POSITIVITY IN ALGEBRAIC GEOMETRY

Exercise 1. *Show that if $f\colon X \to Y$ is a surjective morphism with connected fibres and $\mathrm{Pic}(X) = \mathbb{Z}$ then Y is either a point or it is isomorphic to X.*

Exercise 2. *Find an example of divisors D_1, D_2 on a surface such that $\mathcal{O}_X(D_1 + D_2)$ is not isomorphic to $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$.*

Exercise 3. *Let X be a smooth surface. Show that if C_1 and C_2 are curves in X then*

$$C_1 \cdot C_2 = C_2 \cdot C_1$$

(recall that we defined $C_1 \cdot C_2 = \deg \mathcal{O}_X(C_2)|_{C_1}$.)

Exercise 4. *Let Y be a smooth surface and let $\phi\colon X \to Y$ be the blow-up at a point $p \in Y$ with exceptional divisor $E = \phi^{-1}(p)$. Show that $E^2 = -1$ and, in particular, $|mE| = \{mE\}$ for any positive integer m.*

Exercise 5. *Let $X \subset \mathbb{P}^N$ be a projective variety. Show that $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_X$ is very ample.*

Exercise 6. *Let $f\colon X \to Y$ be a fibration and let L be an ample line bundle on Y. Showt hat f^*L is semi-ample.*

Exercise 7. *Let $X = \mathbb{P}^2$ and let Z be six points in $\mathbb{P}^2$ in general position. Consider $L = \mathcal{O}_X(3) = \mathcal{O}_X(1)^{\otimes 3}$ and let*

$$W = \{s \in H^0(X, L) \mid s|_Z = 0\} \subset H^0(X, L) \simeq \mathbb{C}^{10}.$$

Show that $\dim W = 4$. As in the lecture, define

$$\phi_W\colon X \dashrightarrow Y := \overline{\phi_W(X)} \subset \mathbb{P}(W^*) \simeq \mathbb{P}^3$$

on the open set $X \setminus Z$. Show that Y is a cubic surface which is isomorphic to the blow up of $\mathbb{P}^2$ along Z.

Exercise 8. *Let D be an ample divisor on a smooth projective variety X. Show that $D \cdot C > 0$ for all the curves $C \subset X$.*

Exercise 9. *Show that if D is an ample divisor on a smooth projective variety X then D is big. Recall that D is said to be big if there exists $C > 0$*

$$\dim H^0(X, \mathcal{O}_X(mD)) \geq C m^{\dim X}$$

for any sufficiently divisible positive integer m.

Viceversa, show that if D is a big divisor on X then there exists a positive integer m such that

$$mD \sim A + B$$

where A is ample and $B \geq 0$.

Exercise 10. *Show that if L is a semiample divisor on a projective variety X then $L|_Z$ is semi-ample for any subvariety $Z \subset X$.*

Exercise 11. *Let C be a curve of genus $g > 1$ and let $X = C \times C$. Let $p\colon X \to C$ be the projection onto the first factor. Let*

$$L = p^* K_C + \Delta.$$

Show that $\mathbb{E}(L) = \Delta$ and that $L|_\Delta \sim 0$.