Blow ups

(over $\mathbb{C}$)

Reminder on affine varieties / Spec

"Ring" := finitely generated unital $\mathbb{C}$-algebra

$$
\begin{array}{ccc}
\text{Affine schemes (varieties)}/\mathbb{C} & \longleftrightarrow & \text{Rings (without nilpotents)} \\
V & \longmapsto & \mathcal{O}(V) = \{\text{polynomials on } V\} \\
\operatorname{Spec} R := \{\text{max ideals in } R\} & \longleftarrow\!\shortmid & R \\
x \in V & \longleftrightarrow & \text{max ideal } \mathfrak{m}_x \subset R \\
x \in V & \longleftrightarrow & \mathrm{ev}_x \colon R \to \mathbb{C} \\
\text{affine space } W & \longleftrightarrow & \operatorname{Sym} W^* \\
\text{in coords } \mathbb{C}^n & \longleftrightarrow & \mathbb{C}[x_1, \ldots, x_n] \\
\{p_1(x) = 0 = \ldots = p_k(x)\} & \longleftrightarrow & \mathbb{C}[x_1, \ldots, x_n]/(p_1, \ldots, p_k)
\end{array}
$$

"Reminder" on projective varieties / Proj

Replace rings by graded rings $\Longleftrightarrow$ rings with $\mathbb{C}^*$-action.

$\Longleftrightarrow$ Spec R has a $\mathbb{C}^*$-action $\Longleftrightarrow$ it is a cone.

Proj R is the lines through the origin in Spec R.

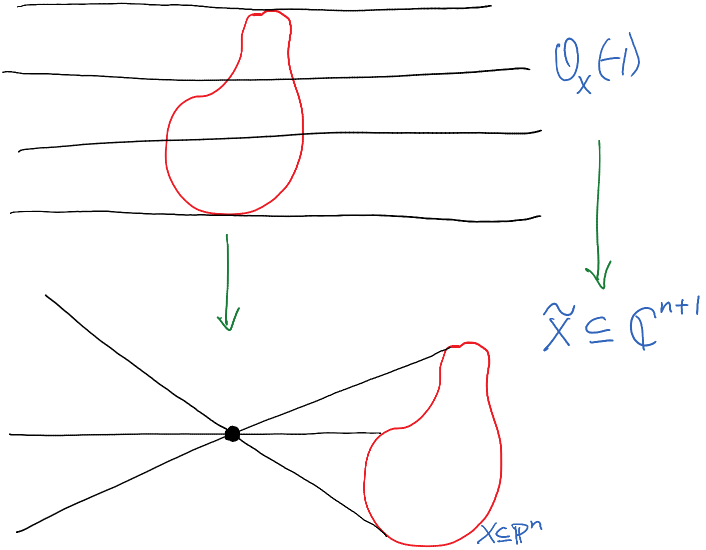
Picking (homogeneous) generators and relations,
$R = \mathbb{C}[x_0, \dots, x_n]/\big(p_1(\underline{x}), \dots, p_k(\underline{x})\big)$, we have

$$\widetilde{X} \ := \ \text{Spec } R \ = \ \big\{p_1 = 0 = \cdots = p_k\big\} \ \subseteq \ \mathbb{C}^{n+1}$$

and $X :=$ Proj R is the set of lines in this, inside the set $\mathbb{P}^n$ of lines in $\mathbb{C}^{n+1}$. In coordinates Proj $R \subset \mathbb{P}^n$ is

$$\big\{[\underline{x}] \in \mathbb{P}^n \ : \ p_i(\underline{x}) = 0 \ \forall i\big\}.$$

The picture


$\mathcal{O}_X(-1)$
$\tilde{X} \subseteq \mathbb{C}^{n+1}$
$X \subseteq \mathbb{P}^n$

Coordinates

The linear functions x_i downstairs pull back to give functions on the (total space of the) tautological line bundle $\mathcal{O}_X(-1)$ upstairs.

They're linear on the fibres, i.e. sections of its dual $\mathcal{O}_X(1)$.

Similarly homogeneous degree d polynomials $f_d(\underline{x})$ in the x_i pullback to functions on $\mathcal{O}_X(-1)$ which have degree d on the fibres, so they're sections of $\mathcal{O}_X(1)^{\otimes d} =: \mathcal{O}_X(d)$.

Thus the graded ring $R = \bigoplus_{d \geq 0} R_d$ is the space of sections of powers of the line bundle $L := \mathcal{O}_X(1)$. That is, $R_d = \Gamma(L^d)$.

$$
\begin{array}{ccc}
\text{Projective polarised schemes} & \longleftrightarrow & \text{Graded rings} \\
(X, L) & \longmapsto & \bigoplus_{d \geq 0} \Gamma(L^d) \\
\operatorname{Proj} R := \{\max^* \text{ hom ideals in } R\} & \longleftarrow\!\!\shortmid & R
\end{array}
$$

where $\max^*$ means maximal amongst homogeneous ideals which are not the irrelevant maximal ideal (of the origin) $\bigoplus_{d>0} R_d$.

Blow up: local model

Consider the projection

$$\begin{array}{ccc} \mathbb{C}^2\backslash\{0\} & \longrightarrow & \mathbb{P}^1, \\ (x,y) & \longmapsto & [x:y]. \end{array}$$

Let X, Y be homogeneous coordinates on $\mathbb{P}^1$. (So only defined up to scale. X/Y well defined function but has a pole.)

Equation of graph inside $\left(\mathbb{C}^2\backslash\{0\}\right) \times \mathbb{P}^1$ is ("$X/Y = x/y$")

$$xY = Xy.$$

So closure is

$$\left\{xY = Xy\right\} \subset \mathbb{C}^2 \times \mathbb{P}^1. \qquad (*)$$

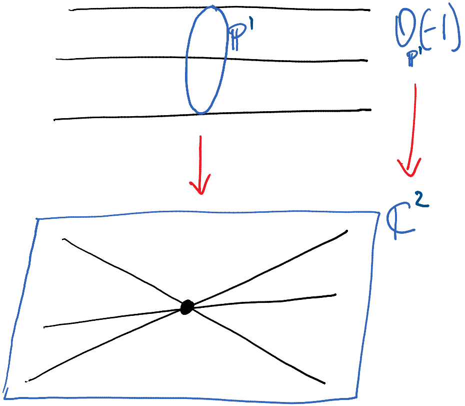
Over each point each point of $\mathbb{P}^1$ we get all points in the corresponding line. I.e. $(*)$ is the tautological line bundle

$$\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

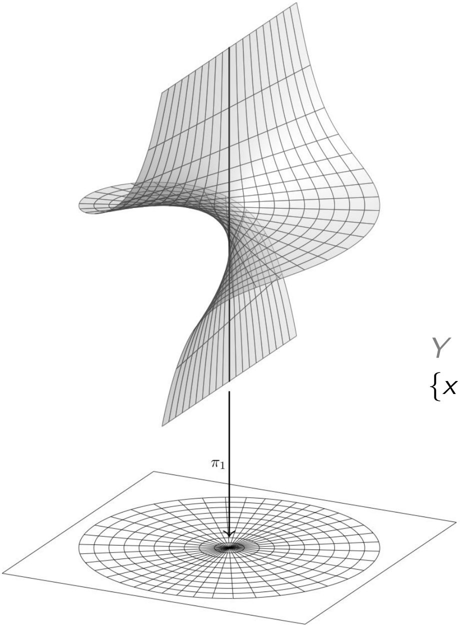
Other projection

$$\mathcal{O}_{\mathbb{P}^1}(-1) \;=\; \{xY = Xy\} \;\subset\; \mathbb{C}^2 \times \mathbb{P}^1.$$

Projecting to $\mathbb{C}^2$ instead of $\mathbb{P}^1$ we find gives an isomorphism away from 0 ($(x, y) \neq (0, 0)$ determines $[X, Y]$) but the whole $\mathbb{P}^1$ over 0.



A more professional picture



$Y = 1 \implies$
$\{x = Xy\} \subset \mathbb{C}^2_{xy} \times \mathbb{C}_X$.

Blow up

$$\mathcal{O}_{\mathbb{P}^1}(-1) \;=\; \{xY = Xy\} \;\subset\; \mathbb{C}^2 \times \mathbb{P}^1$$

is called the blow up of $\mathbb{C}^2$ in the origin.

At the origin it remembers the line you came in on. Therefore separates lines at the origin.

Inverse image $\mathbb{P}^1$ of $0 \in \mathbb{C}^2$ (zero section of $\mathcal{O}_{\mathbb{P}^1}(-1)$) is called the **exceptional divisor** or *exceptional curve* or (-1)-*curve* E.

More generally

$$\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \;=\; \{x_i Y_j = X_i y_j \;\; \forall i, j\} \;\subset\; \mathbb{C}^n \times \mathbb{P}^{n-1}$$

is called the blow up of $\mathbb{C}^n$ in the origin.
Exceptional divisor now $E \cong \mathbb{P}^{n-1}$.

Global

Gluing this model into any complex manifold defines $\mathrm{Bl}_p X$ with exceptional divisor $E \cong \mathbb{P}(T_p M)$.

More generally given a codimension-n submanifold $Z \subset X$ can form $\mathrm{Bl}_Z X$ by a family version of the same construction.

Locally analytically $Z \subset X$ looks like $U \times \{0\} \subset U \times \mathbb{C}^n$.
($U \subset Z$ open.)

The blow up $\mathrm{Bl}_Z X$ is then locally $U \times \mathrm{Bl}_0 \mathbb{C}^n$ and these glue on overlaps as U covers Z. Exceptional divisor is now

$$\begin{tikzcd} E \cong \mathbb{P}(N_Z X) \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Z X \arrow[d] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

Soon we will see a quicker, more direct construction that blows up any *schemes* $Z \subset X$.

Exercises

Ex: Show topologically same as: remove small ball $0 \in B^{2n} \subset \mathbb{C}^n$, divide its boundary by scalar action of $S^1 \subset \mathbb{C}^*$ to collapse S^{2n-1} to $\mathbb{P}^{n-1}$ by Hopf map.

Ex: Show topologically same as connect sum at $0 \in \mathbb{C}^n$ with $\overline{\mathbb{P}^n}$
(Note opposite orientation on $\mathbb{P}^n$ turns normal bundle of hyperplane $\mathbb{P}^{n-1}$ from $\mathcal{O}(1)$ to $\mathcal{O}(-1)$.)

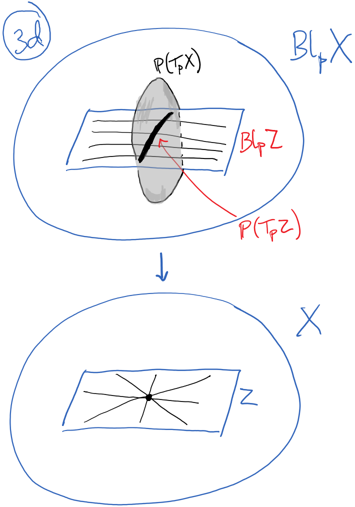
Ex: Do real blow up of $0 \in \mathbb{R}^2$. Remove small disc $0 \in D^2 \subset \mathbb{R}^2$, divide boundary S^1 by ± 1 antipodal map. (So in and out get flipped $\implies$ not oriented.)
Show same as gluing in a Möbius band along its boundary S^1.

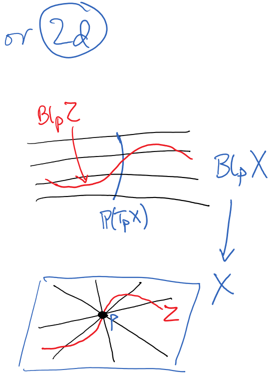
Functoriality

Fix $p \in Z \subset X$ complex manifolds.

Key property: $\mathrm{Bl}_p Z$ is the proper transform of Z in $\mathrm{Bl}_p X$,

$$\mathrm{Bl}_p Z \;=\; \overline{Z \setminus \{p\}} \;\subset\; \mathrm{Bl}_p X.$$





"Blow up remembers direction Z comes into p"

Proper transform

Enough to prove this locally analytically for

$$0 \in \mathbb{C}^m = \{x_{m+1} = 0 = \ldots = x_n\} \subset \mathbb{C}^n.$$

We want to describe $\overline{\mathbb{C}^m \backslash \{0\}} = \overline{\{x_{m+1} = 0 = \ldots = x_n\}}$ inside

$$\mathrm{Bl}_0\,\mathbb{C}^n = \{x_i X_j = x_j X_i \ \ \forall i,j\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}.$$

At any point of $\mathrm{Bl}_0\,\mathbb{C}^n$ at least one X_i is nonzero (X_1 say), so

$$x_i = \frac{x_1}{X_1} X_i$$

so $(X_{m+1} = 0 = \ldots = X_n) \implies (x_{m+1} = 0 = \ldots = x_n)$.

Similarly **away from** $0 \in \mathbb{C}^n$ one x_i is nonzero (x_1 say), so

$$X_i = \frac{X_1}{x_1} x_i,$$

so $(x_{m+1} = 0 = \ldots = x_n) \implies (X_{m+1} = 0 = \ldots = X_n)$.

Proper transform II

So on all of $\mathrm{Bl}_0\,\mathbb{C}^n$ we have

$$(X_{m+1} = 0 = \ldots = X_n) \implies (x_{m+1} = 0 = \ldots = x_n) \qquad (1)$$

while on $\mathbb{C}^n \backslash \{0\} \ = \ \mathrm{Bl}_0\,\mathbb{C}^n \backslash E$ we have

$$(x_{m+1} = 0 = \ldots = x_n) \implies (X_{m+1} = 0 = \ldots = X_n). \qquad (2)$$

By (2), $X_{m+1} = 0 = \ldots = X_n$ on $\mathbb{C}^m \backslash \{0\} \subset \mathrm{Bl}_0\,\mathbb{C}^n$ and therefore also on $\overline{\mathbb{C}^m \backslash \{0\}} \subset \mathrm{Bl}_0\,\mathbb{C}^n$.

By (1) then, $\overline{\mathbb{C}^m \backslash \{0\}} \subset \mathrm{Bl}_0\,\mathbb{C}^n$ is precisely $X_{m+1} = 0 = \ldots = X_n$.

This just cuts $\mathbb{P}^{n-1}$ down to $\mathbb{P}^{m-1}$ and leaves the same blow up equations $x_i X_j = x_j X_i$ intact, giving $\mathrm{Bl}_0\,\mathbb{C}^m$ as claimed.

Upshot is we can define $\mathrm{Bl}_p\,Z$ by (1) embedding Z in some ambient space X (like $\mathbb{C}^N$ or $\mathbb{P}^N$) then (2) taking the proper transform $\overline{Z} := \overline{Z \backslash p}$ inside $\mathrm{Bl}_p\,X$.

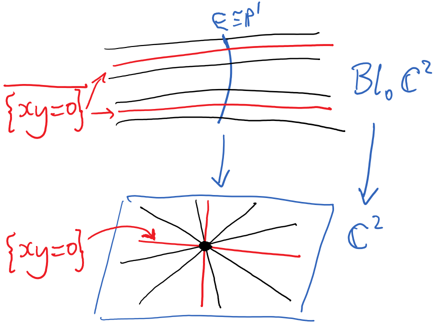
Singularities

So for now we can define blow up of singular varieties by embedding and proper transform.

So to blow up the node $\{xy = 0\} \subset \mathbb{C}^2$ we take

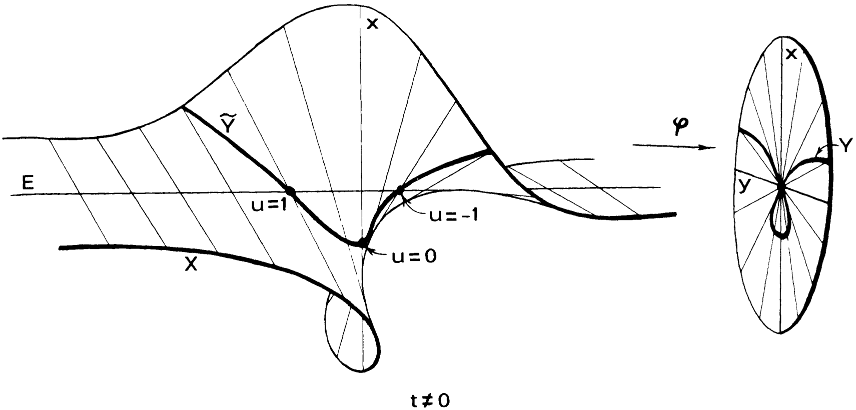
$$\overline{\{xy = 0\}\backslash\{0\}} \ \subset \ \mathrm{Bl}_0 \, \mathbb{C}^2.$$

Ex: As before ($xy = 0 \implies XY = 0$) while, away from origin, ($XY = 0 \implies xy = 0$). So proper transform is $XY = 0$.



"Resolution of singularities"
Note (pullback of) $xy = 0$ contains $2E$.

Hartshorne's picture



$$Y = 1 \implies \quad \{x = Xy\} \subset \mathbb{C}^2_{xy} \times \mathbb{C}_X.$$

Ex: What is proper transform $\overline{Y}$ of black curve $Y = \{y^2 = (x-1)x^2\}$?

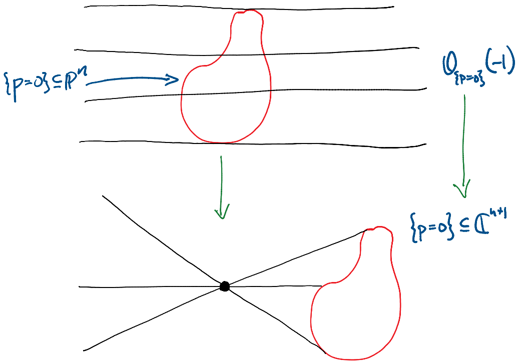
More singularities

More generally consider the cone (singular at 0) $\{p = 0\} \subset \mathbb{C}^n$, where p is a homogeneous polynomial.

Blowing this up (taking proper transform in $\mathrm{Bl}_0\,\mathbb{C}^n$) gives

$$\mathcal{O}(-1) \longrightarrow \Big\{ \{p = 0\} \subset \mathbb{P}^{n-1} \Big\}.$$

"Cylinder resolution of cone on $\{p = 0\} \subset \mathbb{P}^{n-1}$."



Yet more singularities

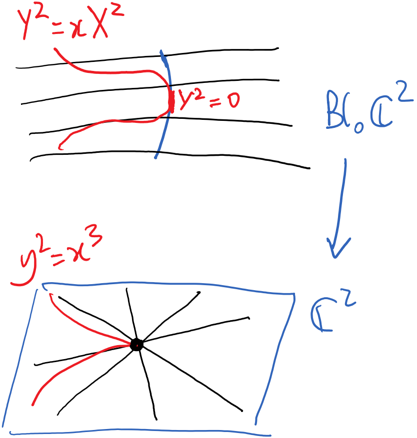
Ex: Do this for $\{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3$. What is the exceptional curve? What is its normal bundle or self-intersection?

For nonhomogeneous p let P denote its leading order homogeneous part. (E.g. $p = x^2 + y^3 \implies P = x^2$.)

Then blow up of $\{p = 0\}$ in 0 need not be smooth but
Ex: its exceptional divisor is $\{P = 0\} \subset \mathbb{P}^{n-1}$.

More curve singularities

E.g. Blowing up the cusp $y^2 + x^3 = 0$ gives exceptional divisor the double point $\{Y^2 = 0\} \subset \mathbb{P}^1$.



Ex: Blow up is $Y^2 = xX^2$ and is smooth.

Resolution of singularities

Often have to blow up many times to **resolve** singularity (i.e. get something smooth).

Ex: Draw $y^2 = x^4$ and its blow up. Show resolved by two blow ups.

Ex: Invent your own curve singularities and resolve them by iterated blow ups in points.

Theorem (Hironaka) Given any variety X we may iteratively blow it up in **smooth** centres ($Z_1 \subset X$, then $Z_2 \subset \mathrm{Bl}_{Z_1} X$, then $\ldots$) so that after a finite number of steps the result is smooth.

Functions

Back to local model

$$\begin{tikzcd} E \cong \mathbb{P}^{n-1} \arrow[r, hook] \arrow[d] & \mathrm{Bl}_0\, \mathbb{C}^n \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{C}^n. \end{tikzcd}$$

Functions on $\mathbb{C}^n$ which vanish at 0
(ideal $\mathcal{I}_0 = (x_1, \ldots, x_n) \subset \mathbb{C}[x_1, \ldots, x_n]$)
pull back to give functions on $\mathrm{Bl}_0\, \mathbb{C}^n$ which vanish on E
(global sections of ideal sheaf $\mathcal{I}_E$.)

Ex: This is all of them: $\Gamma(\mathcal{I}_E) = \mathcal{I}_0$. More generally $\Gamma(\mathcal{I}_E^k) = \mathcal{I}_0^k$.
(Hint: compare short exact sequences $0 \to \mathcal{I}_E \to \mathcal{O}_{\mathrm{Bl}_0 \mathbb{C}^n} \to \mathcal{O}_E \to 0$ and $0 \to \mathcal{I}_0 \to \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}_0 \to 0$.)

Sections of line bundles

"Recall" the line bundle-divisor correspondence $E \longleftrightarrow (\mathcal{O}(E), s_E)$.
Sections of $\mathcal{I}_E$ are the same as sections of the line bundle $\mathcal{O}(-E) := \mathcal{O}(E)^*$:

$$\Gamma(\mathcal{I}_E) \underset{\cdot s_E}{\overset{\cdot/s_E}{\rightleftarrows}} \Gamma(\mathcal{O}(-E)).$$

Combining the two we get an isomorphism

$$\Gamma(\mathcal{O}_{\mathrm{Bl}_0 \mathbb{C}^n}(-kE)) \;\cong\; \mathcal{I}_0^k.$$

(This is "familiar" from the Proj lecture. $\mathcal{O}(-kE)$ is the pullback from $\mathbb{P}^{n-1}$ of $\mathcal{O}_{\mathbb{P}^{n-1}}(k)$ and $\Gamma\big(\mathcal{O}_{\mathbb{P}^{n-1}}(k)\big) = \big\langle X_1^k, X_1^{k-1}X_2, \ldots, X_n^k\big\rangle$. Therefore sections of its pullback are the product of these with the functions $\mathbb{C}[x_1, \ldots, x_n]$ on the fibres. The result is isomorphic to $\big\langle x_1^k, x_1^{k-1}x_2, \ldots, x_n^k\big\rangle\mathbb{C}[x_1, \ldots, x_n] = (x_1, \ldots, x_n)^k = \mathcal{I}_0^k$.)

Proj construction

But sections of (all powers of an ample) line bundle $\mathcal{O}(-E)$ on $\mathrm{Bl}_0\,\mathbb{C}^n$ determine $\mathrm{Bl}_0\,\mathbb{C}^n$ by the Proj construction, so

$$\mathrm{Bl}_0\,\mathbb{C}^n \;=\; \mathrm{Proj}\bigoplus_{k\geq 0}\Gamma(\mathcal{O}_{\mathrm{Bl}_0\,\mathbb{C}^n}(-kE)) \;=\; \mathrm{Proj}\bigoplus_{k\geq 0}\mathcal{I}_0^k.$$

So this gives a global general way to define a blow up

$$\mathrm{Bl}_Z\,X \;:=\; \mathrm{Proj}\bigoplus_{k\geq 0}\mathcal{I}_Z^k.$$

(Really a relative version of Proj from the last lecture, over base X.)

(Since $\mathcal{I}_Z/\mathcal{I}_Z^2 = N_Z^*$ we see $\mathcal{I}_Z^k$ giving sections $\mathrm{Sym}^k N_Z^*$ of $\mathcal{O}_{\mathbb{P}(N_Z)}(k)$ on fibres of exceptional divisor $\mathbb{P}(N_Z)\to Z$.)

Let's unpack this formal definition in a simple example.

Example

Take $Z \subset X$ to be $\{0\} \subset \mathbb{C}^2$. What is Proj $\bigoplus_{k \geq 0} \mathcal{I}_Z^k$?

$\mathcal{I}_Z$ is generated by x, y over $\mathbb{C}[x, y]$; call these generators X, Y.

Then $\mathcal{I}_Z^k$ is $\langle X^k, YX^{k-1}, \ldots, Y^k \rangle \mathbb{C}[x, y]$.

Therefore $\bigoplus_{k \geq 0} \mathcal{I}_Z^k$ is generated by x, y (degree 0) and X, Y (degree 1) subject to the only relation $xY = Xy$,

$$\bigoplus_{k \geq 0} \mathcal{I}_Z^k = \mathbb{C}[x, y][X, Y]/(xY - Xy).$$

So we can read off Proj $\bigoplus_{k \geq 0} \mathcal{I}_Z^k$ to be

$$\{xY = Xy\} \subset \mathbb{C}^2 \times \mathbb{P}^1,$$

the blow up we started with.

Another example

Take $Z \subset X$ to be $\{0\} \subset \{y^2 = x^3\}$ (all in $\mathbb{C}^2$).

$\mathcal{I}_Z$ has generators $X := x$, $Y := y$ over $\mathbb{C}[x, y]/(y^2 - x^3)$ subject to the relations $xY = Xy$ and $Y^2 = xX^2$. Therefore

$$\bigoplus_{k \geq 0} \mathcal{I}_Z^k \;=\; \frac{\mathbb{C}[x, y]}{(y^2 - x^3)} \frac{[X, Y]}{(xY - Xy, Y^2 - xX^2)}\,.$$

So we can read off Proj $\bigoplus_{k \geq 0} \mathcal{I}_Z^k$. We times by $\mathbb{P}^1$, impose $xY = Xy$ so it only appears at the origin, then impose $Y^2 = xX^2$ so over the origin we only get the double point $Y^2 = 0$.

Notice $X \neq 0$ so can set it to 1. Proj becomes Spec of ring with $X = 1$. Thus can discard $x = Y^2$ and then $y = xY = Y^3$ to give

$$\text{Spec } \mathbb{C}[Y] \;=\; \mathbb{C}.$$

So blow up of cusp $\{y^2 = x^3\}$ is $\mathbb{C}$ mapping to cusp by $Y \mapsto (Y^2, Y^3)$. (Cf. an exercise from Spec lecture.)

Exercise: Castelnuevo Criterion

From last time: S projective $\iff$ has a (ample) line bundle $L \to S$ such that $S = \operatorname{Proj} \bigoplus_{k \geq 0} \Gamma(L^k)$ (Pullback of $\mathcal{O}(1)$ under $S \hookrightarrow \mathbb{P}^N$.)

Smooth projective surface $S \supset E \cong \mathbb{P}^1$ with self-intersection -1 (normal bundle $N_E = \mathcal{O}_{\mathbb{P}^1}(-1)$).

Define $d := \deg L|_E = \int_E c_1(L)$ – i.e. $L|_E \cong \mathcal{O}_{\mathbb{P}^1}(d)$.

Ex: Show $L(dE)$ is trivial on E and its sections contract E but nothing else.

I.e. $S \to \operatorname{Proj} \bigoplus_{k \geq 0} \Gamma(L^k(kdE))$ blows E down to a point p; opposite of blow up at p.

Exercise: weighted blow ups

These use **weighted projective spaces**.

E.g. consider $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ with weights $(1,2)$ (I.e. $\lambda \in \mathbb{C}^*$ acts as $\lambda(x,y) = (\lambda x, \lambda^2 y)$) and form (the space of orbits $y = ax^2$)

$$\mathbb{P}(1,2) \ := \ \big(\mathbb{C}^2 \backslash \{0\}\big)/\mathbb{C}^* \ = \ \mathrm{Proj}\ \mathbb{C}[X,Y]$$

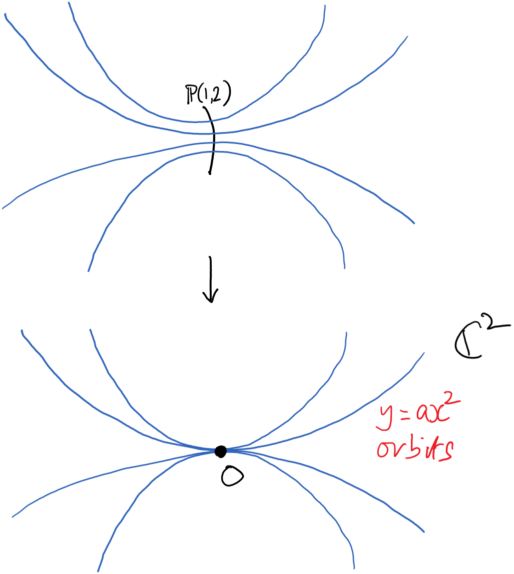
where $\deg X = 1$, $\deg Y = 2$ $(a = Y/X^2)$.

Corresponding $(1,2)$-weighted blow up of $\{0 \in \mathbb{C}^2\}$ remembers which orbit we come in on (instead of which line):

$$\{X^2 y = x^2 Y\} \ \subset \ \mathbb{P}(1,2) \times \mathbb{C}^2.$$

Ex: Write as a Proj.
Relate it to usual blow up of $\mathbb{C}^2$ in ideal (x^2, y) of fat point.
Show it's a blow down of $\mathrm{Bl}_{[1:0]}\,\mathrm{Bl}_0\,\mathbb{C}^2$.

Picture: weighted blow ups



Ex: What happens to straight lines?

Exercise: 3-fold ordinary double point

Ex: Let $X := \{xy = zw\} \subset \mathbb{C}^4$.
Blow up $\{x = 0 = z\}$; what do you get?
Do by proper transform and by Proj.

What's the exceptional locus?

Describe in terms of graph of map $X \setminus \{0\} \to \mathbb{P}^1$ given by $\frac{x}{z} = \frac{w}{y}$.

Repeat for $\{x = 0 = w\}$ and graph of $\frac{x}{w} = \frac{z}{y}$.

Ex: For $Z \subset X$ define $kZ \subset X$ to be subscheme with ideal sheaf $\mathcal{I}_{kZ} := \mathcal{I}_Z^k \subset \mathcal{O}_X$.
Show $\mathrm{Bl}_{kZ} X \cong \mathrm{Bl}_Z X$ (but with a different line bundle on it).

Exercise: all regular birational maps are blow-ups

Suppose $\pi\colon X \to Y$ is a regular map which is an isomorphism on a Zariski open subset of Y.
Suppose X is projective and Y is *normal*.
We want to show π **is a blow up**.

Ex: Pick ample line bundle L on X and $N \gg 0$.
Show $\pi_* L^N = M \otimes \mathcal{I}_Z$ for some line bundle $M \to Y$ and ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_Y$. (Hint: show $M := (\pi_* L^N)^{**}$ is locally free of rank 1.)
Show $\mathrm{Bl}_Z\, Y = X$.