GIT Examples
AC, 29th January 2026

(1) Consider the obvious action of $G = \mathrm{SL}_2$ on the variety $X = (\mathbb{P}^1)^n$ of n **ordered** points $x_1, x_2, \ldots, x_n \in \mathbb{P}^1$. Let $U^\star \subset X$ be the open subset where at least 3 of the points are distinct. For $1 \leq j < j < k \leq n$, denote by $U_{ijk} \subset U^\star$ the open subset where the three points x_i, x_j, x_k are distinct.

For $n = 4$:

(a) Draw a picture of the "thing" $Y = U^\star/G$ obtained by gluing the U_{ijk}/G in the obvious way. (Note: Y is proper but not separated. In my day we used to call a thing like this a "pre-variety.")

(b) Describe explicitly the action of the symmetric group $\mathfrak{S}_4$ on Y.

(c) Show that there is no Zariski open subset $U \subset U^\star$ that is invariant under both G and $\mathfrak{S}_4$, and such that the geometric quotient U/G is proper and separated.

(2) (Harder.) Notation as in the previous question. For $n = 5$, draw a picture of the geometric quotient $Y = U^\star/G$. Show that there **is** a Zariski open subset $U \subset U^\star$ that is invariant under both G and $\mathfrak{S}_5$, and such that the geometric quotient U/G is proper and separated.

In terms of GIT, what is the key difference with the $n = 4$ case that explains this difference in behaviour?*

*The answer is that if n is odd, there are no strictly semistable points.

(3) (a) Let a group G with multiplication $m\colon G \times G \to G$ act on a variety X by $\Psi\colon G \times X \to X$. Consider the three morphisms:

$$p_{23}\colon G \times G \times X \to G \times X;$$
$$(m, p_3)\colon G \times G \times X \to G \times X;$$
$$(p_1, \Psi)\colon G \times G \times X \to G \times X$$

Let's say that a G-linearised coherent sheaf on X is the same as a coherent sheaf $\mathcal{F}$ on X, together with an isomorphism:

$$\varphi\colon p_2^\star \mathcal{F} \to \Psi^\star \mathcal{F}$$

of coherent sheaves on $G \times X$ such that

$$(m, p_3)^\star(\varphi) = p_{23}^\star(\varphi) \circ (p_1, \Psi)^\star(\varphi)$$

Show that a G-linearised line bundle in this sense is the same as a G-linearised line bundle as I defined it in the lecture.

(b) Suppose that a (finite, say) group G acts (from the left) on an affine variety $X = \operatorname{Spec} A$. This induces a left action of G on A as follows. For all $g \in G$, denote by $l_g\colon X \to X$ the (left) action of g. The left action of G on A is defined as follows:

$$g(a) = l^\sharp_{g^{-1}}(a)$$

Show that a G-linearised coherent sheaf on X is the same as a finitely generated A-module M, and a k-linear representation of G on M such that for all $a \in A$, $m \in M$, $g \in G$:

$$g(am) = g(a)g(m)$$

Equivalently, let $A \star G$ be the *twisted group ring*: elements are formal sums $\sum_{g \in G} a_g g$ and the multiplication is induced by the rule:

$$(a_1 g_1)(a_2 g_2) = a_1 g_1(a_2) g_1 g_2$$

Show that a G-linearised coherent sheaf on X is the same as a finitely generated $A \star G$-module.

(4) (Rank 2 toric varieties)[†] In this question we fix a $2 \times n$ integer matrix:

$$X = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

which we interpret as a group homomorphism $\mathbb{T} = \mathbb{G}_m^2 \to \mathbb{G}_m^n$, inducing an action of $\mathbb{T}$ on $\mathbb{A}^n = \operatorname{Spec} K[x_1, \dots, x_n]$.

Below we assume that the vectors:

$$\chi_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \dots, \chi_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \in \mathbb{Z}^2$$

are in counter-clockwise order and that collectively they span a strict[‡] cone $E = \langle \chi_1, \chi_n \rangle_+ \subset \mathbb{R}^2$ (two or more of these vectors may be proportional or coincide). If $\chi, \chi' \in E$ we say $\chi \leq \chi'$ if the ray $\langle \chi \rangle_+$ points to a *later* time than the ray $\langle \chi' \rangle_+$.[§]

A character

$$\chi = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}^2 = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_\mathrm{m})$$

gives a $\mathbb{T}$-linearised line bundle $\mathcal{O}(\chi)$ on $\mathbb{A}^n$ such that the group of $\mathbb{T}$-invariant sections $H^0(\mathbb{A}^n, \mathcal{O}(\chi))^{\mathbb{T}}$ consists of bihomogeneous polynomials $f \in K[x_1, \dots, x_n]$ of bi-degree u, v. For example, tautologically $x_i \in \mathcal{O}(\chi_i)$.

(a) For all $\chi \in E$ prove that the set of unstable points in the corresponding linearisation is the variety of the ideal

$$I = (x_i \mid \chi_i \leq \chi)\,(x_j \mid \chi \leq \chi_j)$$

(b) The GIT quotient $\mathbb{A}^n /\!/ \mathbb{T}$ is covered by $\mathbb{T}$-invariant affine charts $U_{ij}/\mathbb{T}$ ($\chi_i < \chi < \chi_j$) and $U_k/\mathbb{T}$ ($\chi_k = \chi$) where

$$U_{ij} = \{a = (a_1, \dots, a_m) \in \mathbb{A}^n \mid a_i \neq 0 \text{ and } a_j \neq 0\}$$

and, similarly, $U_k = \{a_k \neq 0\}$.

[†]There is a similar treatment for toric varieties of higher rank but it would take me too far to discuss it.

[‡]A cone is *strict* if it contains no vector subspace (other than $\{0\}$). The assumption here is equivalent to the GIT quotients being projective. It is possible to work with cones that are not strict but it is confusing, there are additional layers of detail.

[§]Two useful additional assumptions that I don't discuss are: (1) the row span of X is saturated; for all i, denoting by $X_{\hat{i}}$ the matrix obtained by removing the i^{th} column, the row span of $X_{\hat{i}}$ is saturated.

(c) If for some i $\chi_i < \chi$ and $\chi < \chi_{i+1}$ then all semistable points are stable (so there are no strictly semistable points), and the quotient is covered by affine open subsets each isomorphic to $\mathbb{A}^{n-2}/G$ where G is a finite group.

(d) If for some k $\chi_k = \chi$ then the open subset U_k always contains some strictly semistable points.

(5) ($n \times n$ matrices.) Consider the action of $G = SL_n$ on $n \times n$ matrices T by conjugation (change of basis $g, T \mapsto g^{-1}Tg$), and consider a GIT setup with the trivial line bundle endowed with the trivial G-linearization. T is unstable if and only if it is nilpotent; it is stable if and only if it is semisimple with distinct eigenvalues.

(6) (Examples of the Hilbert–Mumford criterion) Use the Hilbert–Mumford criterion as stated in the lecture to prove the following statements about the action of SL_{n+1} on the space of hypersurfaces of degree d in $\mathbb{P}^n$.

(a) A plane cubic is unstable if and only if it has a triple point or it is the union of a conic and a line tangent to it, or it has a cusp; it is stable if and only if it is nonsingular.

(b) A plane quartic is unstable if and only if it has a triple point or it is the union of a cubic and an inflectional tangent line; it is stable if and only if it has only ordinary double points or ordinary cusps. (The remaining curves with a tacnode – locally analytically $x^2 = y^4 + \text{hot}$ – are strictly semistable.)

(c) A cubic surface is unstable if and only if it is not normal, or it has a triple point, or a double point strictly worse than A_2; it is stable if it has only A_1-singularities.

(7) Consider[¶] the action of $G = SL_2$ on $\mathbb{P}^n$, thought of as the projective space of degree n homogeneous polynomials in two variables X_0, X_1:

$$f(X_0, X_1) = a_0 X_0^n + a_1 X_0^{n-1} X_1 + \cdots + a_n X_1^n$$

[¶] For this question I advise that you use a computational algebra system.

The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ comes with a natural G-linearisation, and hence G acts naturally on the graded ring $K[a_0, \dots, a_n]$.

For $n = 4$:

(a) The homogeneous polynomials:

$$P = -\frac{1}{3}a_2^2 + a_1a_3 - 4a_0a_4$$

and

$$Q = -\frac{8}{3}a_0a_2a_4 + a_0a_3^2 + a_1^2a_4 - \frac{1}{3}a_1a_2a_3 + \frac{2}{27}a_2^3$$

are G-invariant.

(b) Denote by D the discriminant of the polynomial $a_0X_0^4 + a_1X_0^3X_1 + \dots + a_4X_1^4$ (so, for example, $D = 0$ if and only if the polynomial has a double root). Then $D = \pm(4P^3 + 27Q^2)$.

(c) (Harder. You should probably look this up somewhere.) $K[a_0, \dots, a_4]^G = K[P, Q]$.

(d) A polynomial is semistable if and only if it has at most double roots. Denote by U_{ss} the (Zariski open) subset of semistable polynomials, and consider the quotient morphism

$$\pi \colon U_{ss} \to \mathbb{P}^1 = \operatorname{Proj} K[P, Q]$$

If $D(x) \neq 0$ then $\pi^{-1}(x)$ consists of one orbit of stable points, closed in U_{ss}. If $D(x) = 0$, then $\pi^{-1}(x)$ consists of two orbits; one 3-dimensional consisting of polynomials with one double root, and one two-dimensional consisting of polynomials with two double roots.

References

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