INTRODUCTION TO EINSTEIN MANIFOLDS

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1. Introduction

- M^n: smooth, closed, connected, oriented
- Riemannian metric g
- Is there a *best* Riemannian structure on M?
- (Cartan, Weyl) local diffeos invariants in terms of curvature and its covariant derivatives

1.1. **The case $n = 2$.**

- M^2 surface of genus $\gamma \geq 0$
- Unique curvature invariant: Gaussian curvature $K_g \colon M \to \mathbb{R}$
- Uniformization Theorem: existence of g with $K_g \equiv \mathit{const}$
- Sign of constant K_g constrained by Gauss–Bonnet: $\int_M K_g \, \mathrm{dv}_g = 2\pi\chi(M) = 2\pi(2-2\gamma)$
- For $\gamma \geq 1$ such constant curvature g is not unique, but there is a good Teichmüller/moduli space, which is a finite dimensional manifold/orbifold

Exercise 6.1

1.2. **The Einstein equation.** (M^n, g) with $n \geq 3$.

- Riemannian curvature: $\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$
- Sectional curvature: 2-plane $\Pi \subset T_mM$ with o.n. basis $e_1, e_2 \rightsquigarrow K(\Pi) = \langle \mathrm{Rm}(e_1,e_2)e_2, e_1\rangle$
- Constant sectional curvature $\Longrightarrow (M^n, g)$ locally isometric to $\mathbb{S}^n, \mathbb{R}^n$ or $\mathbb{H}^n$
- Ricci curvature: $\mathrm{Ric}(X,Y) = \sum_{i=1}^n \langle \mathrm{Rm}(e_i, X)Y, e_i\rangle$
- Einstein equation: $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$

Exercise 6.2

- Scalar curvature: $\mathrm{Scal} = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$
- (Aubin) Every M admits a metric of constant negative scalar curvature
  - Can modify any $\tilde{g}$ on M in a neighbourhood of a point so that $\int_M \mathrm{Scal}_{\tilde{g}} \, \mathrm{dv}_{\tilde{g}} < 0$
  - Construct u so that $g = e^{2u}\tilde{g}$ has unit volume and constant scalar curvature equal to the Yamabe invariant of conformal class $[\tilde{g}]$

$$Y(M, [\tilde{g}]) = \inf_{g \in [\tilde{g}]} \frac{\int_M \mathrm{Scal}_g \, \mathrm{dv}_g}{\mathrm{Vol}(M,g)^{\frac{n-2}{2}}} < 0$$

2. The Hilbert–Einstein functional

- $\mathfrak{Met}(M)$: space of smooth Riemannian metrics on M; $\mathfrak{Met}_1(M)$: normalised volume
- $\mathfrak{Diff}(M)$: diffeomorphisms, acting on $\mathfrak{Met}(M)$; $\mathfrak{Diff}_0(M)$: diffeos isotopic to the identity
- $T_g\mathfrak{Met}(M) = \Gamma(\mathrm{Sym}^2 T^*M)$

Exercise 6.3

Exercise 6.4

- $T_g\left(\mathfrak{Diff}(M) \cdot g\right) = \mathrm{im}\, \delta^*$
- $T_g\mathfrak{Met}(M) = \mathrm{im}\, \delta^* \oplus^{\perp_{L^2}} \ker \delta$
- $\mathfrak{Diff}(M)$–invariant functional $\mathcal{F} \colon \mathfrak{Met}(M) \to \mathbb{R}$

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- L^2–gradient: g_t with $g_t = g$ and $\dot{g}_t = h$ at $t = 0 \rightsquigarrow \frac{d}{dt}\mathcal{F}(g_t)|_{t=0} = \langle \mathrm{grad}_g \mathcal{F}, h\rangle_{L^2}$
- $\mathfrak{Diff}(M)$–invariance $\Longrightarrow \delta(\mathrm{grad}_g \mathcal{F}) = 0$

Exercise 6.5

- Hilbert–Einstein functional: $\mathcal{S}(g) = \int_M \mathrm{Scal}_g \, \mathrm{dv}_g$
- $\mathrm{grad}_g \mathcal{S} = -\left(\mathrm{Ric}_g - \frac{1}{2}\mathrm{Scal}_g \, g\right)$
- $g \in \mathfrak{Met}_1(M)$ is a critical point of $\mathcal{S}|_{\mathfrak{Met}_1(M)}$ if and only if g is Einstein

Exercise 6.6

3. Moduli

- Einstein equation elliptic in harmonic coordinates: $\mathrm{Ric}_{ij} = -\frac{1}{2}\sum_{p,q} g^{pq}\partial^2_{pq} g_{ij} + l.o.t.$
  $\Longrightarrow$ Einstein metrics real analytic in harmonic coordinates
- Elliptic deformation complex: if $\mathrm{Ric}(g) = \lambda g$

$$0 \to \Omega^1(M) \xrightarrow{\delta^*} \Gamma(\mathrm{Sym}^2 T^*M) \xrightarrow{d_g\mathrm{Ric}-\lambda} \Gamma(\mathrm{Sym}^2 T^*M) \xrightarrow{B_g} \Omega^1(M) \to 0$$

  where $d_g\mathrm{Ric}(h) = \frac{1}{2}\triangle_L h - \delta^*\delta h - \frac{1}{2}\nabla d\,\mathrm{tr}_g h$ and $B_g(h) = \delta h + \frac{1}{2} d\,\mathrm{tr}_g h$

Exercise 6.7

4. Examples

Almost all known Einstein metrics either have a large symmetry group (in the closed case the Einstein constant must then be positive) or have special holonomy (or are related to some holonomy reduction).

Exercise 6.8

4.1. **Homogeneous spaces.**

- G compact Lie group, K closed subgroup $\rightsquigarrow M = G/K$
- G/K reductive: K–invariant splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
- $TM = G \times_K \mathfrak{p}$
- G–invariant metrics on $M \overset{1:1}{\longleftrightarrow}$ K–invariant positive definite inner products on $\mathfrak{p}$

4.1.1. *Isotropy irreducible homogeneous spaces.*

- If $\mathfrak{p}$ is an irreducible K–representation then there exists a unique K–invariant symmetric bilinear form on $\mathfrak{p}$ and therefore there is a unique G–invariant metric on M, which must be Einstein (see also Exercise 6.5)
- $G = \mathrm{SU}(n+1)$, $K = \mathrm{U}(n) \rightsquigarrow M = \mathbb{CP}^n$ with Fubini-Study metric

4.1.2. *The canonical variation.*

- $K \subset H \subset G$
- $M = G/K \to G/H = B$ fibre bundle with fibre $F = H/K$
- G/H and H/K reductive isotropy irreducible $\rightsquigarrow$ Einstein metrics g_B and g_F with scalar curvature s_B and s_F respectively
- 1-p family of G–invariant metrics on M up to scale

$$g_t = g_B + t g_F$$

- Restrict normalised Hilbert–Einstein functional

$$\frac{\mathcal{S}(g_t)}{\mathrm{Vol}(M, g_t)^{\frac{n-2}{n}}} \propto t^{\frac{\dim F}{n}}\left(\tfrac{1}{t}\mathrm{s}_F + \mathrm{s}_B - t|\mathrm{curv}|^2\right) =: \mathcal{S}(t)$$

  where curv = curvature of the connection on H–bundle $G \to B$ coming from $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_B$
- Palais' Principle of Symmetric Criticality: $\mathcal{S}'(t) = 0 \Longrightarrow g_t$ Einstein

Exercise 6.9

4.2. **Kähler–Einstein metrics.**

- M^{2n}, J almost complex manifold
- J-compatible non-degenerate 2-form $\omega \rightsquigarrow g = \omega(\,\cdot\,, J\,\cdot\,)$
- (M, J, ω, g) Kähler $\Leftrightarrow N_J = 0$ and $d\omega = 0 \Leftrightarrow \nabla J = 0 \Leftrightarrow \nabla\omega = 0$
- Ricci form $\rho_\omega = \mathrm{Ric}_g(J\,\cdot\,,\,\cdot\,)$: $d\rho_\omega = 0$ and $[\rho_\omega] = 2\pi c_1(M, J)_{\mathbb{R}} \in H^2(M, \mathbb{R})$
- Calabi–Yau Theorem: (M, J, ω) Kähler such that $2\pi c_1(M, J)_{\mathbb{R}} = \lambda[\omega]$ for some $\lambda \in \mathbb{R}_{\leq 0}$ $\Longrightarrow \exists\, u \in C^\infty(M)$ such that $\omega_u := \omega + i\partial\overline{\partial}u$ is Einstein with Einstein constant λ.
- K3 surfaces
  - unique simply connected smooth M^4 supporting Kähler structures with $c_1(M, J) = 0$ (quartic in $\mathbb{CP}^3$, double cover of $\mathbb{CP}^2$ branched over a sextic, ...)
  - Kähler Ricci-flat metrics on M that are hyperkähler: triple of parallel g–compatible (J_1, J_2, J_3) with $J_1 J_2 = J_3$ ($\rightsquigarrow$ triple of Kähler forms $\omega_1, \omega_2, \omega_3$)
  - 57-dimensional moduli space of hyperkähler metrics of unit volume on M

Exercise 6.10

5. Obstructions

Exercise 6.11

- $n = 4$: $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$
- curvature operator $\mathcal{R}\colon\ \Lambda^2 \to \Lambda^2$

$$\mathcal{R} = \left(\begin{array}{c|c} \frac{1}{12}\mathrm{Scal} + \mathrm{W}^+ & \overset{\circ}{\mathrm{Ric}} \\ \hline \overset{\circ}{\mathrm{Ric}} & \frac{1}{12}\mathrm{Scal} + \mathrm{W}^- \end{array}\right)$$

- Chern–Gauss–Bonnet Theorem: Euler characteristic $\chi(M) = \sum_i (-1)^i b_i(M)$

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{1}{24}\mathrm{Scal}^2 + |\mathrm{W}^+|^2 + |\mathrm{W}^-|^2 - \frac{1}{2}|\overset{\circ}{\mathrm{Ric}}|^2\right) \mathrm{dv}_g$$

- Hirzebruch Signature Theorem: signature $\tau(M) = b_+(M) - b_-(M)$

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|\mathrm{W}^+|^2 - |\mathrm{W}^-|^2\right) \mathrm{dv}_g$$

- Hitchin–Thorpe Inequality: (M^4, g) Einstein $\Longrightarrow 2\chi(M) \geq 3|\tau(M)|$
- Moreover $2\chi(M) \pm 3\tau(M) = 0$ iff $\Lambda^\pm M$ is flat

Exercise 6.12

6. Exercises

Exercise 6.1. You are going to prove that the set of flat 2-tori up to isometries and homotheties (*i.e.* change of the metric of the form $g \mapsto \lambda^2 g$ for some $\lambda \in \mathbb{R} \setminus \{0\}$) is

$$\mathcal{M} = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \geq 1, x \in [0, \tfrac{1}{2}], y > 0\}.$$

Every flat 2-torus can be presented as $(\mathbb{R}^2/\Lambda, g_\Lambda)$, where Λ is a lattice of full rank in $\mathbb{R}^2$ and g_Λ is the Riemannian metric on $\mathbb{R}^2/\Lambda$ induced by the standard Euclidean metric on $\mathbb{R}^2$. It will therefore suffice to classify lattices Λ up to the action of $\mathrm{O}(2)$ and homotheties.

(i) Let u_1 be the shortest non-zero vector in Λ. Up to rotations and homotheties we can assume that $u_1 = (1, 0)$ (in particular $\|u_1\| = 1$). Let u_2 be the shortest vectors of $\Lambda \setminus \mathbb{Z}u_1$. Then u_1 and u_2 are linearly independent (over $\mathbb{R}$). Show that $\Lambda = \mathbb{Z}u_1 + \mathbb{Z}u_2$. (Hint: if not there would exist $u \in \Lambda$ such that $u = \lambda_1 u_1 + \lambda_2 u_2$ with $2|\lambda_i| < 1$. But then we would have $\|u\| < \|u_2\|$.)

(ii) Write $u_2 = (x, y)$. Up to reflections along the coordinate axis we can assume that u_2 lies in the first quadrant, *i.e.* $x, y \geq 0$. Also $x^2 + y^2 \geq 1$ and $y > 0$ since u_2 is longer than u_1 and u_1, u_2 are linearly independent. Prove that $x \leq \frac{1}{2}$. (Hint: if not consider $u_2 - u_1$.)

Exercise 6.2. Show that when $n = 3$ g Einstein $\Longrightarrow$ g has constant sectional curvature. Deduce that $S^2 \times S^1$ does not admit any Einstein metric. (Hint: you might want to consider the curvature operator $\mathcal{R}\colon \Lambda^2 \to \Lambda^2$ defined by $\mathcal{R}(x \wedge y) = \frac{1}{2}\sum_{i,j} \langle \mathrm{Rm}(x, y)e_j, e_i\rangle\, e_i \wedge e_j$ and observe that $\Lambda^2 \simeq \Lambda^1$ when $n = 3$.)

Exercise 6.3. Let $\{g_t\}_{t\in(-\epsilon,\epsilon)} \subset \mathfrak{Met}(M)$ be a 1-parameter family of metrics on M depending smoothly on t and set $h = \frac{d}{dt} g_t|_{t=0}$. Show that $\frac{d}{dt}\mathrm{dv}_{g_t}|_{t=0} = \frac{1}{2}(\mathrm{tr}_g h)\,\mathrm{dv}_g$. Deduce that

$$T_g\mathfrak{Met}_1(M) = \{h \in \Gamma(\mathrm{Sym}^2 T^*M) \mid \int_M \mathrm{tr}_g h\,\mathrm{dv}_g = 0\}.$$

Exercise 6.4. For a 1-form ξ let $\delta^*\xi$ denote the symmetrisation of $\nabla\xi$, *i.e.*

$$\delta^*\xi(X, Y) = \tfrac{1}{2}\big((\nabla_X\xi)(Y) + (\nabla_Y\xi)(X)\big)$$

for every pair X, Y of vector fields.

(i) Show that $\delta^*\xi = -\frac{1}{2}\mathcal{L}_{\xi^\sharp} g$.

(ii) Let $\delta\colon \Gamma(\mathrm{Sym}^2T^*M) \to \Omega^1(M)$ denote the formal L^2–adjoint of $\delta^*\colon \Omega^1(M) \to \Gamma(\mathrm{Sym}^2T^*M)$. Show that $\delta(ug) = -du$ for every function u.

Exercise 6.5. Let $\mathcal{F}\colon \mathfrak{Met}(M) \to \mathbb{R}$ be a $\mathfrak{Diff}(M)$–invariant functional and assume that $g \in \mathfrak{Met}(M)$ is homogeneous, *i.e.* there exists a Lie group G acting transitively on M and preserving g. Fix a point $p \in M$ and denote by H the stabiliser of p in G. By differentiation, H therefore acts on T_pM as a subgroup of $\mathrm{SO}(T_pM, g_p)$. Assume that the H–representation T_pM is irreducible.

(i) By restricting to the action of $H \subset \mathfrak{Diff}(M)$ on $\mathfrak{Met}(M)$, show that there exists $\lambda \in \mathbb{R}$ such that $\mathrm{grad}_g\mathcal{F}|_p = \lambda g_p$. (Hint: use Schur Lemma.)

(ii) Use the G–action to deduce that $\mathrm{grad}_g\mathcal{F} = \lambda g$ everywhere on M.

Exercise 6.6. Assume that $n \geq 3$. You are going to show that the critical points of the Hilbert–Einstein functional restricted to metrics with unit volume are the Einstein metrics.

(i) Show that

$$\mathrm{grad}_g\mathcal{S} = -\left(\mathrm{Ric}_g - \tfrac{1}{2}\mathrm{Scal}_g\, g\right).$$

(Hint: you can take for granted the following formula: if g_t is a smooth path in $\mathfrak{Met}(M)$ starting at g in the direction of h then

$$\frac{d}{dt}\mathrm{Scal}_{g_t}|_{t=0} = \triangle(\mathrm{tr}_g h) + d^*(\delta h) - \langle \mathrm{Ric}_g, h\rangle.)$$

(ii) Use the invariance under diffeomorphisms of the Hilbert–Einstein functional to deduce that

$$\delta\mathrm{Ric} + \tfrac{1}{2}d\mathrm{Scal} = 0.$$

(iii) Show that $g \in \mathfrak{Met}_1(M)$ is a critical point of $\mathcal{S}|_{\mathfrak{Met}_1(M)}$ if and only if there exists a function $\lambda \in C^\infty(M)$ such that $\mathrm{Ric} = \lambda g$.

(iv) Use part (ii) to show that λ above is constant and therefore g is Einstein.

Exercise 6.7. Suppose that $\mathrm{Ric}(g) = \lambda g$ and let $h \in \Gamma(\mathrm{Sym}^2T^*M)$ be an infinitesimal variation of g. Show that the symmetric tensor $d_g\mathrm{Ric}(h) - \lambda h$ lies in the kernel of B_g, where $B_g(h') = \delta h' + \frac{1}{2}d\,\mathrm{tr}_g h'$ is the Bianchi operator. (Hint: differentiate the Bianchi identity $\delta\mathrm{Ric} + \frac{1}{2}d\mathrm{Scal} = 0$ along a path starting at g in the direction of h.)

Exercise 6.8. Let X be a Killing vector field, *i.e.* a vector field such that $\mathcal{L}_X g = 0$. Equivalently ∇X is a skew-symmetric $(1, 1)$ tensor.

(i) Show that

$$\triangle\left(\tfrac{1}{2}|X|^2\right) = -|\nabla X|^2 + \mathrm{Ric}(X,X).$$

(ii) Show that a closed (M,g) does not carry any Killing field if $\mathrm{Ric} < 0$.
(iii) Show that a closed (M,g) does not carry any Killing field if $\mathrm{Ric} = 0$ and $b_1(M) = 0$. (Hint: show that $X^\flat$ is a harmonic 1-form and use the fact that $\triangle = \nabla^*\nabla$ on 1-forms if $\mathrm{Ric} = 0$.)

Exercise 6.9. You are going to apply the formalism introduced in Section 4.1.2 to produce an Einstein metric on S^7 that does not have constant curvature. We consider $K \subset H \subset G$ with $K = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, $H = \mathrm{Sp}(1)$ and $G = \mathrm{Sp}(2)$.

(i) Show that $M = S^7$, $B = S^4$ and $F = S^3$. (Hint: you might want to use the double covers $\mathrm{Sp}(2) \to \mathrm{SO}(5)$ and $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$.)
(ii) Verify that B and F are isotropy irreducible homogeneous spaces. Normalise the resulting Einstein (constant curvature) metrics so that $s_B = 12$ and $s_F = 6$.
(iii) Calculate $|\mathrm{curv}|^2$. (Hint: you can use the fact that g_1 is the standard round metric on $\mathbb{S}^7$ with scalar curvature 42.)
(iv) Deduce the existence of a critical point $t_* \neq 1$ of $\mathcal{S}(t)$.
(v) Show that the Einstein metric g_{t_*} does not have constant curvature.
(vi) Show that g_1 and g_{t_*}, normalised to have the same volume, cannot be connected by a path of Einstein metrics. (Hint: compare the values of the Hilbert–Einstein functional.)

Exercise 6.10. Recall that the total Chern class of an almost complex manifold X^{2n} is $c(X) = 1 + c_1(X) + \cdots + c_n(X)$. For example, $c(\mathbb{CP}^3) = (1+h)^4$ (modulo $h^4 = 0$), where h is the generator of $H^2(\mathbb{CP}^3;\mathbb{Z})$. Now, let M be a quartic surface in $\mathbb{CP}^3$.

(i) Consider the exact sequence

$$0 \to TM \to T\mathbb{CP}^3|_M \to \mathcal{O}(4)|_M \to 0.$$

(a) Taking determinants, show that $c_1(M) = 0$.
(b) Show that M has Euler characteristic $\chi(M) = 24$. (Hint: the Euler class of M is $c_2(M)$.)

(ii) Show that $b_2(M) = 22$. (Hint: M is simply connected by the Lefschetz Hyperplane Theorem.)
(iii) Use parts (i.a) and (ii) to show that $h^{2,0}(M) = h^{0,2}(M) = 1$ and $h^{1,1}(M) = 20$.
(iv) Show that $b^+(M) = 3$ and $b^-(M) = 19$. (Hint: use Exercise 6.11 below.)

Exercise 6.11. Let V be a 4-dimensional vector space endowed with a positive definite inner product and a volume form $\mathrm{dv} \in \Lambda^4 V^*$.

(i) Using dv and the wedge product define a non-degenerate pairing q on $\Lambda^2 V^*$. Show that q has signature $(3,3)$. Let $\Lambda^\pm V^*$ be maximal positive/negative subspaces of $(\Lambda^2 V^*, q)$.
(ii) Show that the induced action of $\mathrm{SL}(V) \simeq \mathrm{SL}(4,\mathbb{R})$ (*i.e.* the matrices that preserve dv) on $\Lambda^2 V^*$ defines a double cover $\mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$. Restricting to compact subgroups, we see that $\mathrm{SO}(4) \to \mathrm{SO}(3)^+ \times \mathrm{SO}(3)^-$ is a double-cover; here $\mathrm{SO}(3)^\pm$ is the induced action of $\mathrm{SO}(4)$ on $\Lambda^\pm V^*$.
(iii) Identify V with the quaternions $\mathbb{H}$ and $\mathrm{SU}(2)$ with the unit sphere $\mathbb{S}^3 \subset \mathbb{H}$. Define a map $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{H} \to \mathbb{H}$ by $(q_1, q_2, x) \mapsto q_1 x \overline{q_2}$. Show that this defines a double cover $\mathrm{SU}(2)^+ \times \mathrm{SU}(2)^- \to \mathrm{SO}(4)$.
(iv) Show that this induces a double cover of $\mathrm{U}(1) \times \mathrm{SU}(2)^- \to \mathrm{U}(2)$, where $\mathrm{U}(1) \subset \mathrm{SU}(2)^+$ is the subgroup of diagonal matrices.
(v) Show that $\mathrm{U}(2)$ acts on $\Lambda^- V^*$ as $\mathrm{SO}(3)^-$ and on $\Lambda^+ V^*$ as the subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)^+$ preserving the standard Kähler form ω on $\mathbb{H} \simeq \mathbb{C}^2$.

(vi) Deduce that on a Kähler surface (M,ω), $\Lambda^+ M = [\![\Lambda^{2,0}M]\!] \oplus \mathbb{R}\omega$ and $\Lambda^- M = [\![\Lambda^{1,1}_0 M]\!]$, where $\Lambda^{1,1}_0 M$ is the space of $(1,1)$-forms orthogonal to ω. Here for a complex vector space W we denote by $[\![W]\!]$ the real vector space such that $W \oplus \overline{W} = [\![W]\!] \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 6.12. This exercise is about non-existence and uniqueness of Einstein metrics in dimension 4.

(i) Use the Chern–Gauss–Bonnet Theorem to show that $M = S^3 \times S^1$ does not admit any Einstein metric. (Hint: by Bieberbach's Theorem any compact flat manifold is finitely covered by a flat torus.)

(ii) Let $M_{k,\ell} = k\,\mathbb{CP}^2 \sharp \ell\,\overline{\mathbb{CP}^2}$, where $\overline{\mathbb{CP}^2}$ denotes $\mathbb{CP}^2$ with the opposite orientation. For which (k,ℓ) can't $M_{k,\ell}$ carry any Einstein metric?

(iii) Let g be an Einstein metric on the smooth 4-manifold underlying a complex K3 surface. Show that g is hyperkähler. (Hint: use the fact that every flat bundle on a simply connected manifold can be trivialised by a basis of orthonormal parallel sections.)