

COMPLEX AND KÄHLER MANIFOLDS

1. ALMOST HERMITIAN GEOMETRY

Three subgroups of $\mathrm{GL}(2n, \mathbb{R})$:

- $\mathrm{GL}(n, \mathbb{C})$: preserves multiplication by i on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$

$$J_0: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}, \quad J_0 = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}, \quad J_0^2 = -1_{2n}$$

- $\mathrm{Sp}(2n, \mathbb{R})$: preserves the non-degenerate skew-symmetric bilinear form

$$\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

- ω_0 non-degenerate: $v \mapsto v \lrcorner \omega_0$ defines isomorphism $V \rightarrow V^*$
- $\frac{1}{n!} \omega_0^n$ standard volume form on \mathbb{R}^{2n}

Recall: differential k -form $\alpha \in \Lambda^k V^$ on a vector space $V = k$ -multilinear alternating map*

$$\alpha: V \times \cdots \times V \rightarrow \mathbb{R}.$$

Basis $\{e_1, \dots, e_m\}$ of $V \leftrightarrow$ basis $\{\varepsilon^1, \dots, \varepsilon^m\}$ of $V^ \leftrightarrow$ basis $\{\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \mid i_1 < \cdots < i_k\}$ of $\Lambda^k V^*$ defined in terms of minor determinants*

- $\mathrm{U}(n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{R})$: preserves compatible $(J_0, \omega_0) \rightsquigarrow$ positive definite symmetric bilinear form $g_0(\cdot, \cdot) = \omega_0(\cdot, J_0 \cdot)$

Exercise 5.1

Definition 1.1. Let M be a smooth manifold of dimension $2n$.

- (i) A Riemannian metric on M is a smooth section g of $\mathrm{Sym}^2 T^*M$ which is positive definite at every point.
- (ii) An almost complex structure on M is a smooth section J of $\mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}_{TM}$.
- (iii) A non-degenerate 2-form on M is a smooth section of $\Lambda^2 T^*M$ that is non-degenerate at every point.

We say that (M^{2n}, g, J, ω) is an almost Hermitian manifold if the Riemannian metric g , almost complex structure J and non-degenerate 2-form ω are related by $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$.

2. INTEGRABLE ALMOST COMPLEX STRUCTURES AND COMPLEX MANIFOLDS

M^{2n} with almost complex structure J

- decomposition $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ into $\pm i$ -eigenspaces of J
- $T^*M \otimes \mathbb{C} = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M \rightsquigarrow du = \partial u + \bar{\partial}u$ for every $u \in C^\infty(M; \mathbb{C})$
- Cauchy–Riemann operator: u J -holomorphic if $\bar{\partial}u = 0$
- $\Lambda^2 T^*M \otimes \mathbb{C} = \Lambda^{2,0}T^*M \oplus \Lambda^{1,1}T^*M \oplus \Lambda^{0,2}T^*M$
- $\alpha \in \Omega^{0,1}(M) \rightsquigarrow d\alpha = d^{2,-1}\alpha + \partial\alpha + \bar{\partial}\alpha$

Recall: exterior differential $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$, the unique \mathbb{R} -linear map such that

- (i) for every smooth function f , $df \in \Omega^1(M)$ is the standard differential
- (ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$
- (iii) $d \circ d = 0$

- Nijenhuis tensor $N_J: \Lambda^{0,1}T^*M \rightarrow \Lambda^{2,0}T^*M$ such that $d^{2,-1}\alpha = -N_J(\alpha)$

- J integrable if $N_J \equiv 0$

Exercise 5.2

Theorem 2.1 (Newlander–Nirenberg, 1957). *The almost complex structure J is induced by a holomorphic atlas on M iff it is integrable.*

Exercises 5.3 and 5.4

3. SYMPLECTIC AND KÄHLER MANIFOLDS

M^{2n} with non-degenerate 2-form ω

- ω symplectic if $d\omega = 0$
- Darboux Theorem: existence of coordinate charts $\varphi: U \rightarrow \mathbb{R}^{2n}$ such that $(\varphi^{-1})^*\omega = \omega_0$
- Moser Stability Theorem: M closed, $\{\omega_t\}_{t \in [0,1]}$ family of cohomologous symplectic forms $\Rightarrow \exists$ family $\{\psi_t\}_{t \in [0,1]}$ of diffeomorphisms with $\psi_0 = \text{id}$ and $\psi_t^*\omega_t = \omega_0$.

Exercise 5.5

Recall: $d \circ d = 0 \rightsquigarrow$ deRham complex of M^n

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \rightarrow 0$$

and deRham cohomology $H_{dR}^k(M) = \{\alpha \in \Omega^k(M) \mid d\alpha = 0\} / d\Omega^{k-1}(M)$.

Definition 3.1. An almost Hermitian manifold (M^{2n}, g, J, ω) is Kähler iff J is integrable and ω is symplectic. In this case ω is often referred to as the Kähler form.

Exercises 5.6, 5.7, 5.8 and 5.9

- (M^{2n}, J) complex: $N_J \equiv 0 \Rightarrow \bar{\partial} \circ \bar{\partial} = 0 \rightsquigarrow$
- Dolbeault complex

$$0 \rightarrow \Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{p,n}(M) \rightarrow 0$$

and Dolbeault cohomology $H_{\bar{\partial}}^{p,q}(M) = \{\alpha \in \Omega^{p,q}(M) \mid \bar{\partial}\alpha = 0\} / \bar{\partial}\Omega^{p,q-1}(M)$

- Hodge theory for closed Kähler manifolds: $H_{\bar{\partial}}^{p,q}(M) = \overline{H_{\bar{\partial}}^{q,p}(M)}$ and $H_{dR}^k(M; \mathbb{C}) = \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(M)$

Proof. Some global analysis on manifolds needed for this:

- M closed: $H_{dR}^k(M) \xrightarrow{1:1} \mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) \mid \Delta\alpha = (dd^* + d^*d)\alpha = 0\}$
- (M, g, J, ω) Kähler $\Rightarrow J$ parallel with respect to the Levi-Civita connection of $g \Rightarrow \Delta$ preserves type (p, q) -decomposition \square

4. HOLOMORPHIC BUNDLES AND CHERN CONNECTIONS

(M, J) complex manifold + $E \rightarrow M$ smooth complex vector bundle of rank k

- connection: \mathbb{C} -linear map $\nabla: C^\infty(M; E) \rightarrow \Omega^1(M; E) = C^\infty(M; T^*M \otimes E)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s$$

for every $s \in C^\infty(M; E)$ and $f \in C^\infty(E; \mathbb{C})$

- in a local trivialisation (or gauge) $E|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}^k$: $\nabla s = ds + As$ for $A \in \Omega^1(\mathcal{U}; \mathbb{C}) \otimes \mathfrak{gl}(k, \mathbb{C})$
- change of gauge: $g: \mathcal{U} \rightarrow \text{GL}(k, \mathbb{C}) \rightsquigarrow g^*\nabla = d + gAg^{-1} - (dg)g^{-1}$, i.e. $g^*\nabla(s) = g\nabla(g^{-1}s)$
- (E, h) Hermitian bundle: ∇ unitary if $\nabla h = 0 \iff$ in a local unitary gauge $\nabla = d + A$ with $A \in \Omega^1(M) \otimes \mathfrak{u}(k)$
- Cauchy–Riemann operator on E : \mathbb{C} -linear map $\bar{\partial}_{\mathcal{E}}: C^\infty(M; E) \rightarrow \Omega^{0,1}(M; E)$ such that

$$\bar{\partial}_{\mathcal{E}}(fs) = \bar{\partial}f \otimes s + f\bar{\partial}_{\mathcal{E}}s$$

for every $s \in C^\infty(M; E)$ and $f \in C^\infty(E; \mathbb{C})$.

- connection $\nabla \rightsquigarrow$ Cauchy–Riemann operator $\bar{\partial}_{\mathcal{E}} = \nabla^{0,1}$

- Chern connection: Hermitian metric h + Cauchy-Riemann operator $\bar{\partial}_{\mathcal{E}}$ on $E \implies \exists!$ unitary connection ∇ with $\bar{\partial}_{\mathcal{E}} = \nabla^{0,1}$

Proof. In a local unitary gauge $(E, h)|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}^k$ with standard Hermitian metric on \mathbb{C}^k
 – $\bar{\partial}_{\mathcal{E}} = \bar{\partial} + \alpha$ for some $\alpha \in \Omega^{0,1}(\mathcal{U}; \mathbb{C}) \otimes \mathfrak{gl}(k, \mathbb{C})$
 – set $\nabla = d + A$ where $A = \alpha - \alpha^*$ □

Recall: curvature $F_{\nabla} \in \Omega^2(M; \text{End } E)$ of connection ∇

- locally $F_{\nabla} = dA + A \wedge A$ if $\nabla = d + A$
- $F_{\nabla} = d_{\nabla} \circ d_{\nabla}$ where $d_{\nabla}: \Omega^k(M; E) \rightarrow \Omega^{k+1}(M; E)$ is the exterior covariant differential
- curvature as obstruction to integrability: $F_{\nabla} = 0 \implies$ locally $E|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}^k$ with $\nabla = d$

$\mathcal{E} \rightarrow M$ holomorphic vector bundle with underlying smooth complex vector bundle E

- Cauchy–Riemann operator $\bar{\partial}_{\mathcal{E}}$: in local holomorphic trivialisation $\bar{\partial}_{\mathcal{E}} = \bar{\partial}$
- Cauchy–Riemann operator $\bar{\partial}_{\mathcal{E}} \rightsquigarrow$ sheaf of “holomorphic” sections $\mathcal{O}(\mathcal{E}) = \ker \bar{\partial}_{\mathcal{E}}$
- $\bar{\partial}_{\mathcal{E}} \circ \bar{\partial}_{\mathcal{E}} = 0 \iff E$ has the structure of a holomorphic bundle \mathcal{E}

Proof. See §2.2.2 in Donaldson–Kronheimer. □

Exercise 5.10

5. EXERCISES

Exercise 5.1 (More linear algebra). Let V be a finite-dimensional vector space endowed with a positive definite symmetric bilinear form g_0 .

- Show that $A \mapsto g_0(A \cdot, \cdot)$ defines an isomorphism between the vector space $\mathfrak{so}(V, g_0)$ of skew-symmetric endomorphisms of (V, g_0) (i.e. endomorphisms $A: V \rightarrow V$ such that $g_0(Au, v) = -g_0(u, Av)$ for all $u, v \in V$) and $\Lambda^2 V^*$.
- Show that ω_0 is the image of J_0 under the isomorphism defined above.
- Deduce that any two of g_0, J_0, ω_0 determine the third one and that $U(n) = \text{SO}(2n) \cap \text{GL}(n, \mathbb{C}) = \text{SO}(2n) \cap \text{Sp}(2n, \mathbb{R})$.

Exercise 5.2 (The Nijenhuis tensor). Let J be an almost complex structure on M .

- Verify that $d^{2,-1}$ is tensorial, i.e. $d^{2,-1}(f\alpha) = f d^{2,-1}\alpha$ for any function f and $(0, 1)$ -form α . Deduce the existence of the Nijenhuis tensor N_J , i.e. the existence of a section N_J of $\text{Hom}(\Lambda^{0,1} T^* M, \Lambda^{2,0} T^* M)$ such that $d^{2,-1}\alpha = -N_J(\alpha)$ for every $(0, 1)$ -form α .
- Under the isomorphism

$$\text{Hom}(\Lambda^{0,1} T^* M, \Lambda^{2,0} T^* M) \simeq \left(\Lambda^{0,1} T^* M \right)^* \otimes \Lambda^{2,0} T^* M \simeq T^{0,1} M \otimes \Lambda^{2,0} T^* M$$

identify N_J with the skew-symmetric map $N_J: T^{1,0} M \times T^{1,0} M \rightarrow T^{0,1} M$ defined by $N_J(X, Y) = [X, Y]^{0,1}$.

(Hint: use the fact that $d\alpha(X, Y) = X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y])$ for every 1-form α and vector fields X, Y .)

Exercise 5.3 (The Newlander–Nirenberg Theorem). In this exercise you discuss some easy aspects of the proof of the Newlander–Nirenberg Theorem 2.1.

- Show the easy implication of the Newlander–Nirenberg Theorem: if J is induced by a holomorphic atlas then J must be integrable.
- Show that the converse “hard” implication amounts to showing that, assuming J is integrable, for every point $p \in M$ there exists an open set $\mathcal{U} \subset M$ containing p and J -holomorphic functions $z_1, \dots, z_n: \mathcal{U} \rightarrow \mathbb{C}$ such that $(z_1, \dots, z_n): \mathcal{U} \rightarrow \mathbb{C}^n$ is a diffeomorphism onto its image.

Exercise 5.4 (Almost complex structures on spheres). In this exercise you construct almost complex structures on \mathbb{S}^2 and \mathbb{S}^6 . The latter is non-integrable and deciding whether \mathbb{S}^6 carries an integrable almost complex structure is a famous open problem. It can be shown (Borel–Serre, 1953) that spheres of dimensions $\neq 2, 6$ cannot carry almost complex structures.

- (i) Let \mathbb{S}^2 denote the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ in \mathbb{R}^3 . Given $x \in \mathbb{S}^2$ identify $T_x\mathbb{S}^2$ with the plane $x^\perp \subset \mathbb{R}^3$. Show that the formula

$$J_x(u) = x \times u,$$

where \times denotes the cross product on \mathbb{R}^3 , defines an almost complex structure on \mathbb{S}^2 . Show that this almost complex structure is integrable.

(Hint: you can answer the last question without doing any computation.)

- (ii) The existence of the octonions \mathbb{O} implies the existence of a cross product on $\mathbb{R}^7 = \text{Im } \mathbb{O}$, a bilinear alternating map $\times: \mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}^7$ with the properties that $u \times v$ is orthogonal to u and v and has norm $|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2$. One simply sets $u \times v = \text{Im}(uv)$, where uv is octonionic multiplication. Show that the formula of part (i) defines an almost complex structure J on the 6-sphere \mathbb{S}^6 and that the non-associativity of the 7-dimensional cross product makes J non-integrable.

Exercise 5.5 (Moser’s trick). In this exercise you prove the Moser Stability Theorem.

- (i) We begin with some observations about time-dependent vector fields. Let $\psi: [0, 1] \times M \rightarrow M$ be a smooth map such that $\psi_t = \psi(t, \cdot): M \rightarrow M$ is a diffeomorphism for every t . Define a 1-parameter family of vector fields $\{X_t\}_{t \in [0, 1]}$ on M by $X_t = d_{(t, \cdot)}\psi \left(\frac{d}{dt} \right)$. Conversely, if M is closed (or if the vector fields are compactly supported) a family of vector fields $\{X_t\}_{t \in [0, 1]}$ generates a family $\{\psi_t\}_{t \in [0, 1]}$ of diffeomorphisms with $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ and $\psi_0 = \text{id}$. Let $\{\alpha_t\}_{t \in [0, 1]}$ be a smooth 1-parameter family of k -forms on M . Show that

$$\frac{d}{dt}\psi_t^*\alpha_t = \psi_t^* \left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t \right).$$

- (ii) Let $\{\omega_t\}_{t \in [0, 1]}$ be a smooth family of symplectic forms and suppose that $\frac{d}{dt}\omega_t = d\mu_t$ for a smooth family $\{\mu_t\}_{t \in [0, 1]}$ of 1-forms. Construct a family $\{X_t\}_{t \in [0, 1]}$ of vector fields such that $\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0$.
(Hint: recall Cartan’s Magic Formula $\mathcal{L}_X\alpha = d(X \lrcorner \alpha) + X \lrcorner d\alpha$.)
- (iii) Deduce Moser’s Stability Theorem from parts (i) and (ii).
- (iv) Let (M, J) be an almost complex manifold and suppose that ω_0 and ω_1 are two symplectic forms on M compatible with J . Show that $\omega_t = (1-t)\omega_0 + t\omega_1$ is a family of symplectic forms compatible with J .

(Hint: consider the corresponding family of Riemannian metrics using the fact that the set of positive definite bilinear forms on \mathbb{R}^n is convex.)

- (v) Let (M, g_0, J_0, ω_0) be an almost Hermitian manifold with ω symplectic (such manifolds are sometimes called almost Kähler). Consider the sets

$$\mathcal{J}(\omega_0) = \{\omega_0\text{-compatible almost complex structure } J\},$$

$$\mathcal{H}(J_0, [\omega_0]) = \{J_0\text{-compatible symplectic form } \omega \text{ with } [\omega] = [\omega_0] \in H_{dR}^2(M)\}$$

and the subset $\mathcal{Y}(J_0, \omega_0) \subset \text{Diff}_0(M) \times \mathcal{H}(J_0, [\omega_0])$ consisting of pairs (f, ω) where f is a diffeomorphism isotopic to the identity such that $f^*\omega = \omega_0$. Use parts (iii) and (iv) to construct a map $\Phi: \mathcal{Y}(J_0, \omega_0) \rightarrow \mathcal{J}(\omega_0)$.

Exercise 5.6 (Projective manifolds). In this exercise you show that complex projective space $\mathbb{C}\mathbb{P}^n$ and its holomorphic submanifolds are Kähler.

- (i) Show that $\omega_{\text{FS}} = i\partial\bar{\partial} \log(1 + |z|^2)$ is a Kähler form on \mathbb{C}^n . Here $|z|^2 = |z_1|^2 + \dots + |z_n|^2$.
- (ii) Show that if $\varphi(z_1, \dots, z_n) = z_1^{-1}(1, z_2, \dots, z_n)$ then $\varphi^*\omega_{\text{FS}} = \omega_{\text{FS}}$.

(iii) Show that the charts $(\mathcal{U}_i, \varphi_i)$ defined by

$$\varphi_i: \mathcal{U}_i = \{[z_0 : \cdots : z_n] \mid z_i \neq 0\} \rightarrow \mathbb{C}^n, \quad \varphi_i([z_0 : \cdots : z_n]) = z_i^{-1}(z_0, \dots, \check{z}_i, \dots, z_n),$$

where \check{z}_i means that we drop the i th coordinates, form a holomorphic atlas on $\mathbb{C}\mathbb{P}^n$.

- (iv) Show that the formula $\omega = \varphi_i^* \omega_{\text{FS}}$ over \mathcal{U}_i defines a Kähler form on $\mathbb{C}\mathbb{P}^n$, called the Fubini–Study Kähler form.
- (v) Let M be a complex submanifold of $\mathbb{C}\mathbb{P}^n$. Show that M is Kähler.
(Hint: consider the restriction of the Fubini–Study Kähler form to M .)

Exercise 5.7 (Hopf surface). In this exercise you show that there are complex manifolds that cannot be Kähler.

- (i) Let (M, ω) be a closed symplectic manifold. Show that the deRham cohomology class $[\omega] \in H_{dR}^2(M)$ cannot vanish.
(Hint: use Stokes’ Theorem and the fact that ω^n is a volume form.)
- (ii) Fix a real number $s \neq 0$ and let \mathbb{Z} act on $\mathbb{C}^2 \setminus \{0\}$ by $n \cdot (z_1, z_2) = (e^{ns} z_1, e^{ns} z_2)$. Define $M^4 = \mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$. Show that M is a complex manifold.
- (iii) Show that M is diffeomorphic to $S^1 \times S^3$ and conclude that M cannot be Kähler.
(Hint: work in spherical coordinates on $\mathbb{C}^2 \setminus \{0\} \simeq \mathbb{R}^+ \times \mathbb{S}^3$.)

Exercise 5.8 (Kodaira–Thurston manifold). In this exercise you show that there are symplectic manifolds that cannot be Kähler.

- (i) Show that if (M, g, J, ω) is a closed Kähler manifold then $H^1(M; \mathbb{R})$ is even dimensional.
- (ii) Let $(n, m) \in \mathbb{Z}^2$ act on \mathbb{R}^2 by translation $(x, y) \mapsto (x + n, y + m)$ and on \mathbb{T}^2 via the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. Let M be the 4-manifold $M = (\mathbb{R}^2 \times \mathbb{T}^2) / \mathbb{Z}^2$. Show that the standard symplectic form on $\mathbb{R}^2 \times \mathbb{T}^2$ descends to M and defines a symplectic form on M .
- (iii) Calculate the first deRham cohomology group of M and conclude that M cannot carry any Kähler structure.
(Hint: you can either study \mathbb{Z}^2 -invariant closed 1-forms on $\mathbb{R}^2 \times \mathbb{T}^2$ or calculate the fundamental group of M , thus its first homology, and then use the fact that deRham cohomology is dual to homology with real coefficients.)

Exercise 5.9 (Hyperkähler 4-manifolds). A hyperkähler triple on a 4-manifold is a triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} \frac{1}{3} (\omega_1^2 + \omega_2^2 + \omega_3^2).$$

- (i) Work on \mathbb{R}^4 with coordinates (x_0, x_1, x_2, x_3) . Show that the forms

$$\omega_i = dx_0 \wedge dx_i + dx_j \wedge dx_k,$$

where (ijk) is a cyclic permutation of (123) , define a hyperkähler triple. Show also that you can identify \mathbb{R}^4 with the quaternions \mathbb{H} so that the almost complex structures corresponding to ω_1, ω_2 and ω_3 using the standard inner product on \mathbb{R}^4 as in Exercise 5.1 are, respectively, left multiplications by i, j and k .

- (ii) Let $(M^4, \omega_1, \omega_2, \omega_3)$ be a manifold endowed with a hyperkähler triple. Write $\omega = \omega_1$ and $\omega_c = \omega_2 + i\omega_3$. Show that $\omega_c \wedge \omega_c = 0$ and deduce the existence of an almost complex structure J for which a 1-form α is of type $(1, 0)$ if and only if $\alpha \wedge \omega_c = 0$.
(Hint: you can use the fact that $\omega_c \wedge \omega_c = 0$ if and only if ω_c is decomposable, i.e. it can locally be written as $\omega_c = \theta_1 \wedge \theta_2$ for linearly independent complex 1-forms θ_1, θ_2 .)
- (iii) Show that J is integrable.
(Hint: observe that $d\overline{\omega_c} = 0$ and differentiate the relation $\alpha \wedge \overline{\omega_c} = 0$ valid for any $(0, 1)$ -form α to deduce that $d^{2, -1}\alpha = 0$.)

- (iv) Show that (ω, J) are compatible and that, denoting by g the resulting metric, (M, g, J, ω) is Kähler.

(Hint: show that $\omega \wedge \omega_c = 0$ and deduce that ω is of type $(1, 1)$ with respect to J . Positivity of the resulting metric can be shown by arguing that on each tangent space $\omega_1, \omega_2, \omega_3$ must be linearly equivalent to the hyperkähler triple of part (i).)

Exercise 5.10 (Bundles and connections). This problem is a collection of three separate questions about holomorphic bundles and connections.

- (i) Show that any choice of Cauchy–Riemann operator on a complex vector bundle E over a Riemann surface defines a holomorphic structure on E .
- (ii) Let (E, h) be a Hermitian vector bundle over a complex manifold and let ∇ be a unitary connection on E . Decompose the bundle-valued 2-form F_∇ into (p, q) -types: $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1} + F_\nabla^{0,2}$. Show that $\bar{\partial}_\mathcal{E} = \nabla^{0,1}$ satisfies $\bar{\partial}_\mathcal{E} \circ \bar{\partial}_\mathcal{E} = 0$ if and only if $F_\nabla = F_\nabla^{1,1}$.
- (iii) Set $\mathcal{O}_{\mathbb{C}\mathbb{P}^n}(-1) = \{([z], v) \in \mathbb{C}\mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in \mathbb{C}z\}$.
- (a) Show that the projection onto the first factor induces on $\mathcal{O}_{\mathbb{C}\mathbb{P}^n}(-1)$ the structure of a holomorphic line bundle over $\mathbb{C}\mathbb{P}^n$.
- (b) Endow \mathbb{C}^{n+1} with its standard Hermitian metric and denote by h the induced Hermitian metric on $\mathcal{O}_{\mathbb{C}\mathbb{P}^n}(-1)$. Calculate the curvature of its Chern connection.

6. SOME REFERENCES

- J.-P. Demailly, Complex Analytic and Differential Geometry, Sections V.1–7, VIII.8
- A. Cannas da Silva, Lectures on symplectic geometry, Chapters 6–8 and 12–17
- S. Donaldson and P. Kronheimer, The geometry of 4-manifolds, Sections 2.1.5 and 2.2