1. Introduction

Setup: X=(projective) algebraic variety over $\mathbb{C}$ of dimension n.
A fibration $f\colon X \to Y$ is a proper morphism with connected fibres (i.e. $f_*\mathcal{O}_X = \mathcal{O}_Y$).
Zariski: a birational morphism $f\colon X \to Y$ always admits connected fibres.
By the Stein factorization theorem, any morphism $X \to Z$ factors through a fibration $X \to Y$ so that the induced morphism $Y \to Z$ is finite. Indeed

$$Y = \operatorname{Spec} f_*\mathcal{O}_X.$$

Idea: it is usually hard to find fibrations.

Example 1.1. Let $X = \mathbb{P}^n$. Then there are no non-trivial fibrations $f\colon X \to Y$. E.g. if $n = 2$ and $f\colon X \to C$ is a fibration onto a curve C then any two fibres do not meet (contradicting Bezout theorem).
More in general, if $\operatorname{Pic}(X) = \mathbb{Z}$ then there are no non-trivial fibrations $f\colon X \to Y$. Prove it

Let X be a smooth variety. A **divisor** on X is a linear combination of hypersurfaces:

$$D = \sum a_i S_i \qquad \text{with } \dim S_i = n - 1.$$

We say that D is effective ($D \geq 0$) if all the coefficients are positive.
We may define $\mathcal{O}_X(D)$ as a rank 1 sheaf associated to D. Thus, on a projective manifold, we have

Divisors (up to lin. eq.) $\Leftrightarrow$ Invertible sheaves $\Leftrightarrow$ Line bundles

Note that:
$\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ and $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^*$.

not true for singular surfaces. Find example.

If C is a curve (i.e. $n = 1$) and $D = \sum a_i p_i$ is a divisor then $\deg D = \sum a_i$.
If X is a surface (i.e. if $n = 2$), we may define intersection product: $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$

$$C_1 \cdot C_2 = \deg \mathcal{O}_X(C_1)|_{C_2}$$

With the same trick, if $n \geq 2$, D is a divisor (L is a line bundle) and C is a curve, we can define $D \cdot C$ ($L \cdot C$). It turns out that

$$L \cdot C = \int_C c_1(L).$$

Exercise: If $L_i = \mathcal{O}_X(D_i)$ we have

$$D_1.D_2 = \int_{D_1} c_1(L_2) = \int_{D_2} c_1(L_1) = \int_S c_1(L_1).c_1(L_2).$$

Prove it

In particular $H^0(X, \mathcal{O}_X(D))$ is a finite dimensional $\mathbb{C}$-vector space such that

$$\mathbb{P}(H^0(X, \mathcal{O}_X(D)) =: |D| = \{G \sim D \mid G \geq 0\}.$$

is the **linear system** associated to D.

Fact: if $f\colon X \to Y$ is a fibration then

$$H^0(X, f^*L^{\otimes m}) = H^0(Y, L^{\otimes m})$$

for all m.
A divisor D (or a line bundle L) is **base point free**, if for any $x \in X$ there exists $G \in |D|$ such that $x \notin G$ ($s \in H^0(X, L)$ such that $s(x) \neq 0$).
Assume that L is base point free. Then we may define:

$$\phi_L\colon X \to \mathbb{P}(H^0(X, L)^*)$$

by, for any $x \in X$, $\phi_L(x)$ is the hyperplane in $H^0(X, L)$ defined by all the sections vanishing at x.
L is **very ample** if ϕ_L is an embedding. L is **ample** if there exists $m > 0$ such that $L^{\otimes m}$ is very ample. L is **semi-ample** if there exists $m > 0$ such that $L^{\otimes m}$ is base point free.

Note that if X is projective then by definition $X \subseteq \mathbb{P}^N$ for some N. and in particular, $L = \mathcal{O}_X(1)|_X$ is very ample. If $f\colon X \to Y$ is a fibration and L is very ample (resp. ample) on Y then f^*L is base point free (resp. semi-ample) on X.

In general if $H^0(X,L) \neq 0$, then we can still define $\varphi\colon X \dashrightarrow \mathbb{P}(H^0(X,L)^*)$ as a rational map (i.e. defined on a Zariski open subset of X).
Similarly if $W \subseteq H^0(X,L)$ is a subspace, then we can define

$$\phi_W\colon X \dashrightarrow \mathbb{P}(W).$$

Example 1.2. Assume X is a surface (i.e. $n = 2$) and let $C \subseteq X$ be a curve such that $C^2 < 0$ (e.g. let X be the blow-up of $\mathbb{P}^2$ at one point and let C be the exceptional divisor. In this case $C^2 = -1$). Then for all $m > 0$, $|mC| = \{mC\}$ and clearly C is not semi-ample (exercise: check it).

Prove it

Example 1.3. Let $X = \mathbb{P}^2$ and let Z=6 points in general position in $\mathbb{P}^2$. Consider $L = \mathcal{O}_X(3)$ and

$$W = \{s \in H^0(X,L) \mid s|_Z = 0\} \subseteq H^0(X,L) \simeq \mathbb{C}^{10}$$

Then $\dim W = 4$. Thus

$$\phi_W\colon X \dashrightarrow Y := \phi_W(X) \subseteq \mathbb{P}(W^*) \simeq \mathbb{P}^3.$$

It turns out that Y is a cubic surface which is isomorphic to $\mathbb{P}^2$ blown-up along Z (Exercise: Check it).

Prove it

2. How to check if a line bundle (divisor) is ample (semiample)?

2.1. **Algebraic method.** Projective varieties admit a lot of curves. Curves can be used to understand if a line bundle is ample.

Fact: If L is ample then $L \cdot C > 0$ for any curve C. Similarly if L is semi-ample then $L \cdot C \geq 0$ for any curve C.

Example 2.1 (Stupid example)**.** E=elliptic curve. $X = E \times E$, $p, q \in E$ general points. $D = p_1^*(p) - p_1^*(q)$. Then $D \cdot C = 0$ for any curve C. On the other hand, D is not semi-ample (if p, q are general). Actually $H^0(X, mD) = 0$ for all D.

Example 2.2 (More interesting example)**.** Δ=unit disk. Γ irreducible lattice such that $X = \Delta^2/\Gamma$ is a smooth surface (called **Hilbert modular surface**). Note that if Γ is not irreducible then we could have $X = C_1 \times C_2$ for C_1, C_2 of genus > 1. We have $T_X^* = A \otimes B$ where A, B are line bundle such that $A \cdot C > 0$ and $B \cdot C > 0$ for any curve C but again $H^0(X, mA) = H^0(X, mB) = 0$ for any curve C.

Conjecture 2.3 (Abundance Conjecture:)**.** *If $K_X \cdot C > 0$ (resp. ≥ 0) for any curve C then K_X is ample (resp. semi-ample).*

Given a curve $C \in X$ and a point $x \in X$, we denote by $\operatorname{mult}_x C$ the **multiplicity** of C at x ($x \in C$ is a smooth point if and only if $\operatorname{mult}_x C = 1$).

Theorem 2.4 (Seshadri's criterion)**.** *A divisor D is ample if and only if there exists $\epsilon > 0$ such that for any $x \in X$ and for any curve $C \ni x$ we have*

$$D \cdot C > \epsilon \operatorname{mult}_x C.$$

Note that, because of the stupid example, there cannot be any such criterion to check for semi-ampleness.
Seshadri's criterion implies that if L is ample and D is nef then $A + D$ is ample.

Theorem 2.5 (Nakai-Moishezon-Kleiman's criterion)**.** *Let L be a line bundle on a projective manifold X. Then L is ample if and only if*

$$L^{\dim V} \cdot V > 0$$

for any irreducible variety $V \subseteq X$ (including $V = X$).

Note that if L is nef then L is said **big** if $L^{\dim X} > 0$. This, in particular implies that

$$h^0(X, mL) = Cm^{\dim X} + \dots$$

for some $C > 0$. It also implies that there exists $m \gg 0$ such that $mL = A + D$ where A is ample and D is effective (These last two properties make sense even if L is not nef).

2.2. **Some conditions:**

- Given a line bundle L, we define the base locus of L as
$$Bs(L) = \{x \in X \mid x \in D \text{ for any } D \in |L|\}.^{1}$$
Note that $Bs(mL) \subseteq Bs(L)$ for all m. The **stable base** locus of L is the intersection
$$\mathbf{B}(L) := \cap_{m \geq 1} Bs(mL).$$
Then L is semi-ample if and only if $\mathbf{B}(L) = \emptyset$.
- (Zariski) If X is normal, L is big and nef then L is semi-ample if and only if the graded ring
$$R(X, L) = \bigoplus H^0(X, mL)$$
is finitely generated.

Proof. First note that if $R = \oplus R_d$ is a graded ring then R is finitely generated if and only if any of its truncations $R^{(m)} := \oplus R_{md}$ is finitely generated. Assume that L is semi-ample. Let m such that mL is base point free. It is enough to show that $R(X, mL)$ is finitely generated. There exists Y and A very ample on Y such that $R(X, mL)$ is isomorphic to $R(Y, A)$

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[1] Note that this is just a subset of X, not a subscheme.

and this is a quotient of the ring of polynomials which is finitely generated. $\square$

- (Fujita) If X is normal and Z is the stable base locus of L then $L|_Z$ is not ample.
- over $\overline{\mathbb{F}}_p$, it is hard to find counterexamples of line bundles which are not nef but not semi-ample. In particular, we do not know any line bundle on a surface X which is positive on every curve and it is not ample.
- (Keel) in positive characteristic, L is semi-ample if and only if $L|_{\mathbb{E}(L)}$ is semi-ample. Recall that $\mathbb{E}(L) = \cup_{L|_V \text{ is not big}} V$.

  char $K = p > 0 \Rightarrow$ use Frobenius: $B = A + D$ nef with A =ample and $D \geq 0$

  $\Rightarrow mB - D$ is ample and $H^1(X, mB - D)$ is evaporable, i.e. $\forall \alpha \in H^1(X, mB - D)$, $\exists F : Y \to X$ finite morphism s.t. $F^*\alpha = 0$.

- if $X = C \times C$ where C is a curve of genus > 1 and $L = p_1^*\omega_C \otimes \mathcal{O}_X(\Delta)$. Then L is big and nef and $\mathbb{E}(L) = \Delta$. Moreover $L|_\Delta = \mathcal{O}_\Delta$. On the other hand, over $\mathbb{C}$, L is not semi-ample. Indeed $L|_{2\Delta}$ is not torsion.

  Nakamaye's Theorem: over any algebraically closed field, given L nef and A ample, we may define

$$\mathbf{B}_+(L) := \cap_{\epsilon > 0} \mathbf{B}(L - \epsilon A)$$

Then $\mathbf{B}_+(L) = \mathbb{E}(L)$. Note that, in particular, this implies that $\mathbf{B}_+(L)$ is a numerical invariant.

2.3. **Analytic method:** Let X be a compact Kähler manifold and let L be an holomorphicm line bundle on X. We say that L is **positive** if it admits a Hermitian metric with positive curvature. Recall that a metric on L is given by, for any trivialisation $\theta\colon L|_U \to U \times \mathbb{C}$,

$$||\xi|| = |\theta(\xi)| \cdot h(x) \qquad x \in U, \xi \in L_x$$

and the curvature is given by

$$\Theta(x) = \partial\overline{\partial} \log h$$

(check that it is well defined).
In other words, if $c_1(L)$ is represented by a Kähler form.
Fact: L is positive if and only if $L^{\otimes m}$ is positive for some $m > 0$.

Theorem 2.6. *Let X be a compact Kähler manifold and let L be an holomorphic line bundle. Then L is positive if and only if it is ample.*

If L is ample then it is easy to check that it is positive. Indeed there exists $m > 0$ such that $L^{\otimes m}$ is very ample and in particular $L^{\otimes m} = \phi_{L^{\otimes m}}\mathcal{O}_{\mathbb{P}^N}(1)$ and it is enough to take the restriction of the Fubini-study metric on $\mathbb{P}^N$.
Fact: there exist complete and smooth toric varieties (of dimension 3) without any ample line bundle (i.e. they are

not projective). Note that if X is not projective then there does not exist any integral Kähler form in $H^2(X, \mathbb{R})$.
Kähler manifolds also have a Nakai-Moishezon type theorem:

Theorem 2.7 (Demailly-Paun). *Let X be a Kähler manifold. The cone of all Kähler classes in $H^{1,1}(X, \mathbb{R})$ is a connected component of the set of all classes $\alpha \in H^{1,1}(X, \mathbb{R})$ such that*

$$\int_V \alpha^{\dim V} > 0$$

for every irreducible variety $V \subseteq X$.

In particular, if X does not contain any proper subvariety then the Kähler cone of X is a connected component of the set of classes with positive self-intersection.

3. What about K_X?

The easiest example in the study of directed graphs associated to the birational geometry of projective varieties is given by the category of smooth projective surfaces. To this end, we consider the *directed graph* whose vertices are smooth projective surfaces defined over an algebraically closed field k and whose edges are proper birational morphisms. The connected component containing a projective surface X corresponds to the birational class of X. We now look at some easy properties of this component. First, it is easy to check that this graph is a tree. Indeed, if X, Y are non-isomorphic projective surfaces connected by an edge, i.e. if there exists a non-trivial projective morphism $f\colon X \to Y$, then the second Betti number of X is greater than the one of Y. Thus, the claim follows easily.

Note that there are always infinitely many vertices above a vertex associated to a projective surface X, as it is always possible to blow-up an infinite sequence of points to obtain an infinite chain above X. On the other hand, using the inequality on the second Betti number described above, it is easy to check that starting from a vertex X, it is always possible to find an end-point below X. More specifically, there exists no infinite chain starting from X. Thus, we can think of the end-point Y to be a good representative of the connected class of X. We will see that also in higher dimension, one the main goals of the minimal model programme is to find the end-point of a connected component associated to a projective variety X.

We now show that projective surfaces can be divided into two large classes. The same dichotomy is expected to hold also in higher dimension.

First, we assume that X is a smooth projective surface such that $h^0(X, mK_X) > 0$ for some positive integer m. Then the subgraph obtained by considering the vertices below X and the corresponding edges is finite. In addition, there exists a unique vertex which is an end-point for the connected component containing X. Such a vertex Y is called the *minimal model* of X and, by Castelnuovo theorem, it is characterised by the fact that it does not admit any smooth rational curve E of self-intersection -1. Alternatively, Y is the only surface in the connected component of X such that K_Y is nef, i.e. $K_Y \cdot C \geq 0$ for any curve C in Y.

We now assume that X is a smooth projective surface such that $h^0(X, mK_X) = 0$ for all positive integer m. In this case, X is *uniruled*, i.e. it is covered by rational curves. It is possible to show that although the graph below X might be finite, there are always infinitely many end-points for the connected component of X. For example, if $X = \mathbb{P}^2$ is the two-dimensional projective space over the field k, then the connected component containing the vertex associated to X, corresponds to the set of all the smooth *rational surfaces*. Clearly, X is an end-point of such a graph, but also each Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, with $n \in \mathbb{N}$, $n \neq 1$, is such. Finally, note that not all the projective surfaces which admit a Mori fiber space is an end-point for the directed graph we have constructed (e.g. the blow-up of $\mathbb{P}^2$ at one point admits a Mori fiber space, but it corresponds to a vertex which is not an end-point).