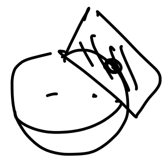
CHERN CLASSES (of vector bundles)

Vector bundles on M : families of vector spaces parametrized (nicely) by points of M.

Examples ① Tangent bundles $T_M = \bigsqcup_{p \in M} T_pM$.

$$T_{S^2} = \left\{ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \begin{matrix} |x| = 1 \\ x \perp v \end{matrix} \right\}.$$

$$T_{S^2} \xrightarrow{p_1} S^2$$



② Trivial vector bundle: $M \times V$

$$M \times V \xrightarrow{p_1} M$$

③ Tautological line bundle on $\mathbb{P}(V)$.

$$\mathbb{P}(V) := \{ \ell \subseteq V \}.$$

$$\mathcal{O}_{\mathbb{P}V}(-1) := \left\{ (\ell, v) \;\middle|\; \begin{matrix} \ell \subseteq V \\ v \in \ell \end{matrix} \right\} \xrightarrow{p} \mathbb{P}V$$

$$(\ell, v) \longmapsto \ell$$

$$p^{-1}(\ell) = \{ v \in \ell \} = \ell.$$

(3') Dual of tautological bundle on $\mathbb{P}V$

$$\underset{\mathbb{P}V}{\mathcal{O}(1)} = \{(\ell \subseteq V, \varphi) \mid \varphi \in \ell^*\} \longrightarrow \mathbb{P}V$$
$$(\ell, \varphi) \longmapsto \ell.$$

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(4) Tautological vector bundle on Grassmannian.

$$\underset{\mathbb{C}^N}{\underset{\shortparallel}{Gr(d,V)}} := \{\pi \subseteq V \mid \pi \text{ is a } d\text{-dim linear subspace}\}.$$

(⚠: Will work only with $\mathbb{C}$-vector spaces, $\mathbb{C}$-vector bundles)

$$\mathcal{E} \longrightarrow Gr(d,V).$$ taut. vector bundle of rank-d.

$$\mathcal{E} = \{(\pi \subseteq V, v \in V) \mid v \in \pi\}.$$

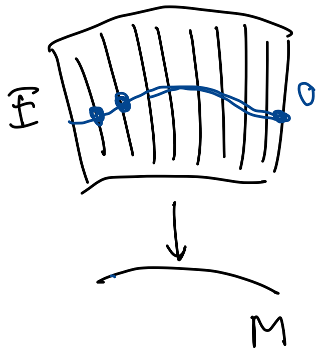
$$\mathcal{E}^* \longrightarrow Gr(d,V)$$
$$\mathcal{E}^* = \left\{(\pi, \varphi) \,\middle|\, \begin{matrix} \pi \subseteq V \\ \varphi \in \pi^* \end{matrix}\right\}.$$

$$Gr(1,V) = \mathbb{P}V$$
$$\mathcal{E}_{Gr(1,V)} = \underset{\mathbb{P}V}{\mathcal{O}(-1)}.$$

Defn of vector bundle $/M$ of rank d:

- A manifold E with a map $p: E \longrightarrow M$.
- On each fiber $p^{-1}(m)$ a structure of dim-d vector space

s.t. $\forall m \in M \ \exists \underset{m}{\underset{\cup}{U}} \subseteq M$ and



$$\begin{tikzcd} \varphi : E|_U \arrow[rr, "\cong"] \arrow[dr, "p|_U"'] & & U \times \mathbb{C}^d \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

linear on fibers

> Prove that the tautological bundles on $\mathbb{P}V$, $Gr(d,V)$ are locally trivial.

(5) Pull-back bundles: $E \xrightarrow{p} Y$ v.bundle.
$X \xrightarrow{f} Y$ map.

$$\left\{ (x \in X, e \in E) \,\middle|\, f(x) = p(e) \right\} =: f^*E$$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "p'"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

> p' is a vector bundle of the same rank as p.

Sections of vector bundles (aka "twisted functions")

$$\Gamma(M,E) := \{\sigma : M \to E \mid p\circ\sigma = \mathrm{id}_M\}.$$

$\shortparallel$

sections of $E/_M$

$$\begin{tikzcd} E \arrow[d, "p"] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

Example: the zero section:

$\sigma_0 : M \longrightarrow E$

$m \longmapsto$ (the zero of the vector space $p^{-1}(m)$).

Example:

$\mathcal{E}^{\vee} \xrightarrow{p} Gr(d,V)$ has nice sections.

$\shortparallel$

$\{(\pi \subseteq V, \varphi \in \pi^*)\}$

Take $\psi \in V^*$. Get a section of $(\mathcal{E}^{\vee}, p)$

$$\begin{tikzcd} \mathcal{E}^{\vee} \arrow[d] & (\pi, \psi|_{\pi}) \\ Gr(d,V) \arrow[u, bend left, "\sigma"] & \pi \subseteq V \arrow[u, mapsto] \end{tikzcd}$$

Can use sections to test whether a vector bundle is trivial

- $\mathbb{R}$-example

Cor T_{S^2} is non-trivial.

A section $\in \Gamma(S^2, T_{S^2})$, aka a vector field on S^2, must vanish at some point (Hairdresser thm), but $S^2 \times \mathbb{R}^2 \longrightarrow S^2$ has never-vanishing sections. $\square$

Another way to test triviality of vector bundles is via **Chern Classes**.

Warning: $\exists$ non-trivial vector bundles with the same Chern classes as the trivial v. bundle.

$\boxed{\text{Find one example!}}$

Chern classes

$$E \longmapsto c_i(E)$$

$$c_i : \mathrm{Vect}^{\mathbb{C}}_d(M) \longrightarrow H^{2i}(M, \mathbb{Z})$$

$$\mathrm{Vect}^{\mathbb{C}}_d(M) = \left\{ \begin{array}{c} \text{iso classes of rank-}d \\ \mathbb{C}\text{-vector bundles on } M \end{array} \right\} \qquad i = 0, 1, \dots, d.$$

Properties:

- $c_0(E) = 1 \in H^0(M, \mathbb{Z})$

$$\boxed{\text{Cor. } c(E^{\text{triv}}) = 1.}$$

- $c_i(f^* E) = f^*(c_i(E))$

$$c(E) = \sum_{i=0}^{d} c_i(E) \in H^*(M) \qquad [\text{Total Chern class}]$$

- $c(E \oplus F) = c(E) \cdot c(F)$ — cup product in H^*.

Concise way for: $c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) \cdot c_{k-i}(F)$.

- $c(\mathcal{O}_{\mathbb{P}V}(1)) = \underset{c_0}{1} + \underset{c_1}{PD(H)} \qquad (H = \{x_0 = 0\} \subseteq \mathbb{P}V.$

- $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(M).$

Define Chern classes in 3 steps:

(1) $\mathrm{Vect}_d(M) \cong [M, Gr(d,\infty)]$ homotopy classes of maps.

$$f^*\mathcal{E} \longleftarrow\!\shortmid f$$

$$\begin{pmatrix} E \\ \downarrow \\ M \end{pmatrix} \longmapsto f_E.$$

$$Gr(d,\mathbb{C}^N) \subseteq Gr(d,\mathbb{C}^{N+1}) \subseteq \cdots$$

$$Gr(d,\infty) = Gr(d,\mathbb{C}^\infty) = \bigcup_{N\to\infty} Gr(d,\mathbb{C}^N).$$

$$\begin{tikzcd} f^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} = \{(\pi,v) \mid v\in\pi\} \arrow[d] \\ M \arrow[r, "f"] & Gr(d,\infty) = \{\pi \subseteq \mathbb{C}^\infty\} \end{tikzcd}$$

another rank-d v. bundle.

rank-d v. bundle.

d-dim

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(2) $H^*(Gr(d,\infty),\mathbb{Z}) \cong \mathbb{Z}[u_1,u_2,\dots,u_d]$

with $u_i \in H^{2i}(Gr(d,\infty))$.

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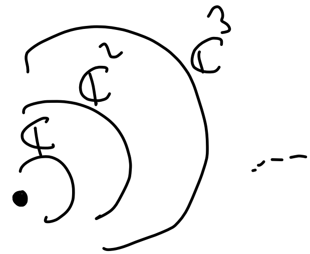
(3) $$c_i\begin{pmatrix} E \\ \downarrow \\ M \end{pmatrix} := (-1)^i (f_E)^*(u_i) \in H^{2i}(M,\mathbb{Z}).$$

Discussion of $H^*(Gr(d,\infty),\mathbb{Z}) \cong \mathbb{Z}[u_1, u_2, \dots, u_d]$.

For $d=1$, $Gr(d,\infty) \cong \mathbb{CP}^\infty$.



$$\mathbb{CP}^\infty = \bigcup_{n=1}^{\infty} \mathbb{CP}^n.$$

$$H^i(\mathbb{CP}^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd.} \end{cases}$$ as a $\mathbb{Z}$-module.

As a ring, $H^*(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}[\alpha]$, $\alpha \in H^2$.

$$H^*(\underset{\{[x_0 : \dots : x_n]\}/\sim}{\overset{\parallel}{\mathbb{CP}^n}}, \mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}.$$

$\alpha = PD(\{x_0 = 0\}) \in H^2$.

$\alpha^j = PD(\{x_0 = x_1 \dots = x_{j-1} = 0\})$

$\alpha^{n+1} = PD(\{x_0 = \dots = x_n = 0\}) = 0$.

Look at the map

$$\underbrace{\mathbb{CP}^\infty \times \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{d \text{ times}} \xrightarrow{\text{can}} Gr(d, \infty)$$

- $(L_1, L_2, \ldots, L_d) \overset{\text{can}}{\longmapsto} (L_1 \oplus \cdots \oplus L_d)$.
- the product v. bundle $\mathcal{O}(-1) \times \cdots \times \mathcal{O}(-1)$ on $(\mathbb{CP}^\infty)^d$.

On H^*, have can^*

$$\begin{array}{ccc}
H^*((\mathbb{CP}^\infty)^{\times d}) & \longleftarrow & H^*(Gr(d,\infty)) \\
\| & & \cong \\
H^*(\mathbb{CP}^\infty)^{\otimes d} & & \mathbb{Z}[u_1, u_2, \ldots, u_d] \\
\| & & \downarrow \cong \\
\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_d] & \supset & \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_d]^{S_d} \\
& & \cong \\
& & \mathbb{Z}\left[\sum \alpha_i, \sum_{i<j} \alpha_i\alpha_j, \ldots, \prod_{i=1}^{d} \alpha_i\right].
\end{array}$$

each α_i has deg 2.

Prove $c(E \oplus F) = c(E) \cdot c(F)$ by using the diagram

$$\begin{tikzcd}
Gr(d,\infty) \times Gr(d',\infty) \arrow[rr] & & Gr(d+d',\infty) \\
(\mathbb{C}P^\infty)^d \times (\mathbb{C}P^\infty)^{d'} \arrow[u, "can"] \arrow[r, phantom, "="] \arrow[urr, phantom, "\circlearrowright"] & {} & (\mathbb{C}P^\infty)^{d+d'} \arrow[u]
\end{tikzcd}$$

Prove $c_1(L \otimes L') = c_1(L) + c_1(L')$ using another map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty,$$

Describe the classes $u_1, \dots, u_d \in H^*(Gr(d,\infty))$ explicitly by PD on each $Gr(d,N)$.

($\Longleftrightarrow$ describe the Chern classes of the $\overset{\text{tautological}}{\underset{\text{dual}}{\wedge}}$ bundles on $Gr(d,N)$).

PD's of cohomology classes on $Gr(d,N)$ are nice subvarieties of $Gr(d,N)$. Defined using Degeneracy loci. $\mathcal{D}_k \subseteq Gr(d,N)$ for $k=1,\dots,d$.

$$\mathcal{D}_k := \left\{ \underset{\substack{\uparrow \\ \dim d}}{\pi} \subset \mathbb{C}^N \;\middle|\; \begin{array}{l} x_1|_\pi, x_2|_\pi, \dots, x_{d+1-k}|_\pi \\ \text{are linearly dependent elements of } \pi^* \end{array} \right\}.$$

$\underline{d=1}$, $\mathcal{D}_1 = \{ \ell \subset \mathbb{C}^N \mid x_1|_\ell \equiv 0 \} \subseteq \mathbb{C}P^{N-1}$ $\supset H$.

$PD(\mathcal{D}_1) = \alpha = u_1.$

$d=2$ $\quad \mathcal{D}_2 := \{ \pi \subset \mathbb{C}^N \mid x_1|_\pi \equiv 0 \} \cong Gr(2,N-1) \subseteq Gr(2,N).$

$$\mathcal{D}_1 := \{ \pi \subset \mathbb{C}^N \mid x_1|_\pi, x_2|_\pi \text{ lin. dep} \}$$

<u>Degeneracy loci</u>. $\mathcal{D}_k \subseteq Gr(d,N)$ for $k=1,\dots,d$.

$$\mathcal{D}_k := \left\{ \underset{\text{dim } d}{\pi} \subset \mathbb{C}^N \;\middle|\; \begin{array}{l} x_1|_\pi, x_2|_\pi, \dots, x_{d+1-k}|_\pi \\ \text{are linearly dependent elements of } \pi^* \end{array} \right\}.$$

$d=2$ $\quad \mathcal{D}_2 := \{ \pi \subset \mathbb{C}^N \mid x_1|_\pi \equiv 0 \} \cong Gr(2,N-1) \subseteq Gr(2,N)$. $\xrightarrow{PD} u_2$

$\mathcal{D}_1 := \{ \pi \subset \mathbb{C}^N \mid x_1|_\pi, x_2|_\pi \text{ lin. dep} \}$ $\xrightarrow{PD} u_1$

$= \{ \pi \subseteq \mathbb{C}^N \mid \dim(\pi \cap \langle e_3, e_4, \dots, e_N \rangle) \geq 1 \}$.

Exercise

$d=3$: looking at $t \subseteq \mathbb{C}^N$, 3d-subspace.

$\mathcal{D}_3 \cong \{ t \subseteq \mathbb{C}^N \mid \dim(t \cap \langle e_2, e_3, \dots, e_N \rangle) \geq 3 \}$ $\quad u_3$

$\mathcal{D}_2 \cong \{ t \subseteq \mathbb{C}^N \mid \dim(t \cap \langle e_3, e_4, \dots, e_N \rangle) \geq 2 \}$ $\quad u_2$

$\mathcal{D}_1 \cong \{ t \subseteq \mathbb{C}^N \mid \dim(t \cap \langle e_4, e_5, \dots, e_N \rangle) \geq 1 \}$. $\quad u_1$

<u>Application</u>: Any smooth cubic surface $Y \subseteq \mathbb{P}^3$ contains exactly 27 lines.

<u>Reformulation</u>: $PD\left(c_4\left(\mathrm{Sym}^3 E\right)\right) = 27$

where $E \longrightarrow Gr(2,4)$ is the dual of the tautological bundle.

$$\underbrace{c_4\left(\mathrm{Sym}^3 E\right)}_{\text{v.bundle of rank } 4} \in H^8\left(Gr(2,4), \mathbb{Z}\right) \overset{PD}{\cong} H_0\left(Gr(2,4), \mathbb{Z}\right) \cong \mathbb{Z}\cdot[pt].$$

$$\boxed{\dim_{\mathbb{C}}\left(Gr(2,4)\right) = 4.}$$

let's compute $c_4(\mathrm{Sym}^3 E)$ in terms of the Chern classes of E: $c_1(E)$, $c_2(E)$.

Suppose $E = L_1 \oplus L_2$; let's compute with this assumption.

$$\mathrm{Sym}^3(L_1 \oplus L_2) = L_1^{\otimes 3} \oplus L_2^{\otimes 3} \oplus (L_1^{\otimes 2} \otimes L_2) \oplus (L_1 \otimes L_2^{\otimes 2})$$

$$c(\mathrm{Sym}^3(L_1 \oplus L_2)) = (1+3x)\cdot(1+3y)\cdot(1+2x+y)\cdot(1+x+2y)$$

$$= 1 + \text{-----} + \boxed{9xy(2x+y)(x+2y)}$$

$$c_1(L_1) = x \qquad c_1(E) = x+y.$$
$$c_1(L_2) = y \qquad c_2(E) = x\cdot y.$$

$$c_4(\mathrm{Sym}^3 E) = 9xy(2x^2+2y^2+5xy) =$$
$$= 9\, c_2(E)\,(2(c_1(E))^2 + c_2(E)).$$

Enough $PD(c_2(E)^2) = 1$, $\quad PD(c_2(E)\cdot(c_1(E))^2) = 1$.

Want $(c_2(E))^2$ is PD to $[\bullet] \in H_0(Gr(2,4))$

$c_2(E)\cdot(c_1(E))^2$ is PD to $[\bullet] \in H_0(Gr(2,4))$.

$$PD(c_2(E)) = \left[\mathcal{D}_2 = \{\pi \subseteq \mathbb{C}^4 \mid \pi \subset \langle e_2, e_3, e_4\rangle\}\right]$$

$$PD(c_1(E)) = \left[\mathcal{D}_1 = \{\pi \subseteq \mathbb{C}^4 \mid \pi \cap \langle e_3, e_4\rangle \text{ contains a line}\}\right]$$

$$PD((c_2(E))^2) = \left[\left\{\pi \subseteq \mathbb{C}^4 \;\middle|\; \begin{array}{l} \pi \subset \langle e_2, e_3, e_4\rangle \\ \pi \subset \langle e_1, e_3, e_4\rangle \end{array}\right\}\right] =$$

$$= \left[\{\pi = \langle e_3, e_4\rangle\}\right] = [\text{a point of } Gr(2,4)].$$

$$PD(c_2(E)\cdot c_1(E)^2) = \left\{\pi \subseteq \mathbb{C}^4 \;\middle|\; \begin{array}{l} \pi \subseteq \langle e_2, e_3, e_4\rangle, \\ \pi \cap \langle e_1, e_2\rangle \text{ contains a line}, \\ \pi \cap \langle e_4, e_1 + e_2\rangle \text{ contains a line}. \end{array}\right\}$$

$$= \{\pi = \langle e_2, e_4\rangle\} = [\text{a point of } Gr(2,4)].$$

Let E, F be vector bundles of rank 2. Compute $c_4(E \otimes F)$ in terms of the Chern classes of E, F.

Let E be a rank-2 vector bundle and L a line bundle. Compute $c_2(E \otimes L)$