The ordinary double point

A beautiful mix of algebraic and symplectic geometry

Smoothness

Varieties are generically smooth.

We expect the "generic" variety to be globally smooth.
(If it has enough deformations, so it can be deformed to be "generic".)

E.g. consider hypersurfaces $\{f = 0\} \subset P$.
(P some smooth ambient space, eg $\mathbb{C}^{n+1}$ or $\mathbb{P}^{n+1}$.)

Singular points are where $f = 0 = df$.

- Locally $f = 0 = \partial_i f$, $i = 1, \ldots, n+1$
- $(n+2)$ equations in $(n+1)$ unknowns
- $\implies$ expect a (-1)-dimensional space of solutions.

I.e. no solutions generically ($\implies \{f = 0\}$ smooth) but finitely many in a 1-parameter family.

"Expect" this for more general varieties too.

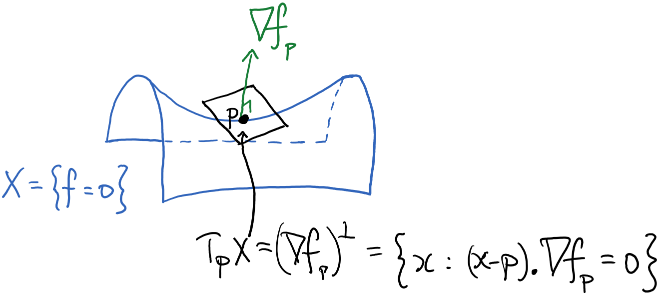
Jacobian criterion

At $p \in \{f = 0\}$ with $df|_p \neq 0$,

$$f(x) \;=\; f(p) + df|_p(x-p) + O(|x-p|^2) \;\sim\; df|_p(x-p)$$

and the implicit function theorem says that, locally analytically,

$$\{f = 0\} \;\simeq\; \{x \in \mathbb{C}^n \colon df|_p(x-p) = 0\}.$$



Therefore $\{f = 0\}$ is smooth near p.

Ordinary double points

Next least bad case: $f(p) = 0 = df|_p$ but second derivative matrix

$$\left(\frac{\partial f}{\partial x_i \partial x_j}\bigg|_p \right)_{i,j=1}^{n+1}$$

non-degenerate. Equivalently, in Taylor expansion about p,

$$f(x) \; = \; \sum_{i,j=1}^{n+1} Q_{ij} x_i x_j + O(|x|^3)$$

the quadratic form Q is non-degenerate.

Equivalently, locally analytically, $f(x) = \sum_{i=1}^{n+1} x_i^2$.

We say that any variety Y (need not be a hypersurface!) has an ordinary double point (ODP/node) at p if *locally analytically* a neighbourhood of $p \in Y$ looks like $0 \in \left\{ \sum_{i=1}^{n+1} x_i^2 = 0 \right\}$.

Examples and 1-parameter families

Ex: Show in 2-dimensions (only!) the ODP is a quotient singularity $\mathbb{C}^2/(\mathbb{Z}/2)$.

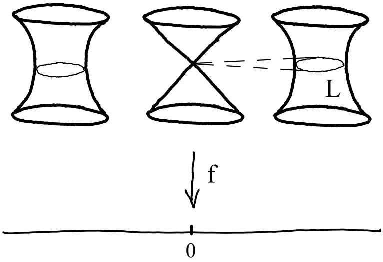
Ex: Draw $\{y^2 = x^2(1-x)\} \subset \mathbb{C}^2$ and $\{y^2 = x^2\} \subset \mathbb{C}^2$. Show both have ODPs at $(0,0)$ (so are analytically equivalent there). Show they are not Zariski locally equivalent.

Ex: Show $\{f = 0\} \subset \mathbb{C}^{n+1}$ has an ODP at $p \in \{f = 0\} \iff df$ has a simple zero at $p \in \mathbb{C}^{n+1}$.

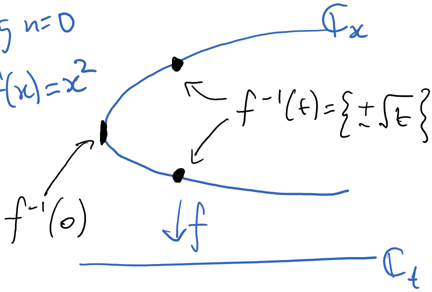
Ex: Compute the number of ODPs (simple zeros of (f, df)) of a generic 1-parameter family of hypersurfaces $\{f + tg = 0\},\ t \in \mathbb{P}^1$ (of degree d in $\mathbb{P}^{n+1}$ say).

Local picture of smoothing

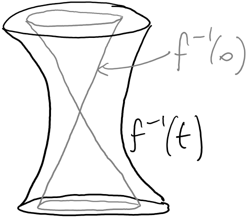
Hypersurface $X_0 \subset P$ given locally by $f = 0$.



Eg $n=0$
$f(x)=x^2$



Alternative picture:



Local model of smoothing

$f\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}, \quad f(x) = \sum_{i=1}^{n+1} x_i^2$ has fibre over t given by

$$\{f = t\} \ = \ \left\{ \sum_{i=1}^{n+1} x_i^2 = t \right\}$$

Write $x_i = a_i + ib_i$ (i.e. $\mathbb{C}_x^{n+1} = \mathbb{R}_a^{n+1} \oplus i\mathbb{R}_b^{n+1}$), suppose $t \in (0, \infty)$ (otherwise rotate real and imaginary parts by writing $x_i = \sqrt{t}(a_i + ib_i)$).

Then taking real and imaginary parts in $\sum_{i=1}^{n+1} x_i^2 \ = \ t$ gives

$$\sum_{i=1}^{n+1} (a_i^2 - b_i^2) = t \qquad \text{and} \qquad \sum_{i=1}^{n+1} a_i b_i = 0$$

$$\iff \quad |\mathbf{a}|^2 \ = \ |\mathbf{b}|^2 + t \qquad \text{and} \qquad \mathbf{a}.\mathbf{b} = 0.$$

So $\left(\frac{\mathbf{a}}{|\mathbf{a}|}, \ |\mathbf{a}|\mathbf{b}\right)$ defines a point of $TS^n \ \cong \ T^*S^n$.

T^*S^n

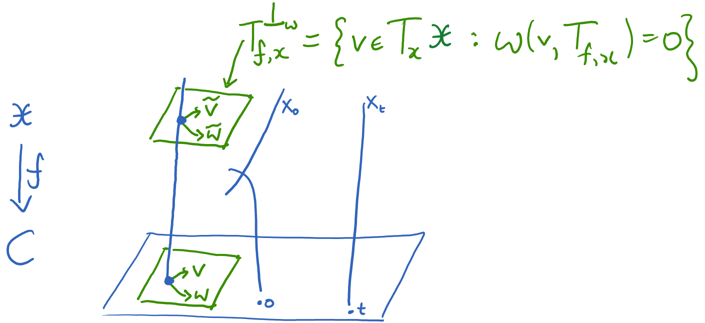
Ex: The above map $\{\sum x_i^2 = t\} \xrightarrow{\sim} T^*S^n$ is a **symplectomorphism**.

Define $L \cong S^n$ to be the zero section $\mathbf{b} = 0$,
I.e. the **real slice** $x_i \in \mathbb{R}\ \forall i$ of $\{\sum x_i^2 = t\}$
$\big(x_i \in \sqrt{t}\mathbb{R}$ in general case when $t \notin (0, \infty)\big)$

Ex: Show L is Lagrangian by using $x_i \mapsto \bar{x}_i$ by checking this takes $\omega \mapsto -\omega$.

L is called the vanishing cycle of the ODP: it is what flows to/collapses down to the origin under parallel transport of fibre (along any path to $t = 0$) of the **symplectic connection** on the fibres of $\sum x_i^2 \colon \mathbb{C}^{n+1} \to \mathbb{C}$ (away from the origin).

Symplectic connection



Ex: Preserves $\omega|_{X_t}$ – fibres X_t symplectomorphic!
Family of Kähler manifolds not locally trivial, but **is** locally trivial as a bundle of symplectic manifolds (Seidel).

Ex: L is what flows to $0 \in \mathbb{C}^{n+1}$ under this connection along a path in base to $0 \in \mathbb{C}$. Use to give another proof that L is Lagrangian.

Curvature

Symplectic connection not flat. Holonomy is a symplectomorphism of the fibre. Curvature is 2-form with values in the hamiltonian vector fields on the fibre.

Parallel transport around an infinitesimal square with sides $v, w \in T_pC$ is infinitesimal motion down the vector field v_h on fibre X_p with hamiltonian

$$h_{v,w} \; = \; \omega(\tilde{v}, \tilde{w}),$$

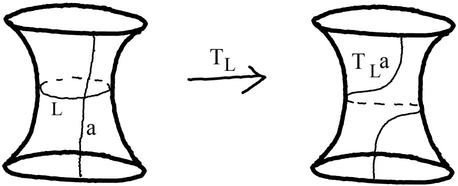
i.e. $v_h \lrcorner \omega = dh$.
So isotopic loops give hamiltonian isotopic monodromies.

Global monodromy $\pi_1(\text{base } C) \to \mathrm{Aut}(X_t, \omega) := \frac{\mathrm{Symp}(X_t,\omega)}{\mathrm{Ham}(X_t,\omega)}$.

Monodromy

Monodromy around path winding once round $0 \in C$?

For very small loop get identity $(h \sim \text{const})$ far away from ODP in $\mathcal{X}$. So monodromy transformation $f^{-1}(t) \circlearrowleft$ concentrated near vanishing cycle L. Called Dehn twist.



Action on homology Picard-Lefschetz reflection

$$\begin{aligned} T_L : \ H_*(X_t) &\longrightarrow H_*(X_t) \\ a &\longmapsto a + (a.L)[L] \end{aligned}$$

Local model

Local model on

$$T^*S^n \;=\; \left\{(\mathbf{a},\mathbf{b}) \in \mathbb{R}^n \oplus (\mathbb{R}^n)^* \colon |\mathbf{a}| = 1,\ \langle \mathbf{b},\mathbf{a}\rangle = 0\right\}$$

is time-π hamiltonian flow of $|\mathbf{b}|$.

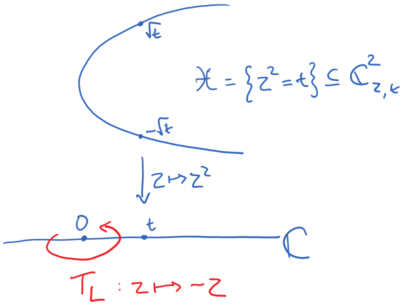
(Discontinuous over zero section $\mathbf{b} = 0$, but continuous after $t = \pi$.)

Equivalently, normalised geodesic flow on

$$TS^n = \left\{(\mathbf{a},\mathbf{b}) \in \mathbb{R}^n \oplus \mathbb{R}^n \colon |\mathbf{a}| = 1,\ \mathbf{b}.\mathbf{a} = 0\right\}$$

along horizontal lift of $\mathbf{b}/|\mathbf{b}|$.

$n = 0$ case:



Families of affine quadrics

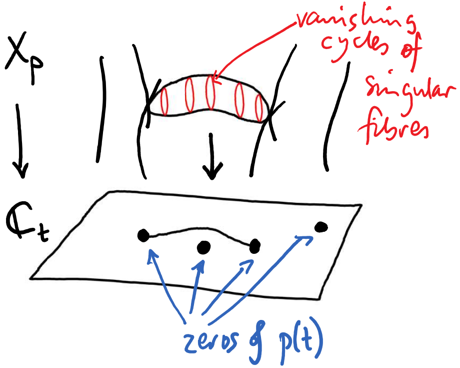
Fix degree d polynomial $p(t)$.

Get n-dimensional $X_p := \{ \sum_{i=1}^{n} x_i^2 = p(t) \} \subset \mathbb{C}^n_{\mathbf{x}} \times \mathbb{C}_t$.

Fibre over $t \in \mathbb{C}_t$ is affine quadric,

- T^*S^{n-1} if $p(t) \neq 0$,
- quadric cone (with ODP) if $p(t) = 0$.

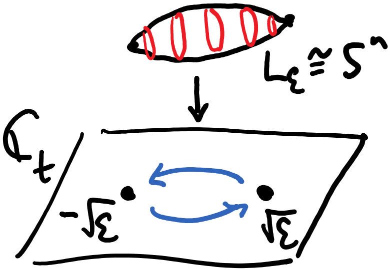
Get Lagrangian S^ns fibred over paths in $\mathbb{C}$ between zeros of p.



Example

E.g. $p(t) = \epsilon - t^2$ gives n-dimensional quadric $\{\sum x_i^2 + t^2 = \epsilon\}$ fibres by $(n-1)$-dimensional quadrics.

Above construction gives the vanishing cycle $L_\epsilon \cong S^n$ of the ODP at $\epsilon \to 0$.

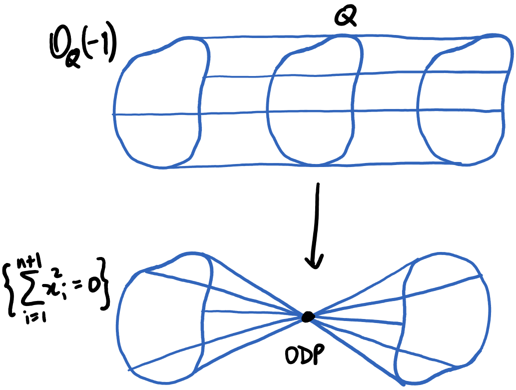


Get Dehn twist monodromy by rotating $\pm\sqrt{\epsilon}$ about each other.
Ex: General case gives representation $B_d \to \mathrm{Aut}(X_p, \omega)$.

Resolution

Ex: Blow up of $\{\sum x_i^2 = 0\} \subset \mathbb{C}^{n+1}$ is the total space of $\mathcal{O}_Q(-1)$, with exceptional divisor Q the quadric $\{\sum x_i^2 = 0\} \subset \mathbb{P}^n$.

Ex: In dimension $n = 2$ we get $\mathcal{O}_{\mathbb{P}^1}(-2) \cong T^*\mathbb{P}^1$ as the resolution.



Dimension 3

In dimension $n = 3$ we have $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Ex: Prove this by rewriting $x_1^2 + \ldots + x_4^2 = ut - vw$.
Reprove by embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}\big(\Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)\big) \cong \mathbb{P}^3$$

by the sections $u := x_1 x_2$, $t := y_1 y_2$, $v := x_1 y_2$, $w := x_2 y_1$.

Can then blow down the full blow up $\mathcal{O}_Q(-1)$ along either ruling to give another resolution with exceptional locus $\mathbb{P}^1$.
(Codimension two! "Small resolution").

More concretely can blow up $X_0 := \{ut = vw\}$ in the Weil divisor $(u = 0 = v)$ to give X^+. (Or blowing up in $(u = 0 = w)$ gives X^-.)

Small resolution of $X_0 = \{ut = vw\}$

Letting U, V denote the homogeneous coordinates on $\mathbb{P}^1$ we get

$$X^+ \ := \ \mathrm{Bl}_{(u,v)} X_0 \ = \ \big\{uV = vU, \ wV = tU\big\} \ \subset \ X_0 \times \mathbb{P}^1.$$

Ex: Show this is what the Proj $\bigoplus_{n \geq 0} (u, v)^n$ construction gives.

Ex: Use this to show X^+ is the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

That is, we plot the graph of

$$\frac{u}{v} = \frac{w}{t} \ : \ X_0 \backslash \{0\} \longrightarrow \mathbb{P}^1$$

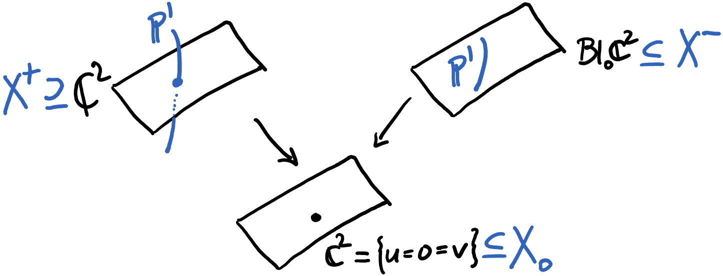
and take its closure. Away from 0 at least one of t, u, v, w is $\neq 0$ so we get a unique point $[\lambda : \mu] \in \mathbb{P}^1$, so $X^+ \to X_0$ is an isomorphism. Over 0 we get exceptional fibre $\mathbb{P}^1$.

(**Note** for a general algebraic X_0 with ODP there may be no algebraic/global Weil divisor looking like $(u = 0 = v)$ locally analytically, so X^+ may not be algebraic.)

The two small resolutions of $X_0 = \{ut = vw\}$

$X^+ = \mathrm{Bl}_{(u,v)} X_0$ and $X^- = \mathrm{Bl}_{(u,w)} X_0$ are **not** isomorphic over X_0.

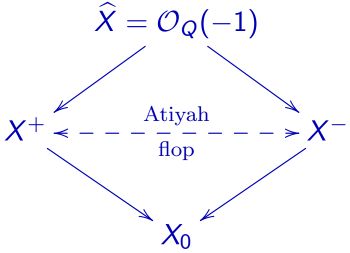
Ex: The proper transform of the plane $\{u = 0 = v\} \cong \mathbb{C}^2_{t,w}$ is again $\mathbb{C}^2$ in X^+, whereas in X^- it is $\mathrm{Bl}_0 \mathbb{C}^2$.



3 blow ups of X_0

So X^+ and X^- are only *birational* blow ups of $X_0 = \{ut = vw\}$. (Atiyah flop).

Blowing either up in their exceptional curve $\mathbb{P}^1$ gives the full blow up $\widehat{X} = \mathrm{Bl}_0 X_0$.



Link

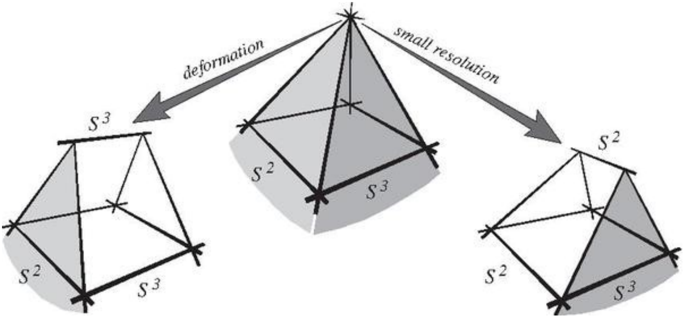
The link of the 3-fold ODP is $S^3 \times S^2$: the cone over $S^3 \times S^2$ is X_0.

The cone over S^2 (times by S^3) is the smoothing T^*S^3.

The cone over S^3 (times by S^2) is a small resolution X^+.

Using the Hopf fibration $S^3 \to S^2$ to express it as $S^3 \times S^2$ in a different way gives the other small resolution X^-.

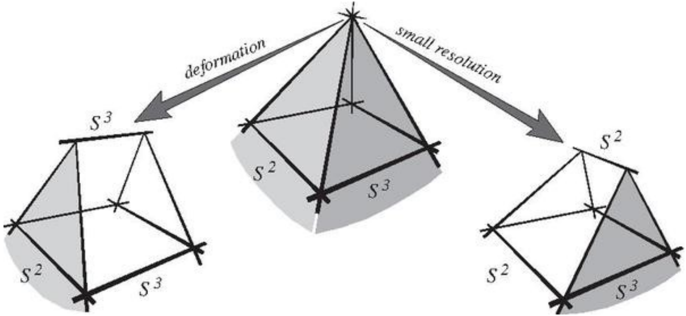
The cone over the S^1 fibre of $S^3 \times S^2 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the full resolution $\widehat{X}$.



Mirror symmetry

Mirror symmetry for Calabi-Yau 3-folds with ODPs tends to give other Calabi-Yau 3-folds with ODPs.

$$\text{smoothing} \overset{\text{MS}}{\longleftrightarrow} \text{small resolution}$$
$$\text{Lagrangian } S^3 \longleftrightarrow \text{exceptional } \mathbb{P}^1$$
$$\text{Dehn twist about } S^3 \longleftrightarrow \text{spherical twist about } \mathcal{O}_{\mathbb{P}^1}$$

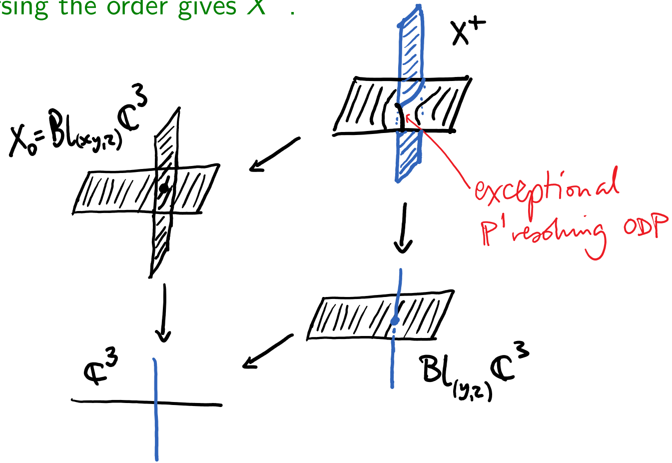


Another local model

Ex: Let X_0 be the blow up $\mathbb{C}^3$ in $\{xy = 0 = z\}$.
Show it has one ODP.

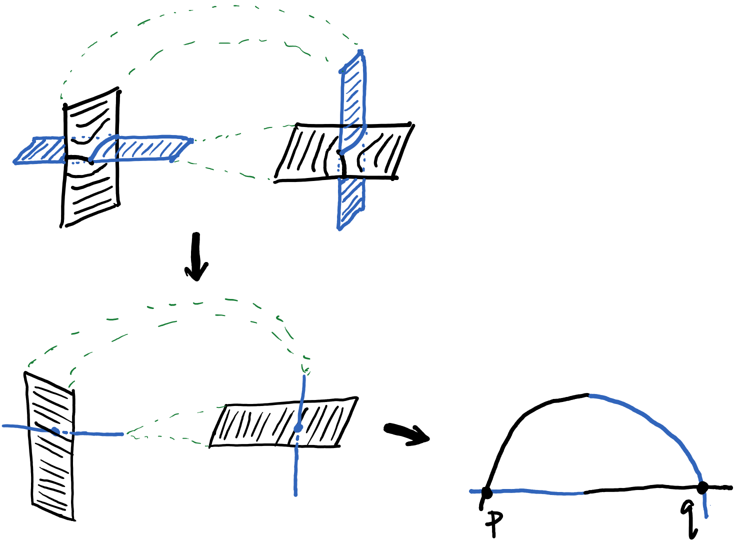
Instead blowing up one branch $\{x = 0 = z\}$ then blowing up the **proper transform** of the other branch $\{y = 0 = z\}$ gives X^+.

Reversing the order gives X^-.

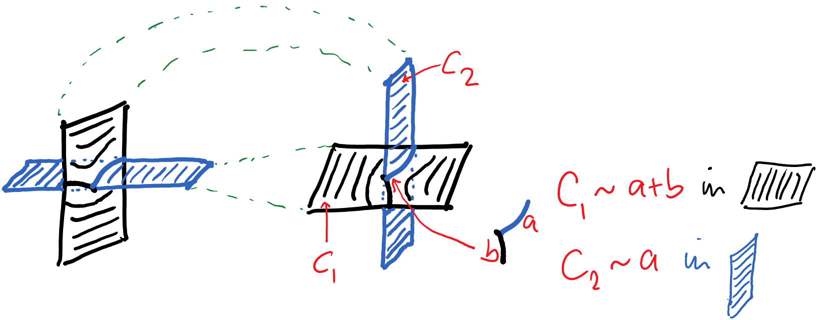


Application: Hironaka's example...

Blow up 2 curves intersecting transversally at 2 points p, q.
Do the X^+ operation at p but the X^- operation at q.



...is analytic, not projective



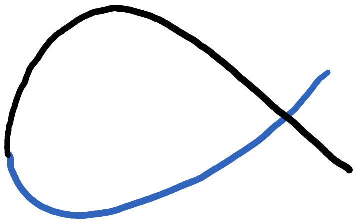
In lower ribbon $a_p \sim C_1$ (over p) and $C_1 \sim a_q + b_q$ (over q).

In upper ribbon $a_p + b_p \sim C_2$ (over p) and $C_2 \sim a_q$ (over q).

Subtracting, in the union of the two we get $C_2 - C_1 \sim b_p \sim -b_q$.
$\implies$ non-Kähler, non-projective.

Hironaka-style exercise

Ex: Do similar with the blow up of a smooth 3-fold in the following curve, treating the two branches differently.



What do you get?

Another model: matrices

The space of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of rank ≤ 1 is the 3-fold ODP:

$$\{ad - bc = 0\} \;=\; X_0.$$

Such matrices can be written $v \otimes f,\; v \in \mathbb{C}^2,\, f \in (\mathbb{C}^2)^*$.

Ex: Show this makes X_0 into the GIT quotient of $\mathbb{C}^2 \oplus (\mathbb{C}^2)^*$ by the $\mathbb{C}^*$ action with weight 1 on $\mathbb{C}^2$ (and so weight -1 on $(\mathbb{C}^2)^*$).

Ex: Change linearisation to produce X^+ by remembering $[v] \in \mathbb{P}^1$.
(Here X^+ will be $\mathcal{O}(-1) \otimes (\mathbb{C}^2)^* \to \mathbb{P}(\mathbb{C}^2)$.)

Or X^- by remembering $[f] \in (\mathbb{P}^1)^*$.
(Here X^- will be $\mathcal{O}(-1) \otimes \mathbb{C}^2 \to \mathbb{P}(\mathbb{C}^2)^*$.)

To get $\widehat{X}$ by remembering $([v],[f]) \in \mathbb{P}^1 \times (\mathbb{P}^1)^*$ we have to quotient $\mathbb{C}^5$ by two copies of $\mathbb{C}^*$ acting with weights $(1,1,0,0,-1)$ and $(0,0,1,1,-1)$.

Global version

Given a map of rank 2 vector bundles $\phi\colon E \to F$ on a 4-fold Y we get a divisor $X_0 \subset Y$ where $\det\phi \in \Gamma(\Lambda^2 E^* \otimes \Lambda^2 F)$ vanishes.

Generically smooth, ODPs where $\phi = 0$.

(For appropriately generic ϕ. Graph of $\phi\colon Y \to \mathrm{Hom}(E, F)$ should be transverse to the rank 0 and 1 loci in this bundle.)

Ex: Show how to define "$\mathbb{P}(\ker\phi) \to Y$" as zeros inside $\mathbb{P}(E) \xrightarrow{\pi} X_0$ of composition

$$\mathcal{O}(-1) \hookrightarrow \pi^* E \xrightarrow{\pi} \pi^* F.$$

Show fibre of $\mathbb{P}(\ker\phi) \to Y$ is empty over $Y \backslash X_0$, a point over $X_0 \backslash \{\mathrm{ODPs}\}$, and $\mathbb{P}^1$ fibre over ODPs. Identify it locally with X^+.

Ex: Replace $\ker\phi$ with $\ker\phi^*$ to get $\mathbb{P}(\operatorname{coker}\phi)^*$ as X^-.

Exercises

Ex: Show double cover $X \to Y$ of a smooth Y, branched over a divisor $D \subset Y$ is smooth if D is smooth, and has ODPs at any ODPs of D.
How do the resolutions match up?

Ex: If $X \to \mathbb{P}^1$ and $Y \to \mathbb{P}^1$ are Lefschetz pencils
(generically smooth maps, but finite number of fibres have ODPs where local model of map is $(x_i)_{i=1}^{\dim X} \mapsto \sum_{i=1}^{\dim X} x_i^2$)
show $X \times_{\mathbb{P}^1} Y \to \mathbb{P}^1$ is a Lefschetz pencil if and only if the two discriminant loci in $\mathbb{P}^1$ are disjoint.

Now move two points of the discriminant locus in $\mathbb{P}^1$ together.
Show the fibre product acquires an ODP.
What is the vanishing cycle?

Simultaneous resolution

Consider $\mathbb{C}^3$ to be a family of affine quadrics over $\mathbb{C}_t$ by $(x, y, z) \mapsto t := x^2 + y^2 + z^2$. Central fibre $t = 0$ is surface ODP.

Ex: Why is there no simultaneous resolution of this family?
(I.e. $Y \to \mathbb{C}^3$ which on each fibre $Y_t \mapsto \{x^2 + y^2 + z^2 = t\}$ is an isomorphism if $t \neq 0$ and the resolution if $t = 0$.)

Ex: Now pull back the family to the $t \mapsto t^2$ double cover of $\mathbb{C}_t$.
("Basechange by $t \mapsto t^2$".)
Show there **is** a simultaneous resolution now.
What does this tell you about the monodromy?

Ex: If you're stuck, first replace $\mathbb{C}^3 \xrightarrow{x^2+y^2+z^2} \mathbb{C}$ by $\mathbb{C} \xrightarrow{x^2} \mathbb{C}$ and do the exercise now.