MORSE THEORY (EXERCISE SHEET)

1. Differential topology

Exercise 1 (Index of a critical point). *Let $M = M^d$ be a smooth manifold and let $f : M \longrightarrow \mathbb{R}$ a smooth function. Assume that $c \in M$ is a critical point i.e. $df(c) = 0$. If $\varphi : \mathbb{R}^d \longrightarrow M$ is an arbitrary chart around c, show that the signature of the Hessian of f at c doesn't depend on the choice of φ*

If the Hessian of f at c is non-degenerate and its signature is (p, q), we say that the index of c is p.

Exercise 2. *Let $M = M^d$ be a smooth manifold and let $f : M \longrightarrow \mathbb{R}$ a smooth function. Assume that $[a, b] \subset \mathrm{Im}(f)$ and that all $x \in [a, b]$ are regular values of f. Show that*

$$f^{-1}([a, b]) \simeq_{diffeo} M_a \times [a, b]$$

where $M_a = f^{-1}(\{a\}))$.

Exercise 3. *Let Σ be a compact surface and assume that there exists $f : \Sigma \longrightarrow \mathbb{R}$ a smooth function which has at most two critical points. Show that Σ is diffeomorphic to the 2-sphere S^2.*

Exercise 4. *Let $f : M \longrightarrow \mathbb{R}$ be a Morse function of a compact manifold M. Show that f has finitely many critical points.*

Exercise 5.
1. *Show that a smooth function on the 2-torus has at least three critical points.*
2. *Find a smooth function on the 2-torus with exactly three critical points (it's a difficult question, finding an example of such a function on the internet and understanding how it is constructed would be a very good way of answering it).*

Exercise 6. *Find an explicit Morse function on the 3-torus.*

2. Morse homology

Exercise 7. *Compute the Morse homology of the sphere S^2 using the standard height function.*

Exercise 8. *Compute the Morse homology of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ using the function $(x, y) \mapsto \cos(x) + \sin(y)$.*

Exercise 9. *Using the fact that Morse homology is equal to singular homology, show that a Morse function on the 2-torus must have at least 4 critical points.*

Exercise 10. *Using the fact that Morse homology is equal to singular homology, find the best possible bound on the number of critical points of a Morse function on $\mathbb{T}^3$.*