Blowing up (in Algebraic Geometry / $\mathbb{C}$)

Homework: Watch last year's lecture by Prof Thomas.

Setup:

$Z \hookrightarrow Y$ closed embedding of algebraic varieties (or schemes).

$Bl_Z Y$, the blow-up of Y along Z,

is a new alg. variety with a map

$$Bl_Z Y \longrightarrow Y.$$

- What is it?

4 points of view:

1) (Algebra) Proj construction
2) (Geometry) taking closure of a graph
3) (Category thry) universal property
4) (Applied maths) eqns on charts

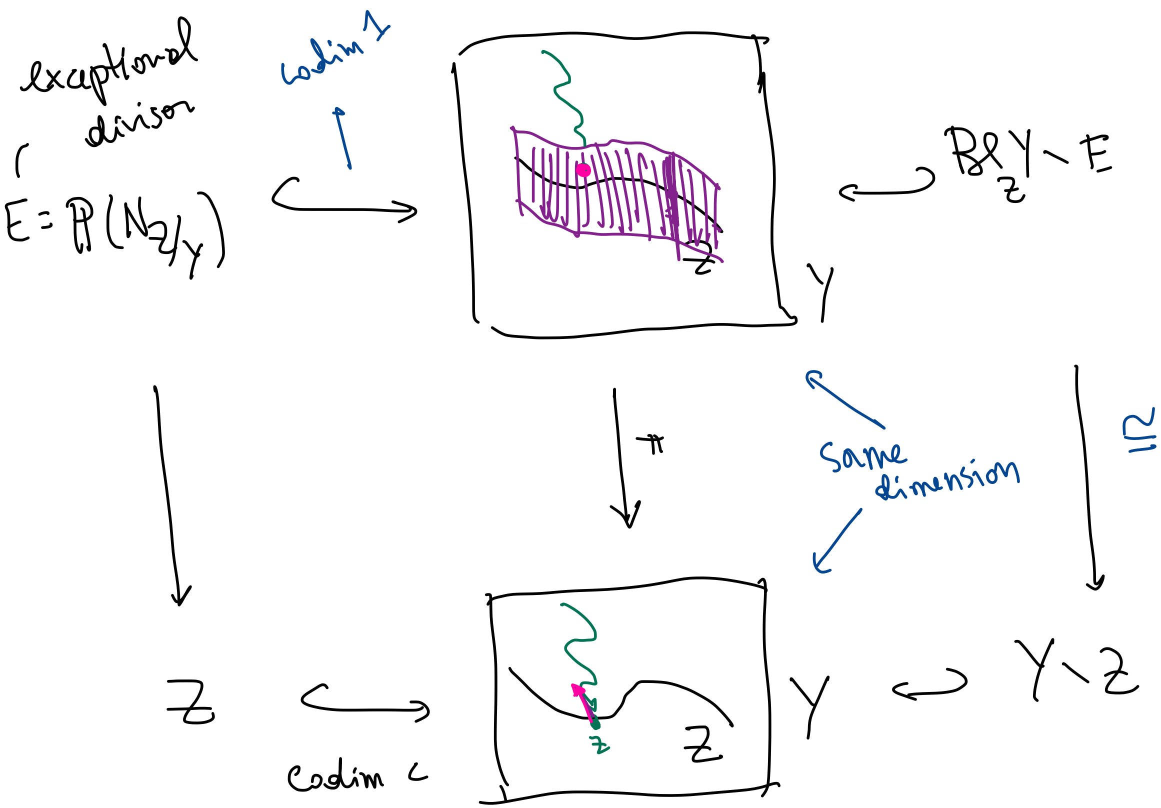
- What is it good for?

- Examples / computations.

Informally: Given $Z \overset{i}{\hookrightarrow} Y$ closed, the blow-up $Bl_Z Y \to Y$ is obtained by replacing Z with $\mathbb{P}(N_{Z/Y})$.

$N_{Z/Y} :=$ normal bundle of $Z \subset Y = (T_Y|_Z)/T_Z$.

$\mathbb{P}(N_{Z/Y}) = \{(z, \ell) \mid z \in Z,\ \ell$ is a direction normal to $Z \subset Y\}$



Blow-up of $Z = 0 \hookrightarrow V = \mathbb{A}^n$.

$$\boxed{\begin{tikzcd} \mathbb{P}V \arrow[r, hook] \arrow[d] & \mathrm{Bl}_0 V \arrow[d] & \mathrm{Bl}_0 V \setminus \mathbb{P}V \arrow[l, hook'] \arrow[d, "\wr\!\wr"] \\ 0 \arrow[r, hook] & V & V \setminus 0 \arrow[l, hook'] \end{tikzcd}}$$

$N_{0/V} = T_{V,0} = V$

$\mathbb{P}(N_{0/V}) = \mathbb{P}V$

$$\mathrm{Bl}_0 V = (V \times \mathbb{P}V)_{inc} \subset V \times \mathbb{P}V$$

$$(V \times \mathbb{P}V)_{inc} = \{(v,\ell) \mid v \in \ell\}, \qquad \mathrm{rk}(v|\ell) \leq 1$$

$$V \times \mathbb{P}V = \left\{(v,\ell) \,\middle|\, \begin{matrix} v \in V \\ \ell \subseteq V \text{ line} \end{matrix}\right\}$$

<u>Check</u>: $\cdot$ $\mathrm{Bl}_0 V\big|_0 = \{(0,\ell) \mid 0 \in \ell\} = \{\ell \subseteq V\} = \mathbb{P}V$.

$$(\mathrm{Bl}_0 V)\big|_{V \setminus 0} = \left\{(v,\ell) \,\middle|\, \begin{matrix} v \neq 0 \\ v \in \ell \end{matrix}\right\} \cong V \setminus 0.$$

<u>Rmk</u>:

$$\boxed{\begin{tikzcd} \mathrm{Tot}(\mathcal{O}_{\mathbb{P}V}(-1)) = \mathrm{Bl}_0 V \arrow[r] \arrow[d, "\pi"'] & \mathbb{P}V \\ V & \end{tikzcd}}$$

is a line bundle

What are blow-ups good for?

- Resolution of singularities (Hironaka)

Any (singular) variety becomes non-singular after finitely many blow-ups.

$$\underbrace{Bl_{Z_n} Y_n}_{\text{non-singular}} \to \cdots \to Bl_{Z_1} Y_1 \to Y_1 = Bl_{Z_0} Y_0 \to \underset{\substack{\uparrow \\ \text{singular}}}{Y_0}$$

Try this for $Y = \{ (x,y) \mid p(x,y) = 0 \}$

with $p(x,y) = x^2 - y^3,\ x^2 - y^4, \ldots$

- Birational geometry

Definitions: 1) Y_1, Y_2 (varieties (so, irreducible)) are birational ($Y_1 \overset{bir}{\sim} Y_2$) if

$$Y_1 \underset{open}{\supseteq} U_1 \xrightarrow[\cong]{\varphi} U_2 \subseteq_{open} Y_2$$

for some U_1, U_2, φ.

2) Y is rational if $Y \overset{bir}{\sim} \mathbb{P}^n$ ($\Leftrightarrow Y \sim \mathbb{A}^m$).

Examples

(1) $Q^3 = \{[x_0 : \cdots : x_4] \in \mathbb{P}^4 \mid x_0x_4 + x_1x_3 + x_2^2 = 0\}$ is rational.

$$Q^3 \supseteq Q^3 \cap \{x_0 \neq 0\} = \{(x_1, x_2, x_3, -x_1x_3 - x_2^2)\} = \mathbb{A}^3 \subseteq \mathbb{P}^3$$

(2) A smooth proj curve Σ_g is rational $\Leftrightarrow g = 0$.

(3) $Bl_Z Y \overset{bir}{\sim} Y$ ($Bl_Z Y \setminus E \cong Y \setminus Z$).

Thm (Week Factorization)

[Abramovich–Karu–Matsuki–Wlodarczyk]

Let X, Y be two smooth projective varieties. Then $X \overset{bir}{\sim} Y$ iff $\exists$ a diagram

$$\begin{tikzcd}[column sep=small] & W_{01} \arrow[dl] \arrow[dr] & & W_{12} \arrow[dl] \arrow[dr] & & \cdots & & W_{n-1,n} \arrow[dl] \arrow[dr] & \\ X = X_0 & & X_1 & & X_2 & \cdots & X_{n-1} & & X_n = Y \end{tikzcd}$$

with each arrow a blow-up.

Examples:

$$\begin{tikzcd}[column sep=small] & Bl_{?}(\mathbb{P}^1 \times \mathbb{P}^1) \cong Bl_{?}(\mathbb{P}^2) \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \mathbb{P}^2 \end{tikzcd}$$

Same for

$$\begin{tikzcd}[column sep=small] & Bl_{pt} Q^n \simeq Bl_{?} \mathbb{P}^n \arrow[dl] \arrow[dr] & \\ Q^n & & \mathbb{P}^n \end{tikzcd}$$

Cor if a quantity is invariant under blow-ups, then it's a birational invariant. (e.g. $\dim H^{p,0}(X)$)

Definition 1 (Algebraic) "what else can it be?"

$Z \xhookrightarrow{i} Y$ closed. Want $Bl_Z Y \longrightarrow Y$.

$$\begin{array}{ccc} Z & \hookrightarrow & Y \\ \| & & \| \\ \operatorname{Spec}(A/I) & & \operatorname{Spec} A \end{array} \qquad I \subseteq A \twoheadrightarrow A/I$$

$Bl_Z Y \longrightarrow Y$ is projective

$$\begin{array}{ccc} Bl_Z Y & \longrightarrow & Y \\ \| & & \| \\ \operatorname{Proj}(R) & \longrightarrow & \operatorname{Spec} A \end{array}$$

R is a graded A-algebra.

$$R = A \oplus I \oplus I^2 \oplus I^3 \oplus \cdots = \bigoplus_{n \geq 0} I^n$$

Called the <u>Rees algebra</u> of I

$x \in I^k$, $y \in I^h$ $\quad x \cdot y \in I^{k+h}$.

$$\boxed{\text{Check the cases } Bl_Y(Y) = \emptyset, \ Bl_\emptyset Y = Y}$$

Verify properties

$$(\mathrm{Bl}_Z Y)\big|_Z = \mathrm{Bl}_Z Y \times_Y Z \overset{?}{=} \mathbb{P}(N_{Z/Y})$$

(*) when Z, Y are smooth.

$$\mathrm{Proj}\Big(\bigoplus_{n\geq 0} I^n\Big) \times_{\mathrm{Spec}\,A} \mathrm{Spec}(A/I) =$$

$$= \mathrm{Proj}\Big(\bigoplus_{n\geq 0}\Big(I^n \otimes_A A/I\Big)\Big) = \mathrm{Proj}\Big(\bigoplus_{n\geq 0} I^n/I^{n+1}\Big).$$

with (*) $\quad I^n/I^{n+1} = \mathrm{Sym}^n_{A/I}(I/I^2)$ $\qquad A/I \curvearrowright I/I^2 \to$ conormal bundle.

now $\mathrm{Proj}\big(\mathrm{Sym}^n_{A/I}(I/I^2)\big) = \mathbb{P}(N_{Z/Y})$

$$\boxed{\begin{array}{l} (\mathrm{Bl}_Z Y)\big|_U \simeq U \quad \text{for any } U \subset Y \text{ open}, \; U \cap Z = \emptyset. \\[2ex] U = \mathrm{Spec}\,B \hookrightarrow \mathrm{Spec}\,A \end{array}}$$

Unpack this definition

Choose generators of I, $I = (f_1, f_2, \dots, f_k)$, so

$Z = Z(f_1, \dots, f_k)$, $f_j : Y \longrightarrow \mathbb{A}^1$.

Then the surjection of A-modules

$$A^{\oplus k} \longrightarrow\!\!\!\!\rightarrow I$$
$$(a_1, \dots, a_k) \longmapsto \sum a_j f_j.$$

gives a closed embedding

$$Bl_Z Y \boxed{\hookrightarrow} \left(Y \times \mathbb{P}^{k-1}\right) \underset{inc}{\hookrightarrow} \left(Y \times \mathbb{P}^{k-1}\right)$$

$$\left(Y \times \mathbb{P}^{k-1}\right) = \left\{ (y, [X_1 : \dots : X_k]) \;\middle|\; rk \begin{bmatrix} X_1 & \dots & X_k \\ f_1(y) & \dots & f_k(y) \end{bmatrix} \leq 1 \right\}$$

1) closure of a graph

2) charts

3) universal property

Description 1 (Geometric)

$$Z \hookrightarrow Y$$

$$\begin{array}{l} Z \\ \shortparallel \\ Z(f_1, \dots, f_k) \end{array}$$

$$Y \setminus Z \xrightarrow{\quad \Phi \quad} \mathbb{P}^{k-1}$$

$$y \longmapsto [f_1(y) : \cdots : f_k(y)]$$

$$\mathrm{Graph}(\Phi) \subseteq (Y \setminus Z) \times \mathbb{P}^{k-1} \underset{\text{open}}{\subseteq} Y \times \mathbb{P}^{k-1}$$

$$Bl_Z Y \cong \text{closure of } \mathrm{Graph}(\Phi).$$

Description 2 (charts)

$$Bl_Z Y \hookrightarrow (Y \times \mathbb{P}^{k-1})_{inc} = \left\{ (y, [X_1 : \cdots : X_k]) \;\middle|\; rk \begin{bmatrix} X_1 & \cdots & X_k \\ f_1(y) & & f_k(y) \end{bmatrix} \leq 1 \right\}$$

Describe the chart $Bl_Z Y \cap \{X_1 \neq 0\}$

$Bl_Z Y \cap \{X_1 \neq 0\}$

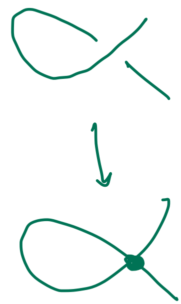
$$= \operatorname{Spec}\left(\left(A[x_2, x_3, \dots, x_k] \Big/ \begin{pmatrix} f_2 = x_2 f_1 \\ f_3 = x_3 f_1 \\ \vdots \\ f_k = x_k f_1 \end{pmatrix} \right) \Big/ (f_1\text{-tors}) \right)$$

$r \in R \qquad (r\text{-tors}) = \{ s \in R \mid s \cdot r^m = 0 \text{ for some } m \geq 1 \}$

What are we doing? We're making $I = (f_1, \dots, f_k)$ into a <u>principal ideal</u> generated by a non-zero divisor.

Example Blow-up

$$Y = \{ (x,y) \in \mathbb{A}^2 \mid y^2 = x^3 + x^2 \} \hookleftarrow Z = \{(0,0)\}$$



$$I = (x,y) \qquad \begin{matrix} x = f_1 \\ y = f_2 \end{matrix}$$

$$Y \setminus 0 \xrightarrow{\Phi} \mathbb{P}^1$$

$$(x,y) \longmapsto [x:y].$$

$$\mathrm{Graph}(\Phi) = \left\{ ((x,y),[X_1:X_2]) \;\middle|\; \begin{matrix} xX_2 - yX_1 = 0 \\ y^2 = x^3 + x^2 \end{matrix} \right\}$$

$$\overline{\mathrm{Graph}(\Phi)} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Chart $X_1 \neq 0$

$$\left(\mathbb{C}[x,y,\underset{X_2/X_1}{\underset{\shortparallel}{b}}] \middle/ \left(\begin{matrix} y = bx \\ b^2x^2 = x^3 + x^2 \end{matrix} \right) \right) \Big/ x\text{-tors.}$$

$$\mathrm{Bl}_0 Y \cap \{X_1 \neq 0\} \cong$$

$$\left\{ (x,y,b) \in \mathbb{A}^3 \;\middle|\; \begin{matrix} y = bx \\ b^2 = x+1 \end{matrix} \right\} = \{(b^2-1, b^3-b, b)\}$$

Observe:

$$\mathrm{Bl}_0 Y \cap \{X_1 = 0\} = \emptyset.$$

Description 3 (Universal property)

A closed subvariety $D \hookrightarrow Y$ is an <u>effective Cartier divisor</u> if $D = Z(s)$ for s a nonzero section of a line bundle on Y.

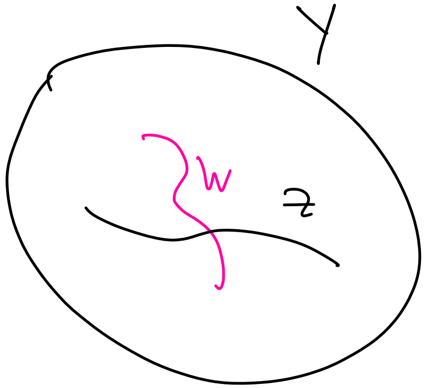
Universal property: Given $S \xrightarrow{\psi} Y$ with $\psi^{-1}(Z) \hookrightarrow S$ is an effective Cartier divisor,

then

$$\begin{tikzcd} & Bl_Z Y \arrow[d, "\pi"] \\ S \arrow[ur, dashed, "\exists !"] \arrow[r, "\psi"'] \arrow[ur, phantom, "\circlearrowright", near end, shift right=3] & Y \end{tikzcd}$$

Corollary

Given



$$\begin{tikzcd} \mathrm{Bl}_{W \cap Z} W \arrow[r, dashed, "\exists!"'] \arrow[d] & \mathrm{Bl}_Z Y \arrow[d] \\ W \arrow[r, hook] & Y \end{tikzcd}$$

$W \cap Z \hookrightarrow W$ closed.

Construct the $\dashrightarrow$ map.

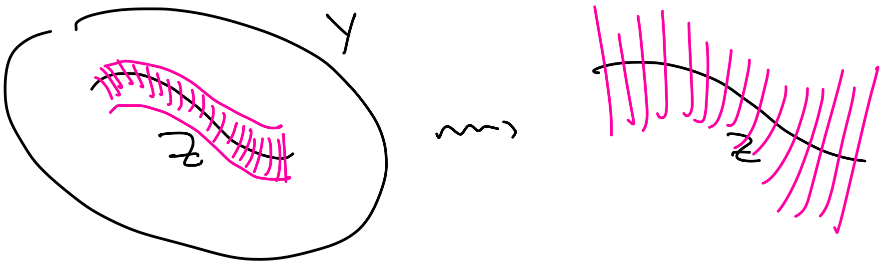
More generally,

$\forall W \to Y$, $\exists!$

$$\begin{tikzcd} \mathrm{Bl}_{W \times_Y Z} W \arrow[r, dashed] \arrow[d] & \mathrm{Bl}_Z Y \arrow[d] \\ W \arrow[r] & Y. \end{tikzcd}$$

Deformation to the Normal cone

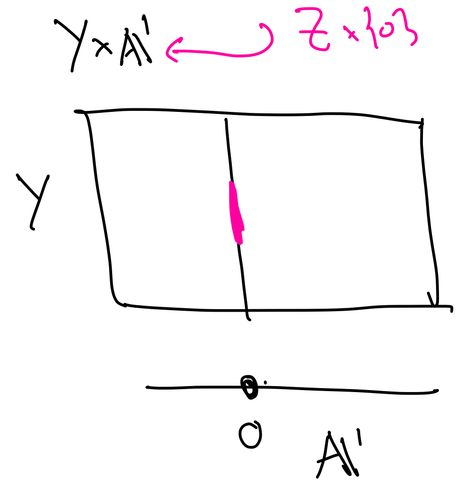
$$Z \hookrightarrow Y \rightsquigarrow Z \hookrightarrow \mathrm{Tot}(N_{Z/Y}).$$



Will construct an $\mathbb{A}^1$-family $\mathcal{D} \to \mathbb{A}^1$ such that

$$\mathcal{D}\big|_{\lambda} \cong \begin{cases} Y & \text{if } \lambda \neq 0 \\ \mathrm{Tot}(N_{Y/Z}) & \text{if } \lambda = 0 \end{cases} \qquad (\lambda \in \mathbb{A}^1)$$

Consider $\mathrm{Bl}_{Z\times 0}(Y\times \mathbb{A}^1)$.



Show $\quad \mathrm{Bl}_Z Y \hookrightarrow \mathrm{Bl}_{(Z\times 0)}(Y\times\mathbb{A}^1)$,

as a closed embedding.

Define $\mathcal{D} := \mathrm{Bl}_{(Z\times 0)}(Y\times\mathbb{A}^1) \setminus \mathrm{Bl}_Z Y$

$$\begin{array}{c}\mathcal{D}\\ \downarrow\\ \mathbb{A}^1\end{array}$$

Show that the open embedding $\mathcal{D} \subseteq \mathrm{Bl}_{(Z\times 0)}(Y\times\mathbb{A}^1)$

corresponds to a special chart of the blow-up

Compute $\mathcal{D}$ for $Y = SL_2 \hookleftarrow Z = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and see that $\mathcal{D}|_0 = sl_2$ (the lie algebra).

Compute $\mathcal{D}$ for $Y = \operatorname{Spec} \mathbb{Z} \hookleftarrow Z = \operatorname{Spec}(\mathbb{F}_p)$.