INTRODUCTION TO GAUGE THEORY NOTES

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1 REFERENCES

The talk was based on Lorenzo Foscolo's excellent notes from the last year:

www.homepages.ucl.ac.uk/~ucaheps/topics/Gauge_theory.pdf

Other great references are Simon Donaldson's LSGNT lectures:

www.lsgnt-cdt.ac.uk/assets/8h00svqzffcqd97tf8oxcokv199scmjb.pdf

and lectures from the MSRI introductory workshop on gauge theory (scroll down to 'Show schedule, notes, and videos')

www.msri.org/workshops/973

2 LIST OF TOPICS DISCUSSED

2.1 *Maxwell's equations and topology*

1. Maxwell's equations in differential forms language
2. Vector potential, gauge invariance
3. Vector potential up to gauge invariance on a compact manifold
4. Aharonov–Bohm effect

2.2 *Connections and curvature*

1. Vector bundle; local trivialization; transition functions
2. Connection (covariant derivative) on a vector bundle
3. Local form of a connection; change of local representative
4. Gauge transformation; gauge group action on the space of connections
5. Parallel transport; example on the sphere
6. Curvature two-form as commutator, curvature as infinitesimal parallel transport
7. Flat connections
8. G-bundles; connections preserving G-bundles
9. Electromagnetism as a $\mathrm{U}(1)$ gauge theory
10. Charge in electromagnetism
11. Example: solution on the two-sphere
12. Generalization of charge: Chern classes

2.3 *Yang–Mills theory*

1. Harmonic forms as minima of the energy functional
2. Yang–Mills functional
3. Yang–Mills equations
4. Moduli space of Yang–Mills connections
5. Self-duality in dimension four
6. Instantons as minima of the Yang–Mills functional
7. Instantons on the four-space
8. Conformal invariance
9. Uhlenbeck compactness theorem; gauge fixing

2.4 *Topology of four-manifolds*

1. Intersection form of four-manifolds
2. Donaldson's diagonalization theorem
3. Idea of proof

3 PROBLEMS

Problem 3.1 (Maxwell's equations)**.**

1. A vector field on $\mathbb{R}^3$ can be identified with a differential one-form using the Euclidean metric. Express the div and curl operators from vector calculus in terms of the exterior differential d and the Hodge star.

2. The electric and magnetic fields E and B are time-dependent vector fields on $\mathbb{R}^3$. Thinking of them as time-dependent one-forms, define the two-form $F = E \wedge \mathrm{d}t + B$ on $\mathbb{R}^4$, where we think of $\mathbb{R}^4$ as $\mathbb{R} \times \mathbb{R}^3$ with t being the first coordinate. Show that Maxwell's equations in the absence of external currents (and without physical constants)

$$\nabla \cdot E = 0, \quad \nabla \times E = -\partial_t B,$$
$$\nabla \cdot B = 0, \quad \nabla \times B = \partial_t E,$$

   are equivalent to

$$\mathrm{d}F = 0, \quad \mathrm{d} * F = 0,$$

   where $*$ is the Hodge star on $\mathbb{R}^4$ induced by the Minkowski metric.

3. Maxwell's equations are invariant under the transformation $E \mapsto B$ and $B \mapsto -E$. This is known as the *electromagnetic duality*. Observe that this corresponds to fact that the equations for F are invariant udner the Hodge star operator acting on two-forms on $\mathbb{R}^4$.

Problem 3.2 (Aharonov–Bohm effect)**.** Construct a flat $\mathrm{U}(1)$ connection on the complement of a line in $\mathbb{R}^3$ with nontrivial holonomy.

Problem 3.3 (Bundles on spheres)**.**

1. Use parallel transport to show that if $E \to B \times [0,1]$ is a G-bundle then the restrictions of E to $B \times \{0\}$ and $B \times \{1\}$ are isomorphic. Conclude that a G- bundle over a contractible space is trivial.

2. Prove that any G-bundle on S^n can be constructed from gluing two trivial bundles over hemispheres using a transition function $S^{n-1} \to G$. Prove that this construction identifies the set of G-bundles on S^n up to isomorphism with the homotopy group $\pi_{n-1}(G)$.

3. Classify complex and real vector bundles on S^2. Show that all of them can be constructed by taking the direct sum of a complex line bundle and trivial bundle.

4. Show that $\mathrm{SU}(2)$ bundles on S^4 are classified by an integer. (This integer is the second Chern class and, in fact, the same is true for all closed four-manifolds.)

5. Show that there is a unique nontrivial $\mathrm{SU}(2)$ bundle on S^5 and construct it using the fact that $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$.

Problem 3.4 (Cauchy–Riemann operators)**.** Let M be a complex manifold and let $E \to M$ be a $\mathrm{U}(n)$ vector bundle, i.e. a complex vector bundle with a Hermitian metric.

1. A *Cauchy–Riemann operator* is a linear operator $\bar{\partial}_E \colon \Gamma(M,E) \to \Omega^{0,1}(M,E)$ satisfying the Leibniz rule

$$\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f \bar{\partial}_E s$$

for a function $f \colon M \to \mathbb{C}$ and section $s \in \Gamma(E)$. Show that if E is a *holomorphic vector bundle*, i.e. it has trivializations for which the transition functions are holomorphic, then it has a canonical Cauchy–Riemann operator in the above sense, defined by taking the standard derivatives $\partial/\partial \bar{z}_i$ with respect to local holomorphic coordinates on M in local holomorphic trivializations. Show that it satisfies

$$\bar{\partial}_E^2 = 0.$$

2. For every $\mathrm{U}(n)$ connection ∇ on E let $\nabla^{1,0}$ and $\nabla^{0,1}$ be the operators obtained by projecting ∇ on the spaces of $(1,0)$ and $(0,1)$ form under the decomposition

$$\Omega^1(M,E) = \Omega^{1,0}(M,E) \oplus \Omega^{0,1}(M,E).$$

Show that $\nabla^{0,1}$ is a Cauchy–Riemann operator and the square of this operator is $F_\nabla^{0,2}$, the $(0,2)$ part of the curvature two-form. Moreover, show that every Cauchy–Riemann operator is of the form $\nabla^{0,1}$ for a unique $\mathrm{U}(n)$ connection ∇.

3. Conclude that if E is a holomorphic vector bundle, then for every Hermitian metric on E there is a unique *Chern connection*: a $\mathrm{U}(n)$ connection ∇ on E such that $\nabla^{0,1} = \bar{\partial}_E$, where $\bar{\partial}_E$ is the canonical Cauchy–Riemann operator. In particular, it satisfies $F_\nabla^{0,2} = 0$.

4. The converse of the above is also true: if ∇ is a $\mathrm{U}(n)$ connection such that $F_\nabla^{0,2} = 0$, then there exists a holomorphic vector bundle structure on E such that ∇ is the corresponding Chern connection. You can find the proof, for example, in the book *Geometry of Four-Manifolds* by Donaldson and Kronheimer.

Problem 3.5 (Flat connections)**.** Let $E \to M$ be a G-bundle. We say that a connection A on E is *flat* if $F_A = 0$.

1. Let $\mathcal{M}$ be the moduli space of flat connections on E up to gauge transformations. Construct a map $\mathcal{M} \to \operatorname{Hom}(\pi_1(M), G)/G$, where G acts on the space of homomorphism by conjugation, in the following way. Fix a point $x \in M$ and a flat connection A. For every loop $\gamma \colon S^1 \to M$ based at x, we can consider the holonomy of A around this loop as an element of G, if we fix a trivialization of E_x and think of G as a subgroup of $\mathrm{GL}(E_x)$ in some local trivialization of E. You need to show that this gives a well-defined map as above.

2. Show that the map $\mathcal{M} \to \operatorname{Hom}(\pi_1(M), G)$ is a bijection by constructing a flat connection from a representation $\pi_1(M) \to G$. To do this, consider the universal cover $\pi \colon \tilde{M} \to M$, which has an action of $\pi_1(M)$, and form the associated bundle $E = (\tilde{M} \times E_x)/\pi_1(M)$, where we use the representation to act by $\pi_1(M)$ on E_x.

3. Let M be the torus, i.e. V/Λ for a vector space V and lattice $\Lambda \subset V$. For $G = \mathrm{U}(1)$, identify the moduli space of flat connections on M with $M^* = V^*/\Lambda^*$, the dual torus.

Problem 3.6 (Abelian Yang–Mills equations)**.** Let M be a compact Riemannian manifold and $L \to M$ a $\mathrm{U}(1)$ bundle, i.e. a complex line bundle with a Hermitian metric. The Yang–Mills equations for a connection A on L are

$$\mathrm{d}^* F_A = 0.$$

(In this case, $\mathrm{d}_A = \mathrm{d}$ because $\mathrm{U}(1)$ is abelian and the action of A on $\operatorname{End}(L)$ valued forms is by taking commutators, so in this case it is trivial.)

1. Fix a connection A_0. Any other connection A is of the form $A_0 + a$ for $a \in \Omega^1(M, i\mathbb{R})$. Show that $F_A = F_{A_0} + \mathrm{d}a$ so that the Yang–Mills equations are equivalent to

$$\mathrm{d}^*\mathrm{d}a = \eta$$

   for a fixed two-form η.

2. Use the Hodge decomposition theorem to show that this equation always has a solution and any other solution is obtained by adding to a a closed 1-form.

3. A gauge transforation of L is of the form $u \colon M \to \mathrm{U}(1)$. Show that $u(A) = A - u^{-1}\mathrm{d}u$. Show that any closed 1-form with integer periods, i.e. whose integral over all closed loops are integer, can be written as $u^{-1}\mathrm{d}u$ for some u.

4. Conclude that the space of Yang–Mills connection on L up to gauge transformations can be identified with the torus $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.

Problem 3.7 (Covariant exterior derivative)**.** Given a connection A on a vector bundle E, we can extend

$$\mathrm{d}_A = \nabla_A \colon \Gamma(M, E) \to \Omega^1(M, E)$$

to an operator

$$\mathrm{d}_A \colon \Omega^k(M, E) \to \Omega^{k+1}(M, E)$$

by the same Leibniz rule that we use to define the usual exterior derivative.

1. Prove the *Bianchi identity* $d_A F_A = 0$.
2. Show that d_A^2 is the algebraic operator obtained by taking the wedge product with the curvature F_A.
3. Conclude that for every flat connection A there are twisted de Rham cohomology groups $H_A^k(M)$ defined by taking the cohomology of the chain complex $(\Omega^\bullet(M, E), d_A)$.

Problem 3.8 (Chern–Weil theory)**.**

1. Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. An *invariant polynomial* is a polynomial $p\colon \mathfrak{g} \to \mathbb{R}$ which is invariant under the adjoint action of G on $\mathfrak{g}$. For $G = \mathrm{U}(n)$ show that all invariant polynomials are functions of the elementary polynomials p_k defined by the relation

   $$\det(it\xi + I) = \sum p_k(\xi)t^k \quad \text{for } \xi \in \mathfrak{u}(n).$$

   Compute p_1 and p_2.
2. Let $E \to M$ be a $\mathrm{U}(n)$ bundle. Use the Bianchi identity $d_A F_A = 0$ to show that for every $\mathrm{U}(n)$ connection the $2k$ form $p_k(F_A)$ is closed. Here $p_k(F_A)$ denotes the form obtained by applying p_k to the curvature form $F_A \in \Omega^2(M, \mathfrak{u}(E))$ where we combine multiplication in $\mathfrak{u}(n)$ with the wedge product on even forms; $\mathfrak{u}(E)$ is the bundle of skew-Hermitian endomorphisms of E.
3. Show that the de Rham cohomology class of $p_k(F_A)$ does not depend on the choice of the connection. In both this and previous exercise you might find it helpful to consider the case of p_1 first. The cohomology class of $p_k(F)$, up to constant, is the k-th Chern class of E.
4. Do the same exercise for $G = \mathrm{SO}(n)$ to define Pontryagin classes and the Euler class. You might find the following books helpful: *Characteristic Classes* by Milnor and Stasheff, and *Geometry of Differential Forms* by Morita.

Problem 3.9. More exercises in Lorenzo Foscolo's notes!