the image of $[m] : P \rightarrow P$, so that $Q_{red} = P_{red} \otimes \bar{k}$. \square

Remark: In fact the same argument proves something more: if $f : X \rightarrow S$ is smooth and projective **and if S is reduced** then there is an abelian scheme $A \rightarrow S$ that is a subscheme of $\text{Pic}_{X/S}^0$ and whose closed fibres are the abelian varieties of the proposition. We shall denote A by $(\text{Pic}_{X/S}^0)_{red}$. The only further thing that requires proof is that the abelian varieties in question have the same dimension, which follows from the fact that the sheaf $R^1 f_* \mathbb{Q}_\ell$ is locally constant on S . I don't know whether this remains true for arbitrary schemes S .

4 The Picard scheme of an abelian scheme

Suppose that $A \rightarrow S$ is an abelian scheme of relative dimension g .

Theorem 4.1 *The Picard scheme $\text{Pic}_A^0 \rightarrow S$ is smooth of relative dimension g .*

PROOF: Assume first that $S = \mathbf{Spec} k$ where k is an algebraically closed field. Put $\mathcal{Q} = (\text{Pic}_A^0)_{red}$. We know that the homomorphism $\phi_D : A \rightarrow \mathcal{Q}$ is finite when D is ample, and so $\dim \mathcal{Q} \geq g$. Since the tangent space $T_0(\text{Pic}_A^0)$ is, by Lemma 2.7, $H^1(A, \mathcal{O}_A)$ it is enough to show that $\dim H^1(A, \mathcal{O}_A) \leq g$.

For any projective k -variety put $H(X) = \bigoplus H^i(X, \mathcal{O}_X)$. Under cup product this is a skew-commutative graded k -algebra and vanishes in all degrees outside the range $[0, \dim X]$. Moreover, the group structure of A defines a graded co-multiplication on $H(A)$; that is, $H(A)$ is a connected graded Hopf algebra, skew-commutative and finite-dimensional over the perfect field k . Therefore $H(A) = \bigotimes_i k[x_i]$ where x_i is homogeneous of positive degree [Bo], Th. 6.1. So $\dim H^1(A, \mathcal{O}_A)$ is the number n , say, of x_i whose degree is 1. From Borel's theorem, $\prod x_i \neq 0$ and has degree $\geq n$, so $n \leq g$ and the theorem is proved when $S = \mathbf{Spec} k$.

Now let S be an arbitrary Noetherian scheme. We know that $\text{Pic}_A^0 \rightarrow S$ is projective and its geometric fibres are smooth; we need to show that it is flat.

We know that for any ample L on A the S -morphism $\phi_L : A \rightarrow \text{Pic}_A^0$ is finite. Over a geometric point $s \in S$ the morphism $\phi_{L,s} : A_s \rightarrow (\text{Pic}_A^0)_s$ of fibres is a finite morphism between regular schemes of the same dimension, so is flat. Then the flatness of $\text{Pic}_A^0 \rightarrow S$ follows from the fibrewise criterion for flatness of [EGA] IV₃, Cor. 11.3.11: suppose given a commutative diagram

$$\begin{array}{ccccc} & & g & & \\ & \curvearrowright & & \curvearrowleft & \\ X & \xrightarrow{f} & Y & \xrightarrow{h} & S \end{array}$$

of morphisms of finite type. Then $(g$ is flat and $f_s : X_s \rightarrow Y_s$ is flat for each geometric point $s \in S$) if and only if $(h$ is flat at every point of $f(X)$ and f is flat). \square

Corollary 4.2 (of the proof) *If L is ample on A then $K(L)$ is flat over S .*

PROOF: By the fibrewise criterion just quoted, ϕ_L is flat. So $K(L) = \phi_L^{-1}(0) \rightarrow \{0\}$ is flat. But $\{0\}$ is just S . \square

Corollary 4.3 (of the proof) *$H(A) = \Lambda(H^1(A, \mathcal{O}_A))$.*

PROOF: In the notation of the proof, we have shown that $n = g$. Moreover, $\prod x_i \neq 0$, so $\sum \deg x_i \leq g = n$. Therefore $\deg x_i = 1$ for all i . \square

Put $\text{Pic}_{A/S}^0 = A^\vee$, the dual of A . Since A^\vee represents the Picard functor there is a universal divisor class $P = P_A$, the *Poincaré class*, on $A \times_S A^\vee$, which is unique when normalized by the constraints that $P|_{A \times \{0_{A^\vee}\}}$ and $P|_{\{0_A\} \times A^\vee}$ should both vanish. Note that P_A , as a family of classes on A^\vee parametrized by A , defines a homomorphism $\psi_A : A \rightarrow (A^\vee)^\vee$.

Theorem 4.4 (the duality theorem) *ψ_A is an isomorphism.*

PROOF: Write $X = A^\vee$. Then P_A defines a morphism $\rho : A \rightarrow X^\vee$ such that $(\rho \times 1_X)^* P_X = P_A$. Say $\ker \rho = K$. So there is a factorization

$$A \begin{array}{c} \xrightarrow{\rho} \\ \eta \searrow \quad \nearrow \bar{\rho} \\ \rightarrow A/K \rightarrow X^\vee \end{array}$$

Say $L = (\tilde{\rho} \times 1_X)^* P_X$, a class on $(A/K) \times X$, so that $(\eta \times 1_X)^* L = P_A$. L defines a morphism $\alpha : X \rightarrow (A/K)^\vee$ and the composite $X \rightarrow (A/K)^\vee \rightarrow A^\vee$ is the isomorphism defined by P_A . Therefore $(A/K)^\vee \rightarrow A^\vee$ is an isomorphism. So K is finite and the dual homomorphism $A \rightarrow A/K$ is an isomorphism. So $K = 0$. \square

Now fix also a torsor $X \rightarrow S$ under A and a line bundle $\mathcal{L} = \mathcal{O}(L)$ on X . There is a homomorphism $\phi_L : A \rightarrow \text{Pic}_X^0$ of g -dimensional abelian schemes given by the theorem of the square.

Proposition 4.5 *There is a natural isomorphism $\text{Pic}_X^0 \rightarrow \text{Pic}_A^0$.*

PROOF: The action isomorphism $A \times X \xrightarrow{\cong} X \times X$ induces a commutative diagram

$$\begin{array}{ccc} \text{Pic}_A^0 \times \text{Pic}_X^0 & \xleftarrow{\cong} & \text{Pic}_X^0 \times \text{Pic}_X^0 \\ & \swarrow \text{pr}_2^* & \uparrow \text{pr}_2^* \\ & & \text{Pic}_X^0 \end{array}$$

Taking quotients by $\text{pr}_2^* \text{Pic}_X^0$ induces the desired isomorphism. \square

5 Albanese schemes

Suppose that $f : X \rightarrow S$ is projective and smooth **and that S is reduced**. Put $A = (\text{Pic}_{X/S}^0)_{red}$, an abelian scheme over S .

Theorem 5.1 *There is a torsor $Y \rightarrow S$ under $A^\vee \rightarrow S$ and an S -morphism $a : X \rightarrow Y$ such that for every abelian torsor $Z \rightarrow S$ and every S -morphism $h : X \rightarrow Z$ there is a unique S -morphism $\pi : Y \rightarrow Z$ such that $h = \pi \circ a$.*

PROOF: Return to the construction of the Picard scheme $\text{Pic}_{X/S} = \text{Pic} = \sqcup \text{Pic}_p$. We constructed a Brauer–Severi scheme $V \rightarrow \text{Pic}_p$ for an appropriate polynomial p whose geometric fibres are the relevant complete linear systems on X . Moreover, we identified $V \rightarrow \text{Pic}_p$ with the universal subscheme $\mathcal{U} \rightarrow G$ of the effective Cartier divisors E on X with $\chi(X, \mathcal{O}_X(-E + nH)) = p(n)$, so that there is a corresponding universal line bundle $\mathcal{L} = \mathcal{O}(-E)$ on $V \times_S X$.

Write $\mathcal{Q} = (\text{Pic}_p)_{red}$, a torsor under A , as in Remark 3.8. (Without the assumption that S is reduced we do not know that the torsor Pic_p can be reduced to a torsor under A .)

Put $V' = V \times_{\text{Pic}_p} \mathcal{Q} \rightarrow \mathcal{Q}$ and let \mathcal{L}' denote the restriction of \mathcal{L} to $V' \times_S X$. Then $V' \rightarrow S$ is smooth and projective and \mathcal{L}' defines a morphism $a : X \rightarrow \text{Pic}_{V'/S}$. The image of a lies in some connected component of $\text{Pic}_{V'/S}$. Since $V' \rightarrow \mathcal{Q}$ is a Brauer–Severi scheme the connected components of $\text{Pic}_{V'/S}$ are isomorphic to those of $\text{Pic}_{\mathcal{Q}/S}$, each of which is a torsor under the abelian scheme $\text{Pic}_{A/S}^0 = A^\vee$. We have now constructed a candidate morphism $a : X \rightarrow Y$ where Y is a torsor under A^\vee . Note that $\text{Pic}_{Y/S}^0 \rightarrow S$ is an abelian scheme, so that the pull back morphism $a^* : \text{Pic}_{Y/S}^0 \rightarrow \text{Pic}_{X/S}^0$ factors through A and the corresponding morphism $\text{Pic}_{Y/S}^0 \rightarrow A$ is an isomorphism. That is, a induces an isomorphism on reduced Picard schemes. Therefore the smallest abelian sub-torsor of Y that contains $a(X)$ is Y itself. In other words, $a(X)$ spans Y .

Assume given an S -morphism $h : X \rightarrow Z$, where Z is a torsor under some abelian scheme $C \rightarrow S$. We must show that this factors through $a : X \rightarrow Y$.

Suppose that we have a morphism $T \rightarrow S$ and a point $x_0 \in X(T)$. Translate by $a(x_0)$; then we can suppose that $Y_T = A_T^\vee$ and $a(x_0) = 0_{A_T^\vee}$, the origin. After translating by $h(x_0)$ we can suppose that $Z_T = C_T$ and $h(x_0) = 0_{C_T}$. Then h is a line bundle \mathcal{M} on $(X \times_C V)_T$, and so defines $C_T^\vee \rightarrow \text{Pic}_{X_T/T}^0$. This factors as $C_T^\vee \rightarrow (\text{Pic}_{X_T/T}^0)_{red} = A_T$. Dualize this to get $\pi : Y_T = A_T^\vee \rightarrow C_T$. By construction, $(h, 1_T) = \pi \circ (a, 1_T)$.

Take $T = X$ and $x_0 = \Delta$, the diagonal. Then there is a morphism $\Pi : Y \times_S X \rightarrow Z \times_S X$ such that $\Pi \circ (a, 1_X) = (h, 1_X)$. That is, Π is a family of S -morphisms $Y \rightarrow Z$ parametrized by X , so is a morphism $X \rightarrow \text{Mor}_S(Y, Z)$. Write $\Pi = (\pi_\eta)_{\eta \in X}$.

For any geometric point $s \in S$ and any two geometric points $\xi, \eta \in X_s$ we have $\pi_\eta(a(\xi)) = h(\xi)$. That is, $\pi_\eta(a(\xi))$ is independent of η . So Π is a family of morphisms that fix a choice of base points y in Y and z in Z . The tangent space to the scheme of based morphisms is $H^0(Y, \mathfrak{m}_y.g^*\mathcal{T}_Z)$. Now \mathcal{T}_Z is a free sheaf, since Z is an abelian torsor, so $H^0(Y, \mathfrak{m}_y.g^*\mathcal{T}_Z) = 0$. Therefore Π is a constant family of morphisms. That is, $\Pi = (\pi, 1_X)$ for some morphism $\pi : \mathcal{Q} \rightarrow Z$, and the theorem is proved except for the uniqueness of π .

The uniqueness of π follows from the fact that $a(X)$ spans Y . \square

6 Riemann–Roch for curves

Following [MM], we derive the the Riemann–Roch theorem for divisor classes D on a curve C over a field k from the construction and functorial properties of the Picard scheme $\text{Pic} = \text{Pic}_C$ of C . The theorem itself is just a special case of Serre duality but, in comparison with the proof of that theorem, the proof given here exhibits the geometry of the situation more clearly, and leads on (though not here) to the relationship between the geometry of the curve and that of its Jacobian (which mostly means the geometry of the theta divisor). In turn, that leads to the relationship between the moduli space of curves and that of principally polarized abelian varieties.

Write $h^i = \dim H^i$, $\chi = \sum (-1)^i h^i$ and $\Omega_C^1 = \mathcal{O}(K_C)$. Define the *genus* g of C by $g = \dim H^1(C, \mathcal{O}_C)$.

Lemma 6.1 (1) $\chi(C, \mathcal{O}(D)) = \deg D + 1 - g$ for any divisor class D on C .
 (2) D is linearly equivalent to an effective divisor if $\deg D \geq g$.
 (3) (Riemann) $h^0(C, \mathcal{O}(D)) = \deg D + 1 - g$ if $\deg D \gg 0$.

PROOF: (1) is a trivial induction and (2) is an immediate consequence. (3) is a special case of the cohomology vanishing theorems used in the construction of Hilbert schemes. \square

Fix $n \in \mathbb{N}$ and let $p_i : C^n \rightarrow C$ denote the projection onto the i th factor. We denote by $C^{(n)} = C^n / \mathfrak{S}_n$ the n th symmetric product and observe that $C^{(n)}$ can be identified with $\text{Hilb}_{C,p}$ where $p(r) = rn + 1 - g$. We shall write $\text{Pic}_p = \text{Pic}^n$, so that $\text{Pic} = \sqcup \text{Pic}^n$, Pic^0 is the connected component of the identity in the group scheme Pic and each Pic^n is a torsor under Pic^0 .

Lemma 6.2 Pic^n is smooth of dimension g and Pic^0 is an abelian variety.

PROOF: For $n \gg 0$ the morphism $\delta = \delta_n : \text{Hilb}_C^n = C^{(n)} \rightarrow \text{Pic}^n$ is a Brauer–Severi scheme, so is smooth. Since $C^{(n)}$ is smooth, Pic^n is smooth too. The fibres of δ are the complete linear systems of degree n , so are of dimension $n - g$, by Lemma 6.1. So Pic^n is smooth of dimension g for all $n \gg 0$. Since Pic^n is a torsor under Pic^0 the same is true of Pic^0 , and then of Pic^n for all n . Pic^0 is now a smooth projective group variety, which is the definition of an abelian variety. \square

For any variety X write $H^{10}(X) = H^0(X, \Omega_X^1)$.

Lemma 6.3 There is an isomorphism $\phi_n : H^{10}(C) \rightarrow H^{10}(C^{(n)})$ that identifies $H^{10}(C) = H^{10}(C^n)^{\mathfrak{S}_n}$, given by $\phi_n(\omega) = \sum p_i^* \omega$.

PROOF: $H^{10}(C^{(n)})$ naturally embeds into $H^{10}(C^n)^{\mathfrak{S}_n}$, since $\pi : C^n \rightarrow C^{(n)}$ is separable, so each element $\sum p_i^* \omega$ of $H^{10}(C^n)^{\mathfrak{S}_n}$ is a rational 1-form on $C^{(n)}$, and we need to check that $\sum p_i^* \omega$ has no poles on $C^{(n)}$.

If it does have a pole then it does so along the discriminant divisor, so that as a 1-form on C^n it has a pole along each diagonal divisor, which is absurd. \square

Fix $m > 0$ and an effective divisor D of degree m . Then D defines a commutative square

$$\begin{array}{ccc} C^{(n)} & \hookrightarrow & C^{(n+m)} \\ \delta_n \downarrow & & \downarrow \delta_{n+m} \\ \text{Pic}_n & \xrightarrow{\cong} & \text{Pic}_{n+m}. \end{array}$$

Lemma 6.4 *This diagram induces a square of isomorphisms on the spaces H^{10} that are independent of the choice of D and are compatible with the isomorphisms ϕ_n .*

PROOF: The horizontal maps on H^{10} are isomorphisms, by Lemma 6.3, so that it is enough to prove that δ_n^* is an isomorphism for $n \gg 0$. In this case δ_n is a Brauer–Severi scheme and the result is clear. \square

So every H^{10} that appears can be identified with $H^{10}(C)$.

Lemma 6.5 *For an effective divisor E of degree n let $\{E\}$ denote the corresponding point of $C^{(n)}$. Then $H^0(C, \Omega_C^1(-E))$ is identified with the subspace $V_{\{E\}}$ of $H^{10}(C^{(n)})$ consisting of sections that vanish at $\{E\}$.*

PROOF: Suppose first that $E = nx$. Take a local co-ordinate t on C at x . Then $\sigma_1, \dots, \sigma_n$ are co-ordinates on $C^{(n)}$ at $\{E\}$, where σ_i is the i th elementary symmetric polynomial in copies t_1, \dots, t_n of t . Write $\sigma_0 = 1$ and $\sigma_r = 0$ for $r > n$. For $i \geq 1$ define $\sigma_m(i) = \sigma_m|_{t_i=0}$, so that

$$\sigma_{m-r} = \sigma_{m-r}(i) + t_i \sigma_{m-r-1}(i).$$

Multiply this by $(-1)^r t_i^r$ and take $\sum_{r=0}^m$; the result is that

$$\sum_{r=0}^m (-1)^r \sigma_{m-r} t_i^r = \sigma_m(i).$$

Multiply this by dt_i and then take the sum $\sum_{i=1}^n$ to get

$$\sum_{r=0}^m (-1)^r \sigma_{m-r} \theta_r = d\sigma_{m+1}$$

for all $m \in \mathbb{N}$, where $\theta_r = \sum_{i=1}^n t_i^r dt_i$. This shows that $(\theta_0, \dots, \theta_{n-1})$ is a local basis of $\Omega_{C^{(n)}}^1$ at $\{E\}$ and that θ_r vanishes at $\{E\}$ for all $r \geq n$.

Expand $\omega \in H^{10}(C)$ as $\omega = \sum_{r \geq 0} a_r t^r dt$; then $\phi(\omega) = \sum_{r \geq 0} a_r \theta_r$. So

$$\omega|_E = 0 \iff \omega = \sum_{r \geq n} a_r t^r dt \iff \phi(\omega) = \sum_{r \geq n} a_r \theta_r \iff \phi(\omega) \in V_{\{E\}}.$$

The proof for arbitrary $E = \sum n_i x_i$ involves a similar calculation at each x_i . \square

Theorem 6.6 (*Riemann–Roch*) $h^0(C, \mathcal{O}(D)) - h^0(C, \mathcal{O}(K_C - D)) = \deg D + 1 - g$.

PROOF: Suppose that D is an effective divisor of degree n . That is, $\{D\} \in C^{(n)}$. Consider the derivative $\delta_* : T_{\{D\}}(C^{(n)}) \rightarrow T_{\mathcal{O}(D)}(\text{Pic}^n)$. By Proposition 2.2 the fibre of δ through $\{D\}$ is, *as a scheme*, exactly $|D|$ and so

$$\dim \ker \delta_* = \dim |D| = h^0(C, \mathcal{O}(D)) - 1.$$

On the other hand, consider the co-derivative

$$\delta^* : T_{\mathcal{O}(D)}(\text{Pic}^n)^\vee = H^0(C, \Omega^1) \rightarrow T_{\{D\}}(C^{(n)})^\vee.$$

Then $\ker \delta^* = H^0(C, \Omega^1(-D))$, by Lemma 6.5. So we have an exact sequence

$$0 \rightarrow \ker \delta_* \rightarrow T_{\{D\}}(C^{(n)}) \rightarrow T_{\mathcal{O}(D)}(\text{Pic}^n) \rightarrow H^0(C, \Omega^1(-D))^\vee \rightarrow 0$$

of vector spaces, from which R–R follows when D is effective. In particular, it is proved when $\deg D \geq g$.

Corollary 6.7 (1) $h^0(C, \Omega^1(-D)) = h^1(C, \mathcal{O}(D))$ when D is effective.

(2) $h^0(C, \Omega^1) = g$.

PROOF: (1) Apply R–R for effective D and then use Lemma 6.1.

(2) Take $D = 0$ and apply (1). □

Assume that $g > 0$; then K_C is effective, by the Corollary just proved.

Corollary 6.8 $\deg K_C = 2g - 2$.

PROOF: Apply R–R for effective divisors. □

If $\deg D \leq g - 2$ then $\deg(K_C - D) \geq g$, so $K_C - D$ is effective. Then

$$h^0(\mathcal{O}(K_C - D)) - h^0(\mathcal{O}(K_C - (K_C - D))) = \deg(K_C - D) + 1 - g$$

and now R–R is proved whenever $g > 0$ and $\deg D \neq g - 1$.

Suppose that $g > 0$ and $\deg D = g - 1$. If either $h^0(\mathcal{O}(D)) > 0$ or $h^0(\mathcal{O}(K_C - D)) > 0$ we can apply the result for effective divisors. If both vanish there is nothing to prove, so R–R is proved in full if $g > 0$.

Finally, if $g = 0$ then $h^0(\mathcal{O}(P)) \geq 2$ for any point P , by Lemma 6.1, and then the complete linear system $|P|$ defines a morphism $C \rightarrow \mathbb{P}^1$ of degree 1, so that $C \cong \mathbb{P}^1$. In this case R–R is rather easy! □

7 The principal polarization of the Jacobian

Suppose that $f : C \rightarrow S$ is a family of curves of genus $g \geq 2$. There is a closed subfunctor θ of $\text{PicFunc}_{C/S}^{g-1}$ that is the locus of effective classes; its sheafification is a divisor Θ on $\text{Pic}_C^{g-1} = \text{Pic}_{C/S}^{g-1}$. We give a proof that Θ defines a principal

polarization on the Abelian scheme Pic_C^0 , which also equals $\text{Aut}_{\text{Pic}_C^{g-1}/S}^0$. For this we can suppose that $S = \mathbf{Spec} R$ is strictly local, so that PicFunc is representable for all smooth projective S -schemes, and even complete; then we can assume that $C \rightarrow S$ is a versal deformation of the closed fibre, so that R is a power series ring over a Cohen ring. In particular, R is regular and its field of fractions k is of characteristic zero. Fix an algebraic closure \bar{k} of k .

For any S -scheme T and any T -point a of Pic_C^0 we let Θ_a denote the translate of Θ by a .

Put $A = \text{Pic}_C^0$ and $P = \text{Pic}_C^{g-1}$.

Theorem 7.1 (*Jacobians are principally polarized*) *The morphism $\phi_\Theta : A \rightarrow \text{Pic}_A^0$ defined by $a \mapsto \Theta_a - \Theta$ is an isomorphism of abelian S -schemes and Θ is relatively ample.*

PROOF: Since the locus in S where this is true is closed in S it is enough to prove this over $\mathbf{Spec} \bar{k}$.

Consider the commutative diagram

$$\begin{array}{ccc} C^{3g-3} & \xrightarrow{\pi} & C^{(3g-3)} \\ & \searrow \alpha & \downarrow \beta \\ & & \text{Pic}_C^{g-1} \end{array}$$

where $\alpha(z_1, \dots, z_{3g-3}) = \mathcal{O}(\sum z_i - K_C)$ and $\beta(D) = \delta_{3g-3}(D) - K_C$.

We shall use a result due to Klein ([K], vol. 3, p. 429).

For any $a \in A$ pick a basis $\sigma_a = (\sigma_{a,i})$ of $H^0(C, 2K_C + a)$ (whose dimension is $3g-3$, by R-R) and consider the $(3g-3) \times (3g-3)$ determinant $D_a = \det(\sigma_a)$, so that $D_a(z) = \det(\sigma_{a,i}(z_j))$ and $(D_a)_0 \sim \sum p_i^*(2K_C + a)$. Put $\Delta = \sum \Delta_{i,j}$, the big diagonal in C^{3g-3} .

Lemma 7.2 (*Klein*) $(D_a)_0 = \alpha^{-1}(\Theta_a) + \Delta$ and $\alpha^*(\Theta_a) \sim \sum p_i^*(2K_C + a) - \Delta$.

PROOF: Suppose $z \in C^{3g-3} \setminus \Delta$. Then the following are equivalent:

- (1) $z \in (D_a)_0$;
- (2) there exists $\sigma \in H^0(C, 2K_C + a)$ such that $z \subseteq (\sigma)_0$;
- (3) $2K_C + a - z$ is effective;
- (4) $K_C + a - \beta(z) \in \Theta$, by the definition of β and Θ ;
- (5) $\beta(z) - a \in \Theta$, since, by R-R, Θ is preserved by the involution $\iota : D \mapsto K_C - D$;
- (6) $\beta(z) \in \Theta_a$.

So $(D_a)_0 = \alpha^{-1}\Theta_a + n\Delta$ for some $n \geq 1$. By continuity n is independent of a , so to calculate n we can take $a = 0$.

The ramification divisor of π is Δ , so that $K_{C^{3g-3}} \sim \pi^*K_{C^{(3g-3)}} + \Delta$. So

$$2\pi^*K_{C^{(3g-3)}} + 2\Delta \sim 2K_{C^{3g-3}} \sim \sum p_i^*(2K_C) \sim \alpha^*\Theta + n\Delta.$$

So $\pi^*(2K_{C^{(3g-3)}}) \sim \alpha^*\Theta + (n-2)\Delta$. But $C^{(3g-3)}$ is a ruled variety, so $n \leq 1$. The second statement follows. \square

Since ϕ_Θ is a homomorphism, by the theorem of the square, it remains to show that the kernel $K(\Theta)$ of ϕ_Θ is trivial.

If $a \in \ker \phi_\Theta$ then $\Theta \sim \Theta_a$. Then, via Klein's lemma 7.2, $\sum p_i^*(2K_C) \sim \sum p_i^*(2K_C + a)$, so that $a = 0$ and $\ker \phi_\Theta = 0$.

To prove that Θ is ample it is enough to show that the linear system $|2\Theta|$ has no base points and defines a finite morphism $X \rightarrow \mathbb{P}^N$ for some N . In fact we shall prove that $|2\Theta|$ defines a finite morphism that is an embedding of the quotient $X/\langle \iota \rangle$ away from those points D such that $2D \sim K_C$.

For any $a \in A$ the divisor $\Theta_a + \Theta_{-a}$ is an element of $|2\Theta|$, by the theorem of the square. For any $u \in X$ we can find $a \in A$ such that $u \pm a \notin \Theta$, and then $u \notin \Theta_a + \Theta_{-a}$. So $|2\Theta|$ has no base points.

Suppose that $x, y \in X$ are identified by $|2\Theta|$. That is, if $D \in |2\Theta|$ and $x \in D$, then also $y \in D$. So, in particular, if $x \in \Theta_a + \Theta_{-a}$, then $y \in \Theta_a + \Theta_{-a}$. So, whenever $x \in \Theta_a$, either $y \in \Theta_a$ or $y \in \Theta_{-a}$. That is, for every $-a \in \Theta_{-x}$, y lies in Θ_a or Θ_{-a} . Since Θ_{-x} is irreducible, *either* $y \in \Theta_{-c}$ for all $c \in \Theta_{-x}$ *or* $y \in \Theta_c$ for all $c \in \Theta_{-x}$.

- (1) $y \in \Theta_{-c}$ for all $c \in \Theta_{-x}$. That is, $y + p - x \in \Theta$ for all $p \in \Theta$. So $y - x$ preserves Θ , so $y - x \in \ker \phi_\Theta = 0$.
- (2) $y \in \Theta_c$ for all $c \in \Theta_{-x}$. Then $y - p + x \in \Theta$ for all $p \in \Theta$. So $y + x$ preserves Θ . That is, $y + x - K_C \in \ker \phi_\Theta = 0$.

So $|2\Theta|$ identifies x and y if and only if $y = x$ or $\iota(x)$. \square

8 The Weil pairing on the Jacobian

Fix a curve C of genus $g \geq 1$. Put $P = \text{Pic}_C^{g-1}$ and $A = \text{Pic}_C^0$. We have the theta divisor Θ in P and we know, by the theorem of the square, that $n\Theta_a \sim n\Theta$ for all $a \in A[n]$. That is, there is a rational function f on P such that $(f) = n\Theta_a - n\Theta$. Define

$$\mathcal{G}_n = \{(f, a) : f \in k(P)^*, a \in A[n] \text{ and } (f) = n\Theta_a - n\Theta\}.$$

Theorem 8.1 \mathcal{G}_n is a group scheme and is a central extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathcal{G}_n \rightarrow A[n] \rightarrow 0.$$

Its commutator pairing gives a perfect alternating pairing

$$e_n : A[n] \times A[n] \rightarrow \mathbb{G}_m.$$

PROOF: Defining the group law and proving the group axioms is an exercise.

The scheme structure follows from the fact that $A[n]$ is a scheme and the projection $\mathcal{G}_n \rightarrow A[n]$ is relatively representable: it is a \mathbb{G}_m -bundle.

That e_n is perfect follows from the fact that ϕ_Θ is an isomorphism. \square

Remark: In other words, the cup product pairing

$$H_{et}^1(C \otimes \bar{k}, \mathbb{Z}/\ell^m) \times H_{et}^1(C \otimes \bar{k}, \mathbb{Z}/\ell^m) \rightarrow H^2(C \otimes \bar{k}, \mathbb{Z}/\ell^m) \cong \mathbb{Z}/\ell^m$$

is perfect; see Deligne's article in [SGA4.5].

9 Curves over \mathbb{C} as Riemann surfaces

The results here are from [TLT].

Suppose that X is a smooth projective curve of genus g over \mathbb{C} .

Proposition 9.1 (Riemann–Hurwitz) (1) If $X \rightarrow \mathbb{P}^1$ is a finite morphism of degree n and ramification divisor R then $2g - 2 = -2n + \deg R$.

(2) $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$.

PROOF: Exercise. \square

The standard orientation on \mathbb{C} gives an orientation on X . In turn this gives an intersection pairing

$$\cap = (- \cdot -) : H_1(X, \mathbb{Z}) \times H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}$$

which is symplectic and unimodular. Then we can choose a so-called *standard symplectic basis* $(A_1, \dots, A_g; B_1, \dots, B_g)$ of $H_1(X, \mathbb{Z})$, so that

$$(A_k \cdot A_j) = (B_k \cdot B_j) = 0 \text{ and } (A_k \cdot B_j) = \delta_{kj} = -(B_j \cdot A_k).$$

Now suppose that ω, η are closed \mathbb{C} -valued C^∞ 1-forms on C . They define classes in $H^1(X, \mathbb{C})$ whose cup product is given by

$$[\omega] \cup [\eta] = \sum_{k=1}^g \left(\int_{A_k} \omega \int_{B_k} \eta - \int_{A_k} \eta \int_{B_k} \omega \right);$$

this is from the duality between homology and cohomology. Moreover,

$$[\omega] \cup [\eta] = \int_X \omega \wedge \eta,$$

by definition (or construction).

Theorem 9.2 (*Riemann's bilinear relations*)

- (1) $H^{10}(X)$ has a unique basis $(\omega_k)_{k \in [1, g]}$ such that $\int_{A_k} \omega_j = \delta_{kj}$.
- (2) Define $\tau_{kj} = \int_{B_k} \omega_j$. Then the matrix $\tau = (\tau_{kj})$ is symmetric.
- (3) Its imaginary part $\text{Im } \tau$ is positive definite.

PROOF: (1) Suppose $\omega \in H^{10}(X)$ such that $\int_{A_k} \omega = 0$ for all i . Then also $\int_{A_k} \bar{\omega} = 0$ for all i , while

$$\int_X \omega \wedge \bar{\omega} = [\omega] \cup [\bar{\omega}] = \sum \left(\int_{A_k} \omega \int_{B_k} \bar{\omega} - \int_{A_k} \bar{\omega} \int_{B_k} \omega \right).$$

So $\int_X \omega \wedge \bar{\omega} = 0$. But if $\omega = f(z)dz$ in a neighbourhood U and if $z = x + iy$ then

$$\int_U \omega \wedge \bar{\omega} = -2i \int_U |f(z)|^2 dx \wedge dy \in i \mathbb{R}_{<0}$$

unless f is identically zero, so that $\omega = 0$. (1) follows at once.

(2) Suppose $j, k \in [1, g]$. Then $\omega_j \wedge \omega_k = 0$, so that

$$\sum_l \left(\int_{A_l} \omega_j \int_{B_l} \omega_k - \int_{A_l} \omega_k \int_{B_l} \omega_j \right) = 0.$$

This gives $\sum_l (\delta_{lj} \int_{B_k} \omega_k - \delta_{lk} \int_{B_j} \omega_j) = 0$, so $\int_{B_j} \omega_k = \int_{B_k} \omega_j$, as required.

(3) Suppose that $\lambda \in \mathbb{R}^g$ is a row vector such that $\lambda \cdot \text{Im } \tau \cdot \lambda \leq 0$. That is, $\lambda(\tau - \bar{\tau})^t \lambda \in i \mathbb{R}_{\leq 0}$. Put $\eta = \sum_j \lambda_j \omega_j$. Then

$$\sum_k \left(\int_{A_k} \eta \int_{B_k} \bar{\eta} - \int_{A_k} \bar{\eta} \int_{B_k} \eta \right) = [\eta] \cup [\bar{\eta}] \in i \mathbb{R}_{\leq 0}.$$

Since $\int_{A_k} \eta = \lambda_k = \int_{A_k} \bar{\eta}$, $\int_{B_k} \eta = \sum_j \lambda_j \tau_{jk}$ and $\int_{B_k} \bar{\eta} = \sum_j \lambda_j \bar{\tau}_{kj}$, we get

$$\lambda(\bar{\tau} - \tau)^t \lambda \in i \mathbb{R}_{\leq 0}.$$

So $[\eta] \cup [\bar{\eta}] = 0$ and then $\eta = 0$, as required. \square

Corollary 9.3 *The homomorphism*

$$H_1(X, \mathbb{Z}) \rightarrow H^{10}(X)^\vee : \gamma \mapsto \left(\omega \mapsto \int_\gamma \omega \right)$$

embeds $H_1(X, \mathbb{Z})$ as a lattice in $H^{10}(X)^\vee$.

PROOF: Suppose that $\lambda, \mu \in \mathbb{R}^g$ such that $\sum \lambda_k \int_{A_k} \omega + \sum \mu_k \int_{B_k} \omega = 0$ for all $\omega \in H^{10}(X)$. Take $\omega = \omega_j$; then $\lambda_j + \sum_k \tau_{jk} \mu_k = 0$ for all j . Then $(\text{Im } \tau)\mu = 0$, so that $\mu = 0$. Then $\lambda = 0$, so the map is an embedding with discrete image. \square

Define $J(X) = H^{10}(X)^\vee / H_1(X, \mathbb{Z})$. Pick a standard symplectic basis of $H_1(X, \mathbb{Z})$ and let ω denote the corresponding standard basis $(\omega_1, \dots, \omega_g)$ of $H^{10}(X)$. Fix also a base point $P_0 \in X$.

Theorem 9.4 (*Abel's theorem*)

(1) For any $n \in \mathbb{N}$ the morphism

$$\int : X^{(n)} \rightarrow J(X) : (P_1, \dots, P_n) \mapsto \sum_i \int_{P_0}^{P_i} \omega \pmod{H_1(X, \mathbb{Z})}$$

factors through $\delta_n : X^{(n)} \rightarrow \text{Pic}_X^n$.

(2) The induced morphism $\text{Pic}_X^n \rightarrow J(X)$ is an isomorphism.

PROOF: (1) It is enough to show that the fibres of δ_n are mapped to points in $J(X)$. Each such fibre is \mathbb{P}^m for some m , which is simply connected. So any holomorphic morphism $\mathbb{P}^m \rightarrow J(X)$ lifts to a morphism $\mathbb{P}^m \rightarrow H^{10}(X)^\vee$, so is constant, since \mathbb{P}^m has only constant global holomorphic functions.

(2) We must show that if

$$\sum_k \int_{P_k}^{Q_k} \omega = 0 \pmod{H_1(X, \mathbb{Z})} \tag{9.5}$$

then $\sum P_k \sim \sum Q_k$. That is, there is a meromorphic function f on X such that $(f)_0 = \sum P_k$ and $(f)_\infty = \sum Q_k$. For this, we can assume that no P is a Q .

Lemma 9.6 (1) Given two distinct points P and Q on X there is a unique $\phi_{PQ} \in H^0(X, \Omega_X^1(P+Q))$ such that $\int_{A_k} \phi_{PQ} = 0$ for all k , $\text{Res}_P \phi_{PQ} = +1$ and $\text{Res}_Q \phi_{PQ} = -1$.

(2) $\int_{B_j} \phi_{PQ} = 2\pi i \int_Q^P \omega_j$.

PROOF: (1) By R-R $\dim H^0(X, \Omega_X^1(P+Q)) = g+1$. The condition that $\int_{A_k} = 0$ for all k imposes g conditions, and then we can rescale uniquely to get $\text{Res}_P = +1$. Since the residues sum to 0, we must have $\text{Res}_Q = -1$.

(2) Cut X into a $4g$ -polygon \tilde{X} and take small discs Δ_P, Δ_Q centred at P, Q , respectively. Put $\tilde{X}^0 = \tilde{X} \setminus (\Delta_P \cup \Delta_Q)$. On \tilde{X} we can write $\omega_j = f_j$ where f_j is the holomorphic function $f_j(x) = \int_{P_0}^x \omega_j$. Now $\omega_j \wedge \phi_{PQ} = 0$ so that

$$\begin{aligned} 0 &= \int_{\tilde{X}^0} \omega_j \wedge \phi_{PQ} = \int_{\partial \tilde{X}^0} f_j \phi_{PQ} \\ &= - \int_{\partial \Delta_P} f_j \phi_{PQ} - \int_{\partial \Delta_Q} f_j \phi_{PQ} + \sum_k \left(\int_{A_k} \omega_j \int_{B_k} \phi_{PQ} - \int_{A_k} \phi_{PQ} \int_{B_k} \omega_j \right) \\ &= -2\pi i f_j(P) + 2\pi i f_j(Q) + \int_{B_j} \phi_{PQ} = -2\pi i \int_Q^P \omega_j + \int_{B_j} \phi_{PQ}. \end{aligned}$$

□

Then define f by $f(x) = \exp(\sum_k \int_{P_0}^x \phi_{P_k Q_k})$. To check the zeroes and poles is a local matter. However, f is, *a priori*, a multi-valued function; to show that

it is single-valued it is enough to check that its value is unchanged if the path of integration is changed, first by an A -cycle, and then by a B -cycle. In turn, this requires the assumption 9.5 and Lemma 9.6. \square

Stay with the standard bases (A, B) and (ω) . Then we identify $H^{10}(X)^\vee = \mathbb{C}^g$, a vector space of row vectors. The lattice $H_1(X, \mathbb{Z}) = \Lambda$ is then generated by the $2g$ rows of the matrix ${}^t(1_g, \tau)$. In these terms we can define the theta function

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i(n \cdot {}^t z + n\tau^t n)$$

as a formal series.

Proposition 9.7 $\theta(z, \tau)$ is absolutely convergent on compact subsets of $\mathbb{C}^g \times \mathfrak{H}_g$ and so defines a holomorphic function on $\mathbb{C}^g \times \mathfrak{H}_g$.

PROOF: It is enough to check that, for any positive real constants α and β , the series converges absolutely on each region defined by

$$\max | \operatorname{Im} z_j | \leq \alpha/\pi \text{ and } \operatorname{Im} \tau \geq \beta/\pi.1_g.$$

For this, note that in this region

$$\begin{aligned} | \exp(\pi i n\tau^t n + 2\pi i n^t z) | &\leq \exp(-\beta \cdot \sum_j n_j^2 + 2\alpha \sum_j |n_j|) \\ &= \prod_{j=1}^g \exp(-\beta n_j^2 + 2\alpha |n_j|) \end{aligned}$$

for each vector n , so that

$$\begin{aligned} \sum_{n \in \mathbb{Z}^g} | \exp(\pi i n\tau^t n + 2\pi i n^t z) | &\leq 2^g \sum_{n \in \mathbb{N}^g} \prod_{j=1}^g \exp(-\beta n_j^2 + 2\alpha n_j) \\ &= 2^g \prod_{j=1}^g \sum_{n_j \in \mathbb{N}} \exp(-\beta n_j^2 + 2\alpha n_j) = 2^g \left(\sum_{m \geq 0} \exp(-\beta m^2 + 2\alpha m) \right)^g. \end{aligned}$$

But

$$\sum_{m \geq 0} \exp(-\beta m^2 + 2\alpha m) = \exp(-\alpha^2/\beta) \times \sum_{m \geq 0} \exp(-\beta(m - \alpha/\beta)^2),$$

which converges like $\int_{-\alpha/\beta}^{\infty} \exp(-\beta x^2) dx$. \square

Lemma 9.8 If $a, b \in \mathbb{Z}^g$ then

$$\theta(z + a + b\tau) = (\exp \pi i(-2b^t z - b\tau^t b))\theta(z, \tau).$$

PROOF: Manipulate the series. \square

So θ is quasi-periodic as a function of $z \in \mathbb{C}^g$, and therefore its zero locus $(\theta)_0$ is a well defined divisor in $\mathbb{C}^g/\Lambda = J(X)$. We shall write $\theta(z, \tau) = \theta(z)$.

Fix a base point $P_0 \in X$ and embed X in $J(X)$ by $j : P \mapsto \int_{P_0}^P \omega$.

Proposition 9.9 Fix $z \in \mathbb{C}^g$ and consider the multi-valued function $f = f_z$ on X defined by $f(P) = \theta(-z + j(P))$. Then f either vanishes identically or has exactly g zeroes.

PROOF: Cut X into a $4g$ -polygon \tilde{X} whose edges are A_k^\pm and B_k^\pm so that A_k^+ is identified with A_k^- when we reconstruct X from \tilde{X} and similarly for B_k^+ and B_k^- . Suppose that $\sum Q_k$ is the zero locus of f and choose a small disc Δ_k centred at Q_k . Put $\tilde{X}^0 = \tilde{X} \setminus (\cup \Delta_k)$. Then

$$\begin{aligned} 0 &= \int_{\tilde{X}^0} d\left(\frac{df}{f}\right) = \int_{\partial\tilde{X}} \frac{df}{f} - \sum \int_{\partial\Delta_k} \frac{df}{f} \\ &= \sum_{k=1}^g \int_{A_k^- - A_k^+} \frac{df}{f} + \sum_{k=1}^g \int_{B_k^- - B_k^+} \frac{df}{f} - \sum \int_{\partial\Delta_k} \frac{df}{f} \end{aligned}$$

According to Lemma 9.8 f is invariant under translation by an A -cycle, so df/f has the same value on B^+ as on B^- . Also df/f increases by $-2\pi i \omega_k$ along B_k , which joins A_k^- to A_k^+ , so that

$$0 = \sum_{k=1}^g \int_{A_k} (2\pi i \omega_k) - \sum \int_{\partial\Delta_k} \frac{df}{f} = 2\pi i g - \sum \int_{\partial\Delta_k} \frac{df}{f}.$$

Since $\frac{1}{2\pi i} \sum \int_{\partial\Delta_k} \frac{df}{f}$ is the number of zeroes of f , we are done. \square

Define $D = D_0 = (\theta)_0$, a divisor in $J(X)$. The proposition says that, for any $z \in J(X)$, the curve $j(X)$ (which we identify with X) either lies in D_z or intersects it in g points. Certainly D_z does not contain X for general $z \in J(X)$, so we have a rational map $\psi : J(X) - \rightarrow X^{(g)} : z \mapsto D_z \cap X$. This is undefined exactly on the locus $\{z \in J(X) | X \subseteq D_z\}$.

Proposition 9.10 The composite $\delta_g \circ \psi : J(X) \rightarrow J(X)$ is an isomorphism.

PROOF: To begin, we can translate by a point in $J(X)$ to ensure that D_0 does not contain X and that the divisor $D_0 \cap X$ is the unique effective divisor in its linear equivalence class. By the theorem of the square,

$$D_{y+z} - D_0 \sim (D_y - D_0) + (D_z - D_0),$$

so that $\delta_g \circ \psi$ is a homomorphism of groups. Suppose that r lies in the kernel. Then the quotient $Q(z) = \theta(z+r)/\theta(z)$, which is a multi-valued meromorphic function on $J(X)$, restricts to a single-valued rational function q on X . That is, q is invariant if the path of integration from P_0 to each P_i is changed by an arbitrary cycle. But the factor of automorphy for q is the same as that for Q , which is non-trivial if $r \neq 0$ (check this for a B -cycle). \square

Corollary 9.11 (Riemann) *There exists $\Delta \in \mathbb{C}^g$ such that for every $z \in \mathbb{C}^g$ the function $\theta(\Delta - z + j(P))$ on X either vanishes identically or has exactly g zeroes Q_1, \dots, Q_g and in this case $z = \sum_k \int_{P_0}^{Q_k} \omega$.*

PROOF: As z varies so the divisor $\sum Q_k$ of zeroes of the function f_z varies, and we have just proved that $z - \sum j(Q_k)$ is constant. \square

Theorem 9.12

- (1) $\theta(e) = 0 \Leftrightarrow \exists P_1, \dots, P_{g-1} \in X$ such that $e = \Delta - \sum_1^{g-1} \int_{P_0}^{P_k} \omega$.
 (2) Θ is a translate of $(\theta)_0$.

PROOF: (1) \Leftarrow : Let $P_g \in X$ and $z = \sum_1^g \int_{P_0}^{P_k} \omega$. Then

$$0 = \theta(\Delta - \sum_1^g \int_{P_0}^{P_k} \omega + j(P)) = \theta(\Delta - \sum_1^{g-1} \int_{P_0}^{P_k} \omega).$$

\Rightarrow : We can assume that $\theta(e) = 0$ and there exists $y \in X$ such that $\theta(e - \int_{P_0}^y \omega) \neq 0$.

Then there exist $P_1, \dots, P_g \in X$ such that $e = \Delta - \sum_1^g \int_{P_0}^{P_k} \omega$.

Consider the function $f(y) = \theta(e - \int_{P_0}^y \omega)$. This has g zeroes, namely, $\sum_1^g P_k$. Since $\theta(e) = 0$ one of these zeroes is P_0 ; we can assume $P_0 = P_g$. Then $e = \Delta - \sum_1^{g-1} \int_{P_0}^{P_k} \omega$.

(2) is a re-statement of (1). \square

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