Complex manifolds and Kähler manifolds

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October 7, 2022

I. COMPLEX MANIFOLDS

Recall that if $U \subset \mathbf{C}^n$ is an open subset a holomorphic map $f : U \to \mathbf{C}^m$ is *holomorphic* if it is C^∞ and the derivative at each point is complex-linear. In a neighbourbood of any point of U the components of f are represented by power series.

A *complex manifold* X can be defined as a C^∞ manifold with an atlas of charts such that the overlap maps are holomorphic.

Other points of view

- An *almost-complex manifold* is a C^∞ manifold M with a complex structure on the tangent bundle TM: that is a bundle map $I : TM \to TM$ such that $I^2 = -1$. Certainly a complex manifold is almost-complex. In complex dimension 1 (Riemann surfaces) an almost-complex structure is equivalent to a complex structure but not in higher dimensions; there is an "integrability condition". An efficient way to explain this is to consider the complexified cotangent bundle $T^*M_{\mathbf{C}}$. An almost complex structure is encoded in a decomposition

  $$T^*M_{\mathbf{C}} = T^*_{1,0} \oplus T^*_{0,1}.$$

  Then the complex-valued differential forms on M have a (p, q) decomposition:

  $$\Omega^* M = \bigoplus_{p,q} \Omega^{p,q}.$$

In the complex case, in local complex co-ordinates z_i, a form in $\Omega^{p,q}$ can be written as

$$\sum_{I,J} a_{IJ} dZ^I d\overline{z}^J,$$

with $|I| = p, |J| = q$, using "multi-index" notation. Hence or otherwise one sees that the exterior derivative is a sum of

$$\partial : \Omega^{p,q} \to \Omega^{p+1,q} \quad , \quad \overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1},$$

and $d^2 = 0$ implies that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. Holomorphic functions are solutions of the equation $\overline{\partial} f = 0$ on $\Omega^{0,0}$.

In the general almost-complex case we can still define $\partial, \overline{\partial}$ by taking the components of d but we do not have $d = \partial + \overline{\partial}$. On $\Omega^{1,0}$ the exterior derivative has an additional component $\Omega^{1,0} \to \Omega^{0,2}$. This is a bundle map defined by contraction with the *Nijenhuis tensor* N of the almost complex structure which is a section of $TM \otimes_{\mathbf{C}} \Lambda^2 T^*_{0,2}$, i.e. an element of $\Omega^{0,1}(T)$. An almost-complex structure is called *integrable* if $N = 0$. The Newlander-Nirenberg Theorem states that in this case the almost-complex structure arises from a complex structure. This gives an alternative point of view which is helpful in connections with differential geometry.

The proof of the Newlander-Nirenberg Theorem is relatively difficult analysis, but it is formally a complex analogue of the Frobenius Theorem (see Exercise 3).

- Another point of view is based on the *sheaf* $\mathcal{O}_X$ of local holomorphic functions on a complex manifold X. (This is helpful when one extends the theory to singular spaces.) We have sheaf cohomology groups $H^p(\mathcal{O}_X)$ which can be described in many ways. One way uses the Dolbeault complex

$$\overline{\partial} : \Omega^{0,q} \to \Omega^{0,q+1}$$

analogous to the de Rham description of the topological cohomology of a manifold. Another uses Cech cohomology.

A good example to understand this (see Griffiths and Harris, p.34): let X be a Riemann surface and $p \in X$ a point and consider the problem of finding a meromorphic function on X with a single pole at p.

For a Cech approach we consider the cover $X = U_1 \cup U_2$ where U_1 is small disc about p and $U_2 = X \setminus \{p\}$. We can certainly write down a meromorphic function f on U_1 with the required pole. The restriction of f to $U_1 \cap U_2$ is holomorphic and defines a 1-cocycle $f|$ in the Cech complex. This is zero in cohomology if and only if we can find holomorphic g_i on U_i such that $g_2 - g_1 = f|$ on the overlap. In that case $f + g_1$ on U_1 and g_2 on U_2 define the required meromorphic function on X. In other words the obstruction to solving our problem lies in $H^1(\mathcal{O}_X)$.

For a Dolbeault approach, let χ be a cut-off function supported in U_1 and equal to 1 near p. Then $\alpha = \overline{\partial}(\chi f)$ is a C^∞ $(0,1)$-form and defines a class in the Dolbeault cohomology $H^{0,1} = \Omega^{0,1}/\text{Im}\, \overline{\partial}$. If this vanishes we can write $\alpha = \overline{\partial} h$ for a smooth function h on X. Then $\chi f - h$ is the desired meromorphic function. In other words the obstruction to solving our problem appears in $H^{0,1}$.

A *holomorphic vector bundle* over a complex manifold X can be defined as a C^∞ complex vector bundle whose total space is a complex manifold and such that the inclusion of fibres and projection to base are holomorphic maps. From a more differential geometric point of view it is a C^∞ bundle E with an operator

$$\overline{\partial}_E : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$$

such that $\overline{\partial}_E^2 = 0$ and, satisfying the Leibnitz rule, for $\lambda \in \Omega^a$:

$$\overline{\partial}_E(\lambda s) = (\overline{\partial}\lambda)s + (-1)^a \lambda \overline{\partial}_E s.$$

Local holomorphic sections of E are solutions $s \in \Omega^0(E)$ of the equation $\overline{\partial}_E s = 0$.

Alternatively, such a bundle can be viewed as a (finite rank) locally free sheaf of modules $\mathcal{O}(E)$ over the structure sheaf $\mathcal{O}_X$. The sheaf cohomology groups $H^*(\mathcal{O}(E))$ can be computed using the $\overline{\partial}_E$ complex. (In dimension 0, $H^0(\mathcal{O}(E))$ is the space of holomorphic sections of E.) In the case when E is the holomorphic vector bundle $\Lambda^p T^* M$ these cohomology groups are the same as the cohomology $H^{p,q}$ defined by the operators $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$.

A basic fact is that for *compact* X all these cohomology groups are finite dimensional. One important tool is the generalised *Riemann-Roch formula* which is a formula for the Euler characteristic (of a holomorphic bundle E over compact X):

$$\sum(-1)^i \mathrm{dim}\ H^i(X;E)$$

in terms of the *Chern classes* of E and TX.

First steps into deformation theory

One feature which distinguishes the study of complex manifolds from C^∞ manifolds is the appearance of continuous deformations.

Classical Example The family of Riemann surface X_λ for λ in the upper half plane defined by the quotient of $\mathbf{C}$ by the lattice $\mathbf{Z} \oplus \mathbf{Z}\lambda$.

In general, consider the set-up $\pi : \mathcal{X} \to D$ where D is a disc in $\mathbf{C}$, $\mathcal{X}$ is a complex manifold of dimension $(n+1)$ and π is a (proper) holomorphic submersion. Then the fibres $X_t = \pi^{-1}(t)$make up a "holomorphically varying" family of n-dimensional complex manifolds—deformations of the central fibre $X = X_0$.

Let E be the restriction of the tangent bundle of $\mathcal{X}$ to the central fibre. It is a rank $n+1$ holomorphic vector bundle over X and fits in an exact sequence

$$0 \to TX \to E \to \mathcal{O}_X \to 0.$$

From the general theory of sheaves we get a long exact sequence in cohomology with boundary map

$$\delta : H^0(\mathcal{O}_X) \to H^1(TX).$$

The Kodaira-Spencer class of the deformation is defined to be $\delta(1) \in H^1(TX)$. As usual we can understand this abstract definition in different ways

- Thinking of X as covered by co-ordinate charts U_i, deformations are given by deforming the overlap maps. Differentiating with respect to t at $t = 0$ we get vector fields v_{ij} on $U_i \cap U_j$ which define a Cech cocycle representing the class in $H^1(TX)$.

- Thinking of X as a C^∞ manifold with integrable almost-complex structure, a deformation of the structure can be represented by a modified $\overline{\partial}$-operator
$$\overline{\partial}_\mu = \overline{\partial} + \mu\partial$$
where $\mu \in \Omega^{0,1}(T)$ (See Exercise 5). The integrability condition is a nonlinear PDE for μ. The linearised equation, for infinitesimal deformations $\dot{\mu}$, is just $\overline{\partial}_{TX}\dot{\mu} = 0$. On the other hand there are trivial deformations given by applying a diffeomorphism to X. At the linearised level these are given by $\overline{\partial}_{TX} : \Omega^0(TX) \to \Omega^{0,1}(TX)$. The conclusion is that the invariantly defined information is the class of $\dot{\mu}$ in
$$\frac{\text{Ker } \overline{\partial}_{TX} : \Omega^{0,1}(TX) \to \Omega^{0,2}(TX)}{\text{Im } \overline{\partial}_{TX} : \Omega^{0}(TX) \to \Omega^{0,1}(TX)},$$
which is the Dolbeault description of the Kodaira-Spencer class in $H^1(TX)$.

One main theorem (Kodaira-Nirenberg-Spencer) is that if X is compact, has no holomorphic automorphisms and if $H^2(TX) = 0$ then small deformations of the complex structure, modulo diffeomorphisms close to the identity, are in 1-1 correspondence with a neighbourhood of 0 in $H^1(TX)$.

II. METRICS

Recall that a *connection* on a C^∞ complex vector bundle $V \to M$ over a C^∞ manifold M can be regarded as a covariant derivative
$$\nabla_V : \Omega^0(V) \to \Omega^1(V).$$
There is then an extension to a coupled exterior derivative
$$d_V : \Omega^p(V) \to \Omega^{p+1}(V),$$
with $d_V = \nabla_V$ on $\Omega^0(V)$. The composite
$$d_V^2 : \Omega^0(V) \to \Omega^2(V)$$
is a bundle map defined by contraction with the curvature of the connection which is a tensor $F_V \in \Omega^2(\text{End}V)$. If V has a Hermitian metric $\langle\ ,\ \rangle$ on the fibres then a connection is said to be compatible with the metric if
$$\nabla(s_1, s_2) = \langle \nabla_V s_1, s_2\rangle + \langle s_1, \nabla_V s_2\rangle$$

for all sections s_1, s_2.

Now consider a holomorphic vector bundle E over a complex manifold X with a Hermitian metric on the fibres. There is a unique connection—the *Chern connection*— on E which is compatible with the metric and also with the holomorphic structure, in the sense that the $(0,1)$ component of ∇_E is $\overline{\partial}_E$. (Exercise 6) This fact is similar in style to the characterisation of the Levi-Civita connection in Riemannian geometry but the proof is easier and the formula for the connection is simpler. In a local holomorphic trivialisation we identify sections of E with vector-valued functions.The metric is represented by a Hermitian matrix-valued function h and the covariant derivative is

$$\nabla_E = \overline{\partial} + \partial + h^{-1}(\partial h), \tag{1}$$

acting on vector-valued functions, where the last term is a matrix-valued 1-form acting algebraically. The curvature is the matrix-valued 2-form

$$F_E = \overline{\partial}(h^{-1}\partial h). \tag{2}$$

Focus now on the case when E is the tangent bundle of the complex manifold X. The real part of a Hermitian metric is a Riemannian metric on X. In fact these are precisely the Riemannian metrics g which are compatible the almost-complex structure in the sense that for tangent vectors ξ, η we have $g(I\xi, I\eta) = g(\xi, \eta)$. Replacing ξ by $I\xi$ this is equivalent to the statement that the form

$$\omega(\xi, \eta) = g(\xi, I\eta)$$

is skew-symmetric, so a 2-form on X. (From another point of view, it is the imaginary part of the Hermitian form). The 2-forms which arise in this way are the "positive" $(1,1)$ form, of the shape

$$\omega = \sqrt{-1} \sum h_{ij} dz_i d\overline{z}_j, \tag{3}$$

where (h_{ij}) is a positive Hermitian matrix.

We now reach the main definition of this Section II. A Hermitian metric on TX is called *Kähler* if its 2-form ω is closed: $d\omega = 0$.

There are several alternative definitions which can be shown to be equivalent.

1. The 2-form ω is covariant constant with respect to the Levi-Civita connection ∇_{LC} of the Riemannian metric: $\nabla_{LC}\omega = 0$.

2. The Chern connection on E is equal to the Levi-Civita connection.

3. For any point p of X we can choose local holomorphic coordinates z_i such that the first derivatives of the metric tensor h_{ij} vanish at p.

Item (1) says that the *holonomy* of the Riemannian metric is contained in the unitary group $U(n) \subset O(2n)$. Item (2) implies in particular that equation (2) gives a formula for the Riemann curvature tensor. Item (3) is the statement that any complex Kähler structure osculates to order 2 to the flat Euclidean model (Griffiths and Harris p.107).

A complex manifold is called Kähler if it admits a Kähler metric. Often one does not distinguish between the metric and the $(1,1)$ form ω. If ω_0 is one Kähler metric then we get an infinite dimensional family of the form

$$\omega_\phi = \omega + i\overline{\partial}\partial\phi$$

for real-valued functions ϕ such that ω_ϕ is positive (which is certainly true if ϕ is small). The $\overline{\partial}\partial$-lemma says that all Kähler forms in a given de Rham cohomology class $[\omega_0] \in H^2(X, \mathbf{R})$ arise in this way.

Examples

- Any complex projective manifold $X \subset \mathbf{CP}^N$ is Kähler. In general if Y is Kähler and $X \subset Y$ is a complex manifold then X is Kähler so we just need to see that$\mathbf{CP}^N$ is Kähler. There is a standard *Fubini-Study* Kähler form on $\mathbf{CP}^N$ which can be seen in various ways. One is to write
$$\mathbf{CP}^N = SU(N+1)/U(N).$$
The action of $U(N)$ on the tangent space at the base point is the standard action on $\mathbf{C}^N$ so there is a unique Hermitian form up to scale. This shows that there is a unique $SU(N+1)$-invariant 2-form ω_{FS} on $\mathbf{CP}^N$, up to scale.

  Another point of view is to regard $\mathbf{CP}^N$ as the "symplectic quotient" of $\mathbf{C}^{N+1}$ under the action of S^1. Another is to write down an explicit formula for ω_{FS}.

- Any complex torus $\mathbf{C}^n/\Lambda$, for a lattice $\Lambda \subset \mathbf{C}^n$, is Kähler.

- Let Z be the quotient of $\mathbf{C}^2\setminus\{0\}$ by the relation $z \sim 2z$—a *Hopf surface.* It is diffeomorphic to $S^1 \times S^3$. For any compact Käher manifold the de Rham cohomology class $[\omega]$ cannot be zero since ω^n is a volume form. Since $H^2(Z) = 0$ the complex manifold Z is not Kähler.

Hodge Theory

For any Riemannian manifold M the Hodge Theorem states that the de Rham cohomology can be represented by harmonic forms, satisfying $\Delta_d \alpha = 0$, where

$$\Delta_d = dd^* + d^* d.$$

(Here d^* is the adjoint of d.) The same proof shows that on any compact complex manifold with Hermitian metric the Dolbeault cohomology groups $H^{p,q}$ can be represented by (p,q)-forms β satisfying $\Delta_{\overline{\partial}} \beta = 0$, where

$$\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}.$$

When the metric is Kähler, calculation shows that $\Delta_d = 2\Delta_{\overline{\partial}}$. It follows from this that the cohomology of a compact Kähler n-manifold X has a decomposition

$$H^k(X; \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}.$$

The left hand side is topological while each summand on the right hand side has a meaning in terms of the complex geometry of X. In addition there are symmetries

$$h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$$

where $h^{p,q} = \mathrm{dim} H^{p,q}$. One simple consequence is that the odd Betti numbers of X are even. (This gives another way to see that the Hopf surface is not Kähler.)

Line bundles, curvature and holomorphic sections

We now focus on the case of bundles of rank 1: line bundles. These are important because they can be used to define maps to projective space. Suppose that $L \to X$ is a holomorphic line bundle and write V for the vector space of holomorphic sections of L. Suppose that for each point $x \in X$ there is a section which does not vanish at x. Then we have a non-zero evaluation map

$$\mathrm{ev}_x : V \to L_x,$$

which defines a point in $\mathbf{P}(V^*)$. This gives a holomorphic map from X to $\mathbf{P}(V^*)$.

Suppose that we have a holomorphic line bundle $L \to X$ with a Hermitian metric. The curvature F is a pure imaginary 2-form of type $(1,1)$. In a local trivialisation the metric is given by a positive function $h = e^{\psi}$ and by (2) the curvature is $F = \overline{\partial}\partial\psi$. We say that the metric has positive curvature if iF is a positive $(1,1)$-form. *i.e.* a Kähler form. The main slogan is

positivity of curvature $\Leftrightarrow$ many holomorphic sections.

One aspect of this appears through "Weitzenbock formulae" of which the following is a simple example. Suppose that L has a metric of positive curvature and use $\omega = iF$ as Kähler metric. Then for any C^∞ section s of L we have an identity

$$\int_X |\partial_L s|^2 - |\overline{\partial}_L s|^2 d\mu = \int_X |s|^2 d\mu. \tag{4}$$

Here $\nabla_L = \partial_L + \overline{\partial}_L$ is the decomposition into $(1,0)$ and $(0,1)$ components and $d\mu$ is the volume form $\omega^n/n!$. The identity (4) is equivalent to the formula

$$\partial_L^* \partial_L = \overline{\partial}_L^* \overline{\partial}_L + 1. \tag{5}$$

(If instead we had a bundle of *negative* curvature, so we would put $\omega = -iF$ the $+1$ would change to -1.)

The proof od identity (4) can be achieved by writing, pointwise,

$$\left(|\partial_L s|^2 - |\overline{\partial}_L s|^2\right) d\mu == (\nabla_L s * \nabla_L s) \wedge \omega^{n-1}, \tag{6}$$

where $*$ is the bllinear form which combines the Hermitian metric on L with wedge product from 1-forms to 2-forms. (See Exercise 9.)

For another aspect of the slogan consider the model case of $\mathbf{C}^n$ with constant Kähler form

$$\omega_0 = \sqrt{-1} \sum dz_i d\overline{z}_i.$$

Take the trivial holomorphic line bundle $\mathcal{L}_0 \to \mathbf{C}^n$ but with the metric such that $h = |\sigma_0|^2 = \exp(-|z|^2)$ where σ_0 is the trivialising holomorphic section. Then

$$i\overline{\partial}\partial \log h = \omega_0.$$

In other words, we have a holomorphic section of $\mathcal{L}_0 \to \mathbf{C}^n$ which *decays* rapidly at infinity. (If instead we wanted negative curvature $-\omega_0$ then we would get a section which *grows* very rapidly at infinity.)

One precise statement in the direction of our slogan is that a holomorphic line bundle L over compact X admits a metric of positive curvature if and only if it is an ample line bundle: *i.e.* if the sections of L^k for large k define a projective embedding of X. This is the Kodaira embedding theorem.

One way to establish existence of holomorphic sections of L^k for L positive and $k >> 0$ goes on the following lines.

Replacing L by L^k replaces the Kähler form ω by $k\omega$ which corresponds to scaling lengths in the Riemannian metric by $\sqrt{k}$. In this rescaled metric, for $k >> 0$ the geometry is close to the flat model at fixed length scales. Pick a point $p \in X$ and a ball B_R of large radius $O(R)$ in the rescaled metric, centred at p. We can identify L^k with $\mathcal{L}_0$ over B_R, so σ_0 gives a holomorphic section s_0 over this ball. Let χ be a suitable cut-off function, so χs_0 is a smooth section over L^k over X. This is not holomorphic but

$$|\overline{\partial}_{L^k} s_0| = O(e^{-R^2}),$$

and so is very small. We have constructed an "approximately holomorphic" section.

The other step is to adjust s_0 slightly to get an exactly holomorphic section. This uses a Weitzenbock formula for the Laplacian

$$\Delta_{\overline{\partial}} = \overline{\partial}^*_{L^k}\overline{\partial}_{L^k} + \overline{\partial}_{L^k}\overline{\partial}^*_{L^k}$$

on $\Omega^{0,1}(L^k)$. The formula has the shape, schematically,

$$\Delta_{\overline{\partial}} = D^*D + (1 - O(k^{-1})).$$

In particular we have, for the L^2 inner product and large enough k,

$$\langle \Delta_{\overline{\partial}} s, s\rangle \geq (1/2)\|s\|^2.$$

This implies that there is an inverse operator $\Delta_{\overline{\partial}}^{-1}$ with L^2 operator norm at most 2. Now a holomorphic section is defined by the formula

$$s = s_0 - \overline{\partial}^*_{L^k}\Delta_{\overline{\partial}}^{-1}(\overline{\partial}_{L^k} s_0) \tag{7}$$

This approach can be developed to give a proof of the Kodaira embedding theorem. It can also be developed to give a proof of a coarse form of the Riemann-Roch Theorem. As $k \to \infty$:

$$\dim H^0(L^k) \sim (2\pi)^{-n} k^n \mathrm{Vol}_\omega(X), \tag{8}$$

where Vol_ω is the volume in the metric ω.

Exercises

1. Show, from the definition $d = \partial + \overline{\partial}$, that in local holomorphic coordinates $z_i = x_i + \sqrt{-1}y_i$:

$$\overline{\partial} f = \sum_i \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i,$$

where $\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$.

2. For $v \in S^{m-1} \subset \mathbf{R}^m$ let $R_v \in O(m)$ be the linear map which takes v to $-v$ and is the identity on the orthogonal complement of v. Fix a basepoint $v_0 \in S^{m-1}$ and define

$$f_m : S^{m-1} \to SO(m)$$

by $f_m(v) = R_v R_{v_0}^{-1}$.

(a) Show that the sphere S^{2n} has an almost-complex structure if and only if the map f_{2n} is homotopic to a map into $U(n) \subset SO(2n)$.

(b) Show that S^4 does not have an almost-complex structure. (Hint: you may want to recall/read up that $SO(4)$ has a double cover $S^3 \times S^3$ in which $U(2) \subset SO(4)$ lifts to $S^1 \times S^3$.)

(In fact the only spheres which have almost-complex structures are S^2 and S^6.)

3. Let M be a C^∞ manifold with a direct sum decomposition $TM = T_+ \oplus T_-$, so the differential forms on M are a sum of terms $\Omega^{p,q}$, where $\Omega^{p,q}$ denotes the sections of $\Lambda^p T_+^* \otimes \Lambda^q T_-^*$.

(a) Show that there is a section N of $T_+ \otimes \Lambda^2 T_-^*$ such that for a 1-form $\alpha \in \Omega^{1,0}$ the component of $d\alpha$ in $\Omega^{0,2}$ is the contraction $N.\alpha$.

(b)Use the Frobenius Theorem to show that $N = 0$ if and only if T_- defines a *foliation* of M. (That is, in a neighbourhood of any point of M there are functions f_i whose derivatives vanish on T_- and form a basis for T_+^*.

4. The Riemann-Roch formula, for a line bundle L over a compact Riemann surface Σ of genus g, is

$$\dim\, H^0(L) - \dim\, H^1(L) = d - g + 1,$$

where d is the degree (or first Chern class) of the line bundle.

Show that, for $g \geq 2$ the deformation space $H^1(T\Sigma)$ has complex dimension $3g-3$. How can you see this same dimension from the description of Σ as the quotient of the upper half-plane by a discrete subgroup of $PSL(2, \mathbf{R})$?

5. Let μ be a real-valued function on $\mathbf{C}$ with $|\mu| < 1$. Show that there is an almost-complex structure on $\mathbf{C}$ in which the forms of type $(1, 0$ have the shape $a(dz + \mu d\overline{z})$ for complex-valued functions a on $\mathbf{C}$. Show that a function f on $\mathbf{C}$ is holomorphic with respect this almost-complex structure if and only if

$$\frac{\partial f}{\partial \overline{z}} - \mu \frac{\partial f}{\partial z} = 0.$$

6. Derive the formulae ((1) in notes) for the Chern connection of a holomorphic bundle with Hermitian metric and ((2) in notes) for the curvature.

7. Verify the equivalence of the three characterisations in the notes of the Kähler condition.

8. (a) Let α be a $(0, 1)$ form on a compact Riemann surface Σ and β a holomorphic 1-form: i.e $\beta \in \Omega^{1,0}$ and $\overline{\partial}\beta = 0$. Use Stokes' Theorem to show that if there is function f with $\overline{\partial} f = \alpha$ then

$$\int_\Sigma \alpha \wedge \beta = 0.$$

(b) Now let Σ be the Riemann surface which is the quotient of $\mathbf{C} \setminus \{0\}$ by the equivalence $z \sim 2z$. Show that the $(0, 1)$-form $d\overline{z}/\overline{z}$ descends to a $(0, 1)$ form α on Σ and that α defines a non-zero element of $H^{0,1}(\Sigma)$.

(c) Let A be the $(0,1)$ form $\overline{\partial}(\log r)$ on $\mathbf{C}^2 \setminus \{0\}$ where $r^2 = |z_1|^2 + |z_2|^2$. Show that A descends to a $(0,1)$ form on the Hopf surface Z which defines a non-zero element of $H^{0,1}(Z)$.

(d) Show that there is no non-trivial holomorphic 1-form on Z, *i.e.* $H^{1,0}(Z) = 0$. (Hint: Hartogs Theorem from several complex variables theory says that, for $n \geq 2$, a holomorphic function on $\mathbf{C}^n \setminus \{0\}$ extends holomorphically over the origin.)

(In fact, for any compact complex surface the first Betti number is $h^{1,0} + h^{0,1}$. In the case of the Hopf surface $h^{1,0} = 0$ and $h^{0,1} = 1$.)

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9. (a) Show that on a complex n-manifold with Hermitian form ω one has, for a 1-form α of type $(1,0)$:

$$i\alpha \wedge \overline{\alpha} \wedge \omega^{n-1} = |\alpha|^2 \omega^n / n,$$

and for a form of type $(0,1)$ the same identity with sign reversed.

(b) Apply Stokes' Theorem to show that if f is a complex-valued function on a compact Kähler manifold then

$$\int |\partial f|^2 d\mu = \int |\overline{\partial} f|^2 d\mu.$$

10. Let U be the "tautological" line bundle over $\mathbf{CP}^n$ (*i.e.* the fibre of U over a line L is that line) and let $\Lambda \to \mathbf{CP}^n$ be the dual line bundle.

(a) Show that, for $k \geq 0$, a homogeneous polynomial of degree k in the co-ordinates of $\mathbf{C}^{n+1}$ defines a holomorphic section of Λ^k.

(b) Show that any holomorphic section of $\Lambda^k \to \mathbf{CP}^n$ defines a holomorphic function f on $\mathbf{C}^{n+1} \setminus \{0\}$ with $f(\lambda z) = \lambda^k f(z)$. Hence show that the holomorphic sections of Λ^k all arise from homogeneous polynomials on $\mathbf{C}^{n+1}$.

(c) Show that dim $H^0(\Lambda^k) \sim k^n / n!$ as $k \to \infty$.

11. What is the image of the holomorphic map $\mathbf{CP}^1 \to \mathbf{CP}^3$ defined by the sections of $\Lambda^3 \to \mathbf{CP}^1$? Show that the image is an algebraic variety (*i.e.* the common zero set of a finite number of homogeneous polynomials).

12. Let f be a real valued function on the ball in $\mathbf{C}^n$ such that $i\overline{\partial}\partial f$ is a strictly positive $(1,1)$ form. Show that f cannot have an interior maximum. Hence show that a holomorphic Hermitian line bundle with *negative* curvature over a compact complex manifold has no non-trivial holomorphic sections.