LECTURE ON GIT TO THE LSGNT, 30 JAN 2026
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Work with varieties over a closed field k (there are additional issues in char =p)

PROBLEM: construct the quotient X/G

NB There is a close relation between this problem and the problem of constructing moduli spaces (classification) but I shall not discuss moduli problems *much* here (it takes me on a different path)

NB In the Cinfinity (or, God forbid, C0) category this problem is a total mess - Cinfinity actions can be very chaotic. It is a remarkable fact that there are answers at all in algebraic geometry

This is the work of David Mumford, maybe the work that he was awarded the Fields Medal for. The first edition of "Geometric Invariant Theory" was published in 1965 (the year that I was born). In particular this is pre-Hartshorne

He did a good job with it, I am not sure anybody tried to rewrite it nicely. (There is the book by MUKAI and some, but not many, introductury lecture notes, for example an excellent one by our very own Richard Thomas.) Reading this thing now, David had some work ethic. He has to speak the language of EGA, I would say that the book (as well as other works by Mumford) is still a great place to learn to speak EGA

I try to keep it simple but I also want to give an overview of the theory. I need to cut corners and some detail will be lost. This theory is subtle and I WILL (inevitably but inadvertently) make mistakes (hopefully smally ones).

REFERENCES

In preparing this lectures I used

Mumford-Suominen "Introduction to the theory of moduli" (Oslo 1970)
Mumford "stability of projective varieties"
Mumford-Fogarty-Kirwan "geometric invariant theory"

and my own research experience with this theory

(I) THE AFFINE CASE

THEOREM Let G be a (linearly) reductive group acting on an affine variety X with closed orbits. Then the geometric quotient (Y,f) exists and Y = Spec k[X]^G is affine

COMMENTS

(1) G REDUCTIVE: the important thing is that G is _linearly_ reductive. There is a Lie-theoretic definition but I am not really qualified to get into it. Examples: Gm^n, GLn, SLn, On. The main problem is unipotent groups, for example Ga. A typical case of interest is Aut (weighted PP). The key consequence is that the ring of invariants in finitely generated. It is not impossible to work with nonreductive G, you just have to be more careful (Kirwan)

(2) General DEFINITION (GEOMETRIC QUOTIENT): f: X->Y and
(i) If y in Y, then f^{-1}(y) is an orbit;

(ii) V subset Y Zariski open if and only if f^{-1}(V)=U is Zariski open
(iii) If V subset Y is open, H^0(V, O) = H^0(f^{-1}V, O)^G

There is a lot going on inside this definition, but it captures a very desirable kind of quotient. One can try to make a more categorical definition but you can not then avoid Grothendieck topologies (and you will have a _different_ concept). For example:

DEF (Y,f) is a UNIFORM CATEGORICAL QUOTIENT if for all flat morphisms Y' -> Y, (Y',f') is a categorical quotient of X' (i.e. it has the expected universal property)

(3) The assumption on closed orbits is essential: just think of Gm acting on CC^n. More to the point, this is sort of a rubbish theorem, in the sense that the assumptions are rarely satisfied. For example think of CC2 with a Gm action with weights 1,-1. The ring of invariants in CC[z] where z=xy so there are plenty of invariants, but they are not enough to separate orbits. (This example is premature, I think.)

(II) THE GENERAL (I.E. NOT NECESSARILY AFFINE) CASE

If X is not affine, you may want to try to construct X/G by perhaps aiming to choose U = X\Z (where Z is some "bad" locus), cover U in affines, and using the theorem above. Indeed this is what you do when you construct the projective space. Rather interestingly, this approach fails (Because of strictly semistable points; but it has to be said that in many cases it works extremely well), and it is absolutely worth discussing this right now

EXAMPLE n ordered points on PP1. In fact, let n=4.
Let's try to do the reasonable thing and glue open subsets Uijk with i<j<k i.e. we try to make the quotient of the open subset U* where at least 3 of the points are distinct.
So U123/G = PP1 via 0,1,&,x (but 0,&,1,x is also OK)
Consider U134 via 0,y,&,1 then the gluing is (xy=1, x not = 0,&; y not =0,&) and it leads to a nonHausdorff space.

So what are we going to do with this? It is clear (n=4) that the quotient must be PP1: if we want a Hausdorff quotient, we must not only throw away some bad orbits, but also give up the notion of separating (all) the orbits.

EXERCISE 1 For n=4 show that there is no SL2 and S4-invariant U open in U* such that the geometric quotient U/SL2 is separated and proper.
However, for n=5 the appropriate U exists! (This is because there are no strictly semistable points.)

I think that on some level the idea is still to follow the affine case. If R = oplus R_n is a graded ring and X = Proj R maybe Y = Proj R^G. Now if G acts on X then it does not (naturally) act on R: before we can even think of something like this we need to lift the action of G to L

IIa. G-LINEARISED LINE BUNDLES

DEFINITION G-linearised line bundle. X, G, L
for all g we want Fg:L->g*L and
L -> (gh)*L is the same as L-> h*L -> g*(L ->h*L)

NB This is not a particularly tricky definition and I should explain it with some pictures

NB A G-linearized line bundle is not the same as a line bundle on X/G. This is pretty dramatic e.g.

X=CC^2, G a finite subgroup of SL_2. k[x,y]^G -> k[x,y] is NOT (faithfully) flat (so ff descent of modules does not apply). This is another instance in life where finite nontrivial stabilizers are an issue

EXERCISE 2
(1) A G-linearised line bundle is a descent datum of a line bundle for G*X=>X
(2) G finite acting on X=Spec A affine. A G-linearised coherent sheaf is the same as a fg A*G module.

EXAMPLE: reps of torus that leads to FLOP. Explain what is a G-linearization in terms of characters. (A first message is: G-linearization is not unique. That is why people study vGIT.)

IIb. STABILITY

DEFINITIONs Given a set up (X, G, L) of an action of G on X and a G-linearized line bundle L on X:

(i) x is SEMI-STABLE if for some n>0 there exists s in H0(L^n)^G, s(x) not 0, AND (not so important) X\Zs affine (Zs = locus where (s=0))

(ii) x is STABLE if it is semistable AND in addition the action of G on X\Zs has closed orbits

IIc. MAIN THEOREM G reductive, X quasiprojective, L ample

A "quotient" (more precisely: uniform categorical quotient) Y= Xss/G exists and it has the following properties:
(i) there is a Zariski open subset Y* of Y and f^{-1}Y*=Xs is the geometric quotient Xs/G
(ii) for x, y in Xss, f(x) = f(y) if and only if the closures of O(x) and O(y) meet somewhere in Xss
(iii) for all y in Y, f^{-1}(y) contains a unique G-orbit that is closed in Xss
(iv) A semistable point x is stable iff the orbit O(x) is closed in Xss and the stabilizer Gx has minimal dimension

EXAMPLE (flop): rep of torus above: work out stable and semistable points for different choices of G-linearization

EXAMPLE: how is this compatible with the discussion of 4 ordered points in PP1? We need to understand stability. Make the necessary statement here with proof postponed until after the Hilbert-Mumford criterion

EXERCISE 4 Work out stable, unstable etc for rank two toric varieties. *Do the same for higher rank toric varieties
EXERCISE 5 Work out stable, unstable for endomorphisms of a vector space

(III) THE HILBERT-MUMFORD CRITERION IN TERMS OF 1PSG

Key questions in real-life situations: (a) how to test for stability? (b) what does stability "mean" intuitively?

Fix a set up X, G, L as above

Assume here that X is a *representation* of G (this is sufficient for the discussion that I have in mind: there is no point in discussing the general definition, especially because then one has to work a lot more to get to useful applications).

DEFINITION If x in X and lambda: Gm -> G is a 1PSG, then one defines the WEIGHTS of x wrt lambda as follows:
In a basis of eigenvectors e_i, on which Gm acts with weights r_i, write x=(a_1,...,a_n). The weights of x are the r_i such that a_i not zero.

THEOREM x is:

stable if: for all 1PSG there are some (strictly) positive and some (strictly) negative weights

semi-stable if: for all 1PSG not all weights are (strictly) positive (equivalently, negative)

EXERCISE 6 (Hard; play with it and then look it up)
(a) stability of plane curves of degree 3 and
(b) plane curves of degree 4 (G=SL3)
(c) stability of cubic surfaces (G=SL4)

EXERCISE 7 (Hard: play with it and then look it up) Study of the ring of invariants & geometry of the quotient of the space of 4 UNordered points in P1 by the action of G=SL2

(III) WHAT I DID NOT TALK ABOUT

(1) The modern theory of K-stability. The notion is a variant of GIT stability: it is not particularly new in algebraic geometry. It always was hard to prove that things that should be stable are stable, e.g. (embarrassingly) stable curves or smooth surfaces of general type

(2) Symplectic reduction (it would not be fair to rush it) and how in some cases it leads to GIT quotient