Spec and Proj

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Affine varieties: Subset of a VS cut out by poly's:

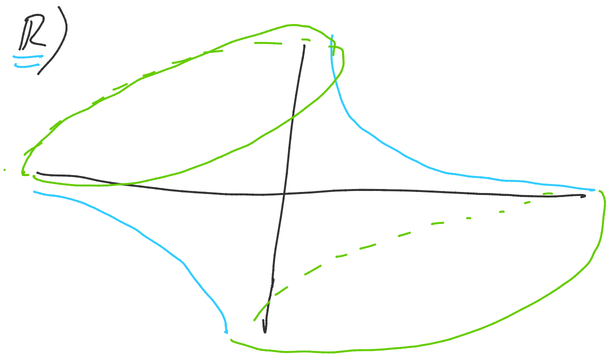
$\mathbb{C}^n \supset \{ f_1(\underset{\sim}{x}) = f_2(\underset{\sim}{x}) = \dots = f_k(\underset{\sim}{x}) = 0 \} = V(f_1, \dots, f_k)$

$f_i \in \mathbb{C}[\underset{\sim}{x}] = \mathbb{C}[x_1, \dots, x_n]$ ring of poly's

e.g: $V(xy-1)$

Over $\mathbb{C}$: $y = \frac{1}{x}$

$= \mathbb{C} \setminus 0$



Ambient VS denoted $\mathbb{A}^n$ "affine space"

- leave field (/ring) arbitrary: $\mathbb{C}^n = \mathbb{A}^n(\mathbb{C})$

  $\mathbb{R}^n = \mathbb{A}^n(\mathbb{R})$ etc...

- allow non-linear co-ord changes (have to be poly!)

e.g. $V(xy - x - 1) = V(x(y-1) - 1)$ same as $V(xy-1)$

Zariski topology

An aff var $V \subset \mathbb{A}^n$ is closed. $\Leftarrow$ allowed "Zariski-closed"

So is e.g. $\{|x|^2 \leq 1\} \subset \mathbb{C}$ ✗ not an aff var. $\Leftarrow$ not allowed.

doesn't make sense over any field.

Still gives a topology "Zariski topology". $\Leftarrow$ Rubbish.

e.g: $\mathbb{A}^1 \supset V = V(f_1(x), \dots, f_k(x))$ is a finite set.

Zar open $\Leftrightarrow$ "cofinite". very large.

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$\Rightarrow$ any 2 intersect.

$\Rightarrow$ $\mathbb{A}^1$ is compact.

<u>Algebra</u> $f_1, \dots, f_k \in \mathbb{C}[\underset{\sim}{x}] \rightsquigarrow V(f_1, \dots, f_k) \subset \mathbb{A}^n$

ideal $J = (f_1, \dots, f_k) \subset \mathbb{C}[\underset{\sim}{x}]$ $\qquad$ $\uparrow$ only depends on J write $V(J)$.

Given $V \subset \mathbb{A}^n$ have an ideal $I_V = \{\text{polys that vanish on } V\}$

If $V = V(J)$ then $J \subset I_V$. But can be $\neq$.

<u>e.g.</u> $J = (x^2) \subset \mathbb{C}[x]$, $V(J) = \{x^2 = 0\} \subset \mathbb{A}^1$, $I_V = (x)$
$= \{0\}$

<u>Defn</u>: Given $\underset{\text{ideal}}{J} \subset \underset{\text{ring}}{R}$

$\mathrm{rad}(J) = \{r \in R \text{ st. } r^n \in J \text{ for some } n \in \mathbb{N}\}$ ideal ✓

Clearly: if $V = V(J)$ then $\mathrm{rad}(J) \subset I_V$

<u>Thm</u>: "Hilbert's Nullstellensatz" Over $\mathbb{C}$ (any alg. closed field).

If $V = V(J)$ then $I_V = \mathrm{rad}\, J$

- Fails over $\mathbb{R}$ (ex).
  or $\mathbb{F}_p$

affine vars in $\mathbb{A}^n$ $\underset{\text{NLSTZ.}}{\overset{\text{bij}}{\longleftrightarrow}}$ radical ideals in $\mathbb{C}[\underset{\sim}{x}]$

Better: quot ring: $\mathbb{C}[\underset{\sim}{x}] / I_V =: \mathbb{C}[V]$

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Say a function $f: V \to \mathbb{C}$ is <u>regular</u> if $f = \hat{f}|_V$ for $\hat{f}$ a poly.

$$\mathbb{C}[\underset{\sim}{x}] \longrightarrow \{\text{reg fns on } V\}$$
$$\hat{f} \longmapsto \hat{f}|_V$$

Ker is I_V $\Rightarrow$ $\mathbb{C}[V]$ (or $\mathcal{O}_V$)

What rings are these?

- quotient of $\mathbb{C}[x_1, \dots, x_n]$ $\longleftrightarrow$ f.g. $\mathbb{C}$-algebra.
  by a radical ideal $\longleftrightarrow$ with no nilpotents "reduced" $\leadsto$ $x^k = 0 \Leftrightarrow x = 0$

$$\boxed{\text{aff vars} \xleftrightarrow{\text{bij}} \text{rings s.t.}}$$

The ring $\mathbb{C}[V]$ knows everything about $V \subset \mathbb{A}^n$.

<u>e.g</u>: points: $p \in V$ gives $ev_p : \mathbb{C}[V] \to \mathbb{C}$
$$f \mapsto f(p)$$

<u>Lemma</u>: Any ($\mathbb{C}$-alg) hom'ism $\mathbb{C}[V] \to \mathbb{C}$ is ev_p for some $p \in V$.

<u>Pf</u>: Pick co-ords. $V \subset \mathbb{A}^n$, $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/I_V$. Given $\psi : \mathbb{C}[V] \to \mathbb{C}$
Look at $\psi(x_i) =: p_i \in \mathbb{C}$. Get $p = (p_1, \dots, p_n) \in \mathbb{A}^n$.
ψ a hom'ism $\Rightarrow$ $\psi = ev_p$.
$I_V \subset \operatorname{Ker} \psi$ $\Rightarrow$ $p \in V$. $\square$

As a set, $V = \{\text{hom'isms } \mathbb{C}[V] \to \mathbb{C}\}$.
$$= \{\text{max ideals in } \mathbb{C}[V]\} =: \operatorname{Specm}(\mathbb{C}[V])$$

Can get Zar topology too. Not enough.....

<u>Def</u>: Given two aff var's $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$, a function $F: V \to W$ is <u>regular</u> if each component is regular $V \to \mathbb{A}^1$.

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$F^*: \mathbb{C}[W] \to \mathbb{C}[V]$ is a ($\mathbb{C}$-alg) hom'ism.

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$$g \longmapsto g \circ F$$

If $V = \{p\}$ then F is a choice of $p \in W$ & $F^* = ev_p$.

<u>Lemma</u>: Any $\mathbb{C}$-alg hom'ism $\mathbb{C}[W] \to \mathbb{C}[V]$ is F^* for a reg map $F: V \to W$. <u>Pf</u>: Ex. //

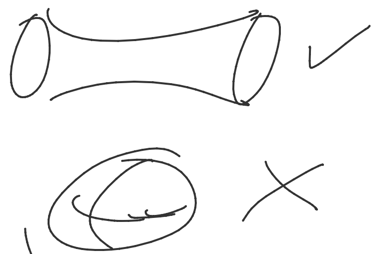
Equiv of categories: (contravariant) affine vars $\Longleftrightarrow$ red. f.g. $\mathbb{C}$-alg's.

$$V \rightsquigarrow \mathbb{C}[V]$$
$$\operatorname{Spec} R \leftsquigarrow R$$

Explicitly: Choose generators for R. So $R = \mathbb{C}[x_1, \dots, x_n] / (f_1, \dots, f_k)$

$$\operatorname{Spec} R = V(f_1, \dots, f_k) \subset \mathbb{A}^n$$

Indep. of choices up to iso'ism.

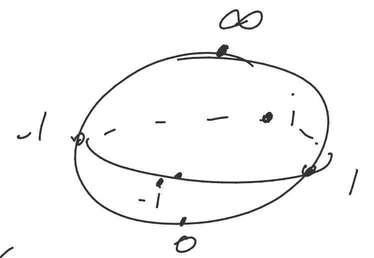


<u>Beyond affine</u>

An aff var $V \subset \mathbb{A}^n$ $(\mathbb{C}^n)$ is not compact (in $\mathbb{C}$-top) (unless finite)

<u>Solution</u>: projective var's:

<u>e.g.</u>: Riemann sphere: $\mathbb{C} \cup \infty$



$= \mathbb{P}^1(\mathbb{C})$

$= \{\text{lines in } \mathbb{C}^2\} = \mathbb{C}^2 \setminus (0,0) / \mathbb{C}^* \quad (x,y) \sim (\lambda x, \lambda y)$

Write $x:y$ for the line through (x,y) or $[x:y]$

$x:y = \lambda x : \lambda y$ for $\lambda \in \mathbb{C}^*$.

Have ambient: $\mathbb{P}^1 \setminus 0{:}1 \xrightarrow{\sim} \mathbb{C}$ so $\mathbb{P}^1 = \mathbb{A}^1 \cup 0{:}1$

Have gradient: $\mathbb{P}^1 \setminus 0:1 \xrightarrow{\sim} \mathbb{C}$ so $\mathbb{P}^1 = \mathbb{A}^1 \cup 0:1$ "∞"

$x:y \mapsto y/x = Y$

$\sim 1 : y/x$

or: $\mathbb{P}^1 \setminus 1:0 \xrightarrow{\sim} \mathbb{A}^1$ so $\mathbb{P}^1 = \mathbb{A}^1_X \cup \mathbb{A}^1_Y$

$x:y \mapsto \frac{x}{y} = X$

$X = 1/Y$ "transition fn"

$$\mathbb{P}^n = \{\text{lines in } \mathbb{C}^{n+1}\}$$
$$= \underbrace{\mathbb{A}^n \cup \dots \cup \mathbb{A}^n}_{n+1 \text{ copies}}$$

Charts: e.g. $\mathbb{P}^2_{x:y:z} \setminus \{0:y:z\} \xrightarrow{\sim} \mathbb{A}^2$

$x:y:z \mapsto (\frac{y}{x}, \frac{z}{x})$

$\sim 1 : \frac{y}{x} : \frac{z}{x}$

$\mathbb{P}^n$ is several affine vars "glued together".

General varieties are

Need a "thing" of which affine var's are a special case

+ can glue "things".

"thing" can be "locally ringed space". or abstract solution....

other solutions.

OR: just use projective var's (or quasi-proj vars).

Can we find "regular fns" on $\mathbb{P}^n$ & take zero locus?

No: reg fn on $\mathbb{P}^n$ = poly on $\mathbb{C}^{n+1}$ which is inv under rescaling

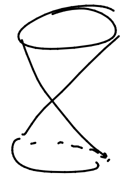
$\Rightarrow$ const.

Only reg fns on $\mathbb{P}^n$ are consts. (bounded hol fns are const)

But can consider homogeneous poly's in $\mathbb{C}[\underline{x}]$

But can consider <u>homogeneous poly's</u> in $\mathbb{C}[\underline{x}]$

"surface ODP"



e.g. $f = xy - z^2$ hom. deg 2 in $\mathbb{C}[x,y,z]$.

If $(x,y,z) \mapsto (\lambda x, \lambda y, \lambda z)$ then $f \mapsto \lambda^2 f$

$\Rightarrow$ $V(f) \subset \mathbb{A}^3$

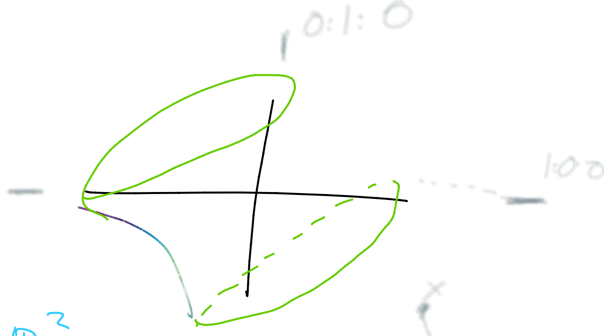
is invariant under rescaling. ie. union of lines, "cone"

defines a subset $V(f) \subset \mathbb{P}^2$. This is a proj. variety.

<u>Defn</u>: A <u>projective variety</u> is a subset of $\mathbb{P}^n$ cut out by hom. polys in $\mathbb{C}[x_1, \dots, x_{n+1}]$.

<u>NB</u>: The poly's are <u>not</u> fns on $\mathbb{P}^n$!

<u>e.g.</u>: $V(xy - z^2) \subset \mathbb{P}^2$



In $z \neq 0$ get $V(xy-1) \subset \mathbb{A}^2$

$\simeq \mathbb{C}^{\times}$

If $z=0$ get $V(xy) \subset \mathbb{P}^1_{x:y} \underset{z=0}{\hookrightarrow} \mathbb{P}^2_{x:y:z}$

$= \{0:1:0,\ 1:0:0\}$ 2 points.

$\mathbb{R}$-points are S^1 $\simeq \mathbb{RP}^1$

$\mathbb{C}$-points $=$ $\mathbb{C}^* \cup 2$ pts.

$\sim S^2$ or $\mathbb{P}^1$

<u>Ex</u>: $V \cong \mathbb{P}^1$ as a proj variety.

$\mathbb{C}[x_1, \dots, x_n]$ is graded, $|x_i^k| = k$

If $f_1, \dots, f_k$ are all hom (can have different degrees)

$\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k)$ is graded.

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$f_1,\dots,f_k$ hom $\rightsquigarrow$ $\mathbb{C}[\underset{\sim}{x}]/(f_i)$ graded ring

$\rightsquigarrow$ $V(f_1,\dots,f_k) \subset \mathbb{P}^n$ proj variety. Not equiv

e.g $xy - z^2 \in \mathbb{C}[x,y,z]$

$\rightsquigarrow$ $\dfrac{\mathbb{C}[x,y,z]}{xy-z^2}$ or $\to$ $V(xy-z^2) \subset \mathbb{P}^2$
$\cong \mathbb{P}^1$

$\not\cong$

Compare $\mathbb{C}[x,y] \longleftrightarrow \mathbb{P}^1$

<u>Proj</u> : graded rings $\rightsquigarrow$ proj varieties
(not an equiv.)

If R is f.g, graded, generated in deg 1, reduced
$\Rightarrow$ $R = \mathbb{C}[x_1,\dots,x_n]/(f_1,\dots,f_k)$ for f_i hom.

Then $\operatorname{Proj} R := V(f_1,\dots,f_k) \subset \mathbb{P}^{n-1}$

<u>In fact</u> $\operatorname{Proj} R$ is a var. <u>plus</u> a line-bundle.

$\mathbb{P}^n$ has a "tautological line bundle".

a point in $\mathbb{P}^n$ "is" a line $\ell \subset \mathbb{C}^{n+1}$.
Stick the lines together. Called $\mathcal{O}(-1) \to \mathbb{P}^n$.
$= \{\ell \subset \mathbb{C}^{n+1},\ p \in \ell\}$

e.g $\mathbb{P}^1$ has $\mathcal{O}(-1) \to \mathbb{P}^1$
$\shortparallel$
$\{x : y \subset \mathbb{C}^2,\ p = (\lambda x, \lambda y) \in \mathbb{C}\}$

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The co-ord fn "x" is a linear fn on $\mathbb{C}^2 \to \mathbb{C}$
$(a,b) \mapsto a$.

gives a linear fn on any line $x:y \to \mathbb{C}$
$(\lambda x, \lambda y) \mapsto \lambda x$

So it's an element of the <u>dual line</u> $\mathrm{Hom}(x:y, \mathbb{C})$

$\Rightarrow$ x is a section of the dual line bundle

$$\mathcal{O}(1) := \mathcal{O}(-1)^* \quad \text{over } \mathbb{P}^1$$

So is y. In fact

$$H^0(\mathbb{P}^1, \mathcal{O}(1)) = \{\text{all alg/holomorphic sections of } \mathcal{O}(1)\} = \langle x, y \rangle$$

Similarly if $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ then

$$H^0(\mathbb{P}^1, \mathcal{O}(k)) = \langle x^k, x^{k-1}y, \dots, y^k \rangle \quad \text{deg } k \text{ poly's.}$$

So $$\mathbb{C}[x,y] = \bigoplus_{k \geq 0} H^0(\mathbb{P}^1, \mathcal{O}(k))$$

If $V \subset \mathbb{P}^n$ a proj variety then all reg fns on V are const.
But we have a line-bundle $\mathcal{L} = \mathcal{O}(1)|_V$. Hence a graded ring

$$R_V = \bigoplus_{k \geq 0} H^0(V, \mathcal{L}^{\otimes k})$$

"Proj" is the equivalence

$$\begin{array}{c} \text{f.g. graded rings} \\ \text{generated in deg } 1 \end{array} \xrightarrow[\sim]{\text{Proj}} \text{Proj var's} + \text{"very ample" line bundle}$$

("very ample" := restriction of $\mathcal{O}(1)$)

<u>e.g.</u>: $\mathbb{P}^1$ with $\mathcal{L} = \mathcal{O}(1)$ gives ring $\mathbb{C}[x,y]$

but $\mathbb{P}^1$ with $\mathcal{L} = \mathcal{O}(2)$ gives the ring

$$\bigoplus_{k \geq 0} H^0(\mathbb{P}^1, \mathcal{O}(2k)) = \mathbb{C}[x^2, xy, y^2] \quad \text{poly's of } \underline{\text{even}} \text{ degree}$$

$$\cong \frac{\mathbb{C}[X, Y, Z]}{XZ - Y^2} \quad \text{as above.}$$