Morse theory

Topics in geometry
22 Oct 2021

Idea: M smooth, closed manifold.

We want to recover the topology of M from a single function

$f: M \to \mathbb{R}$: (1) $\mathrm{Crit}(f) = \{x \in M \mid df_x = 0\}$

(2) Dynamics of ∇f.

Def. $p \in M$ is a critical point of f if $df_p = 0$, equiv. if $\nabla f(p) = 0$.

At $p \in \mathrm{Crit}(f)$, we define the Hessian ($df_p(v) = \langle \nabla f(p), v\rangle$.)

$$d^2 f(X, Y) = \langle \nabla(\tilde{Y} f)(p), X\rangle = X(\tilde{Y} f).$$

$\in T_pM$

directional derivative in the direction Y, extended arbitrarily to $\tilde{Y}$.

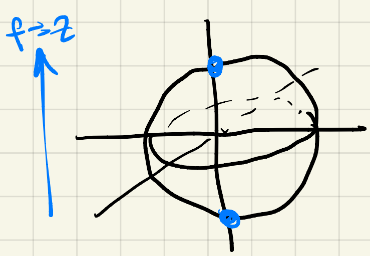
1 (*) Exercise: check that $d^2f(X,Y) = d^2f(Y,X)$. (Note $p \in \mathrm{Crit}(M)$)

local coordinates on $\mathbb{R}^n$ with $p=0$: $d^2 f(\vec{e}_i, \vec{e}_j) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right) = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

Def $d^2 f$ is nondegenerate at $p \in \mathrm{Crit}(f)$ if $\ker(d^2 f) = 0$.

i.e. if $d^2 f(v, x) = 0$ for all $x \in T_pM$ implies $v = 0$.

Ex. $S^2 \subset \mathbb{R}^3$, $f(x,y,z) = z$



At a point $(x,y,z) \in S^2$: $x^2 + y^2 + f^2 = 1$

$\xrightarrow{\partial/\partial x}$ $2x + 2f\frac{\partial f}{\partial x} = 0$, so $\frac{\partial f}{\partial x} = \frac{-x}{f}$.

Similarly $\frac{\partial f}{\partial y} = \frac{-y}{f}$.

Then $\mathrm{Crit}(f) = \{p \in S^2 \mid \nabla f(p) = \langle \frac{-x}{f}, \frac{-y}{f}\rangle = 0\}$

$= \{p \in S^2 \mid (x,y) = (0,0)\}$

$= (0, 0, \pm 1)$.

Hessian: $\frac{\partial^2 f}{\partial x^2} = \frac{-f + x\frac{\partial f}{\partial x}}{f^2} = \frac{-z + x(-\frac{x}{z})}{z^2} = \frac{-x^2 - z^2}{z^3}$

Also $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{-xy}{z^3}$, $\frac{\partial^2 f}{\partial y^2} = \frac{-y^2 - z^2}{z^3}$.

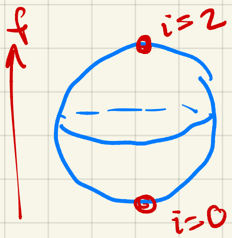
At $(x,y,z) = (0,0,\pm 1)$, $(d^2f)_p = \begin{pmatrix} \mp 1 & 0 \\ 0 & \mp 1 \end{pmatrix} = \mp \mathrm{Id}$.

Both critical points are nondegenerate.

Def. $f: M \to \mathbb{R}$ is Morse if all critical points are nondegenerate.

Morse lemma Let $p \in M$ be a nondegenerate critical point of f.

Then $\exists$ a neighborhood $\mathcal{U}$ of p and a chart $\varphi: \mathcal{U} \xrightarrow{\sim} \overset{\text{open}}{V} \subset \mathbb{R}^n$



such that $\varphi(p) = 0$ and

$$f \circ \varphi^{-1}(x_1, \dots, x_n) = f(p) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

We call i the index of f at p.

Rough sketch: use Taylor's theorem to write

$$f(x_1, \vec{y}) = f_{\vec{y}}(0) + \frac{\partial f_y}{\partial x_1}(0)\, x_1 + \frac{1}{2}\frac{\partial^2 f_y}{\partial x_1^2}(0)\, x_1^2 + o(x_1^2).$$

Reparametrize to eliminate linear term (uses implicit function thm + nondegeneracy),

replace x_1 with $x_1\sqrt{\text{higher order part}}$;

repeat for $x_2, \dots, x_n$. $\square$

2 (*) Exercise: if $f: M \to \mathbb{R}$ is Morse, then all critical points are isolated.

Thm Morse functions are open and dense in $C^\infty(M, \mathbb{R})$.

Existence: embed $M \hookrightarrow \mathbb{R}^N$, take $f_p(x) = \|x - p\|^2$ on M. ($p \in \mathbb{R}^N$)

This is generically Morse, by Sard's theorem.

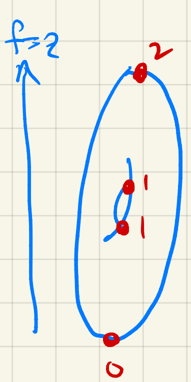
Ex. $T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$

$f(x,y) = \cos(x) + \sin(y)$.

$\Rightarrow \nabla f(x,y) = \langle -\sin(x), \cos(y) \rangle$.

$d^2 f(x,y) = \begin{pmatrix} -\cos(x) & 0 \\ 0 & -\sin(y) \end{pmatrix}$

$p \in \mathrm{Crit}(f)$	$(d^2f)_p$	Index	
$(0, \frac{\pi}{2})$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2	local max
$(0, \frac{3\pi}{2})$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	1	saddle
$(\pi, \frac{\pi}{2})$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	1	saddle
$(\pi, \frac{3\pi}{2})$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0	local min.



Index of each critical point

= # negative eigenvalues of d^2f.

3 (*) Exercise: $f : \mathbb{CP}^n \to \mathbb{R}$, $f([z_0 : \cdots : z_n]) = \dfrac{c_0|z_1|^2 + \cdots + c_n|z_n|^2}{|z_1|^2 + \cdots + |z_n|^2}$

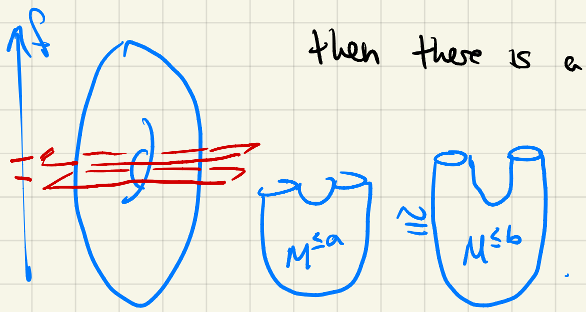
is Morse if the $c_0, \ldots, c_n$ are all distinct.

What are the critical points of f and their indices?

Prop Write $M^{\le a} = f^{-1}([-\infty, a]) = \{p \in M \mid f(p) \le a\}$.

If f has no critical values in $[a,b]$ (i.e. if $p \in \mathrm{Crit}(f)$ then $f(p) \notin [a,b]$),

then there is a diffeomorphism $M^{\le a} \cong M^{\le b}$.



Proof Define $\rho : M \to \mathbb{R}$ as $\begin{cases} -1/\|\nabla f\|^2 & \text{on } f^{-1}([a,b]) \\ 0 & \text{outside a compact neighborhood of } f^{-1}([a,b]). \end{cases}$

Define $\psi_s : M \to M$ as the flow of $\rho \cdot \nabla f$.

$\dfrac{d\psi_s}{ds} = (\rho \cdot \nabla f) \circ \psi_s$.

If $f(\psi_s(x)) \in [a,b]$ then

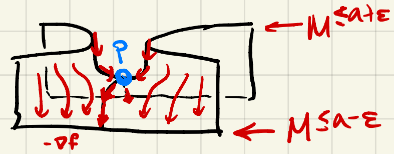
$$\frac{d}{ds}(f \circ \psi_s)(x) = df_{\psi_s(x)}\left(\frac{d\psi_s}{ds}\right)$$
$$= \rho(\psi_s(x)) \cdot df_{\psi_s(x)}(\nabla f(\psi_s(x)))$$
$$= \frac{-1}{\|\nabla f(\psi_s(x))\|^2} \cdot \|\nabla f(\psi_s(x))\|^2 = -1.$$

Conclusion: $\psi_{b-a}(M_b) = M_a$, so ψ_{b-a} is the desired diffeomorphism. ∎

Summary so far: topology of M is controlled entirely by $\mathrm{Crit}(f)$, ∇f near $\mathrm{Crit}(f)$.

Take $p \in \mathrm{Crit}(f)$. Morse lemma: locally $f(x_1, \ldots, x_n) = f(p) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2$.

Suppose we have the standard metric on $\mathbb{R}^n$.



Downward gradient flow equation $\dfrac{d\psi_s(x)}{ds} = -\nabla f(\psi_s(x))$

has an explicit solution $\psi_s(y_1, \ldots, y_n) = (\underbrace{y_1 e^{2s}, \ldots, y_i e^{2s}}_{i = \mathrm{ind}_p(f)}, y_{i+1} e^{-2s}, \ldots, y_n e^{-2s})$.

The topology of M changes from $M^{a-\varepsilon}$ to $M^{a+\varepsilon}$

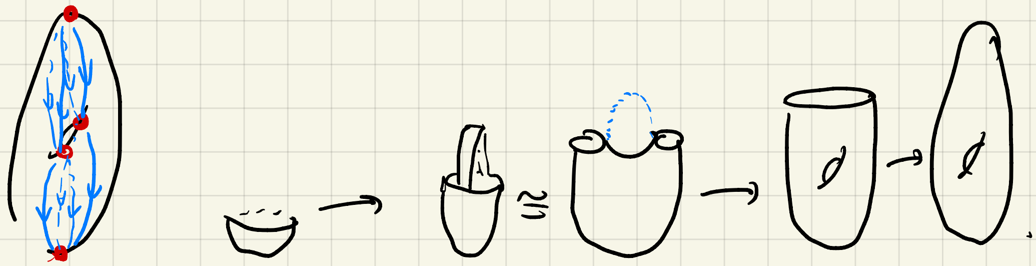
by adding a "handle" near p,

along the "unstable" manifold $U(p) = \{x \in M \mid \lim_{s \to -\infty} \psi_s(x) = p\}$.

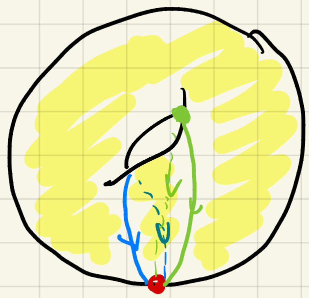
We get a cell structure on M:

- 1 cell per critical point, of dimension $\dim(\mathcal{U}(p)) = \mathrm{ind}_p(f)$.
- Attaching maps are determined by flow of $-\nabla f$.

<u>Ex.</u> T^2

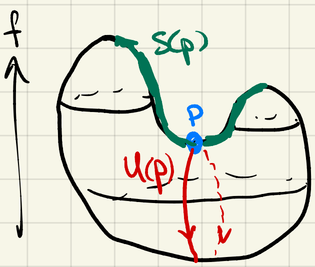


Cell structure:



<u>Morse homology</u>

$f: M \to \mathbb{R}$ Morse. For $p \in \mathrm{Crit}(f)$, define $\varphi_s = \mathrm{Flow}(-\nabla f)$ and



$$\mathcal{U}(p) = \{x \in M \mid \lim_{s \to -\infty} \varphi_s(x) = p\} \quad \underline{\text{unstable manifold}}$$

$$S(p) = \{x \in M \mid \lim_{s \to \infty} \varphi_s(x) = p\} \quad \underline{\text{stable manifold}}.$$

Then $\mathcal{U}(p)$ and $S(p)$ are manifolds of dimension

$\mathrm{ind}(p)$ $\quad n - \mathrm{ind}(p)$.

Given $p \neq q$ in $\mathrm{Crit}(f)$, let $\mathcal{M}(p,q) = \mathcal{U}(p) \cap S(q)$:

space of flows from p to q. If $\mathcal{U}(p)$ is <u>transverse</u> to $S(q)$:

$$\dim \mathcal{M}(p,q) = (\dim \mathcal{U}(p)) - (n - \dim S(q))$$
$$= \mathrm{ind}(p) - \mathrm{ind}(q).$$

<u>Def.</u> $f: M \to \mathbb{R}$ is <u>Morse–Smale</u> if $\mathcal{U}(p) \pitchfork S(q)$ for all $p \neq q$. (Generic)

Then each $\mathcal{M}(p,q)$ is a manifold of dimension $\mathrm{ind}(p) - \mathrm{ind}(q)$.

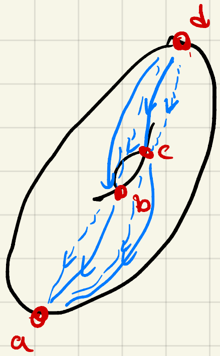
Free action $\mathbb{R} \circlearrowright \mathcal{M}(p,q)$: $\quad s \cdot (\underbrace{\varphi(t)}_{\text{flow line of } -\nabla f}) = \varphi(t+s)$.

Morse complex on M:

$$CM_k(f) := \bigoplus_{\mathrm{ind}(p)=k} \mathbb{Z}/2\mathbb{Z}\langle p\rangle.$$

$$\partial p = \sum_{\mathrm{ind}(q)=\mathrm{ind}(p)-1} \#\left(\mathcal{M}(p,q)/\mathbb{R}\right) q.$$

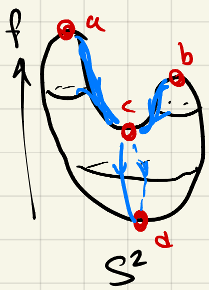
Ex.



$$\begin{tikzcd} CM_2(f) \cong & & d \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dr, shift left] \arrow[dr, shift right, "\partial"] & & \Rightarrow HM_2(f) \cong \mathbb{Z}/2\mathbb{Z} \\ CM_1(f) \cong & b \arrow[dr, shift left] \arrow[dr, shift right] & & c \arrow[dl, shift left] \arrow[dl, shift right, "\partial"] & HM_1(f) \cong (\mathbb{Z}/2\mathbb{Z})^2 \\ CM_0(f) \cong & & a & & HM_0(f) \cong (\mathbb{Z}/2\mathbb{Z}). \end{tikzcd}$$

Observe that $HM_*(f) \cong H_*(\mathbb{T}^2)$.

Ex.



$$\begin{tikzcd} a \arrow[dr] & & b \arrow[dl] \\ & c \arrow[d, shift left] \arrow[d, shift right] & \\ & d & \end{tikzcd}$$

$CM_*(f)$

$\ker(\partial : CM_2 \to CM_1) = \langle a+b\rangle$

$$\frac{\ker(\partial : CM_1 \to CM_0)}{\mathrm{im}(\partial : CM_2 \to CM_1)} = \frac{\langle c\rangle}{\langle c\rangle} = 0$$

$$\frac{\ker(\partial : CM_0 \to CM_{-1})}{\mathrm{im}(\partial : CM_1 \to CM_0)} = \frac{\langle d\rangle}{0} = \langle d\rangle.$$

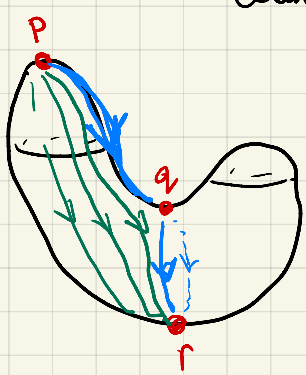
Again, $HM_*(f) \cong H_*(S^2)$.

Thm $HM_*(\underbrace{f : M \to \mathbb{R}}_{\text{Morse-Smale}}) \cong H_*^{cw}(M)$.

Q: Why is $\partial^2 = 0$ in the Morse complex?

A: $$\partial^2 p = \partial\left[\sum_{\mathrm{ind}(q)=\mathrm{ind}(p)-1} \#\hat{\mathcal{M}}(p,q)\cdot q\right] = \sum_{\mathrm{ind}(r)=\mathrm{ind}(p)-2}\left(\sum_q \#\hat{\mathcal{M}}(p,q)\,\#\hat{\mathcal{M}}(q,r)\right) r.$$

Want to show for all p, r that $\sum_q \#\hat{\mathcal{M}}(p,q)\,\#\hat{\mathcal{M}}(q,r) \equiv 0 \pmod 2$.



This is a count of "broken trajectories".

The 1-dimensional space $\hat{\mathcal{M}}(p,r)$ has a compactification

$$\overline{\hat{\mathcal{M}}(p,r)} = \hat{\mathcal{M}}(p,r) \cup \left(\bigcup_q \hat{\mathcal{M}}(p,q)\times\hat{\mathcal{M}}(q,r)\right)$$

as a 1-manifold with boundary. Then

$$\sum_q {}' \#\hat{\mathcal{M}}(p,q)\,\#\hat{\mathcal{M}}(q,r) = \#\partial\left(\overline{\hat{\mathcal{M}}(p,r)}\right) = 0$$

because $\#\partial(\text{compact 1-manifold}) = 0$.

So $\partial^2 = 0$.

4(*) Exercise Let $f: M \to \mathbb{R}$ be Morse–Smale.

Show that $CM_*(f) \cong (CM_{n-*}(-f))^\vee$. ("Poincaré duality")

Instanton homology on 3-manifolds:

$$\left\{\begin{array}{c}\text{3-manifold } Y \\ H_*(Y) \cong H_*(S^3)\end{array}\right\} \rightsquigarrow \left\{\begin{array}{c}\text{infinite-dimensional} \\ \text{space + functional}\end{array}\right\} \xrightarrow{\text{Morse homology}} I_*(Y).$$

Q: how to make sense of this?

Q: why should this give an invariant?

Setup: principal $SU(2)$-bundle $SU(2) \to SU(2) \times Y = P$
$$\downarrow$$
$$Y$$

Define the Chern–Simons functional for a connection A on P by

$$cs(A) = \frac{1}{8\pi^2}\int_{[0,1]\times Y} \mathrm{tr}(F_{\tilde{A}} \wedge F_{\tilde{A}}) \quad \text{where } \tilde{A} = tA \text{ on } [0,1]_t \times Y.$$

(On a closed 4-manifold, this would compute $c_2(P)$.)

Compute $cs(A+sa) = cs(A) + \frac{s}{4\pi^2}\int_Y \mathrm{tr}(F_A \wedge a) + O(s^2)$

$$\Rightarrow \langle \nabla cs(A), a \rangle = \frac{1}{4\pi^2}\int_Y \mathrm{tr}(*(*F_A) \wedge a) = \langle *F_A, a \rangle$$

so $\nabla cs(A) = *F_A$.

In other words, $\mathrm{Crit}(cs) = \{A \mid F_A = 0\}$ = flat connections on P.

Gradient flow equation: $\frac{\partial A_t}{\partial t} = -\nabla cs(A_t)$ (1-parameter family of connections)

$\Longleftrightarrow$ connection $\tilde{A}$ on $\mathbb{R} \times Y$ satisfying $*F_{\tilde{A}} = -F_{\tilde{A}}$.

This is the anti-self-dual equation on $\mathbb{R} \times Y$!

Well-understood: Uhlenbeck, Donaldson, Taubes, –.

$\Rightarrow$ Instanton Floer homology $I_*(Y)$ as

Morse homology of $CS : \dfrac{\{\text{connections}\}}{\text{Aut}(P)} \to \mathbb{R}/\mathbb{Z}$.

"gauge"

Thm (Floer '88) If ∂ counts ASD connections on $\mathbb{R} \times Y$

then $\partial^2 = 0$ and $I_*(Y)$ depends only on Y.

Remark generators of the I_* complex are

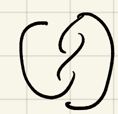
$$\dfrac{\left\{ \text{irreducible flat connections on } \begin{matrix} SU(2)\times Y \\ \downarrow \\ Y \end{matrix} \right\}}{\text{gauge}} \xrightarrow{\text{holonomy}} \dfrac{\left\{ \begin{matrix} \text{nontrivial representations} \\ \pi_1(Y) \to SU(2) \end{matrix} \right\}}{\text{conjugation}}.$$

So $I_*(Y) \neq 0 \Rightarrow \exists$ nontrivial rep $\pi_1(Y) \to SU(2)$

$\Rightarrow \pi_1(Y) \neq 1$. (i.e. $Y \not\cong S^3$).

Thm (Property P conjecture, Kronheimer–Mrowka '03)

Take a knot $K \subset S^3$, remove a tubular nbhd, glue



back in by some nontrivial automorphism of $\partial N(K)$. ("Dehn surgery")

Then if K is knotted, this never produces a homotopy S^3.

Pf idea show that $I_*(\text{surgery}) \neq 0 \Rightarrow \pi_1(\text{surgery}) \neq 1$.