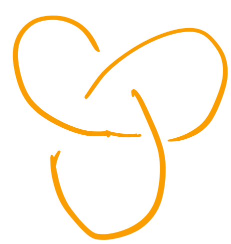
An Introduction to 3-manifolds

A **3-manifold** is a Hausdorff second countable topological space s.t. every point has a nbhd U and a homeo $\phi: U \to \mathbb{R}^3$ onto an open subset $\phi(U)$ of $\mathbb{R}^3$.

The 3-manifolds we consider are **piecewise linear** (pL) or **smooth**. This means that the change of coordinate maps are pL or smooth.

Example Let K be a knot in $S^3 = \mathbb{R}^3 \cup \{\infty\}$; i.e. K is a smooth (or pL) embedding of S^1 in S^3.

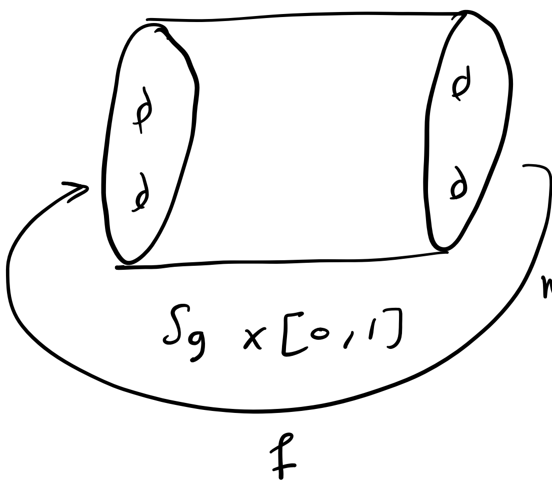
Then $S^3 \setminus K$ is a (non-compact) manifold.



K = trefoil

We can also take a tubular neighbourhood $N(K)$ of K in S^3 and consider $X := S^3 \setminus \mathring{N}(K)$, which is a compact 3-mfld with torus boundary.

Example Let S_g be a surface of genus g and $M^3 = S_g \times S^1$. More generally let $f: S_g \to S_g$ be a diffeomorphism and define $M = M_f$ as the mapping torus of f



$$M_f = \left(S_g \times [0,1]\right)/\sim$$

quotient topology

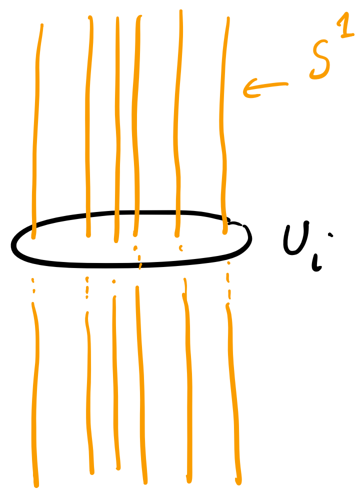
where $(x,1) \sim (f(x), 0)$

Example Let M be the total space of a fibration over S_g with fiber S^1. This means that there is a cover $\{U_i\}$ of S_g and a submersion $\pi: M \to S_g$ such that for each U_i there is a diffeo $\pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times S^1$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[dr, "\pi"'] & U_i \times S^1 \arrow[d, "pr_1"] \\ & U_i \end{tikzcd}$$

where pr_1 is projection onto the first factor.

Intuitively an S^1-bundle over S_g is a family of S^1 (fiber) parametrised by points in S_g (base).

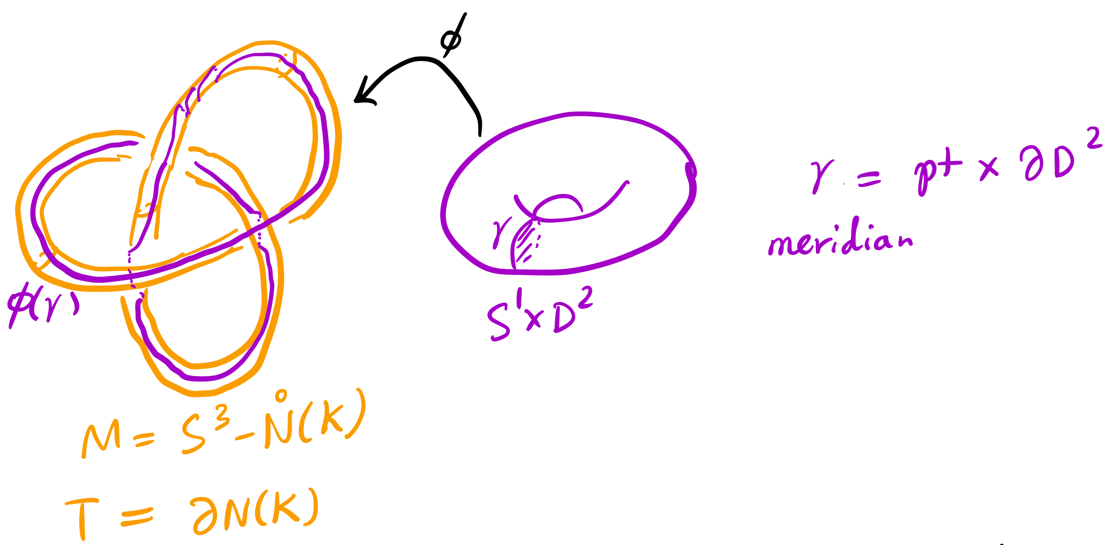


So above each point in the base (S_g) we have a copy of the fiber (S^1).

Note that $S_g \times S^1$ is a special case of an S^1-bundle over S_g, where we can take $\{U_i\} = \{S_g\}$.

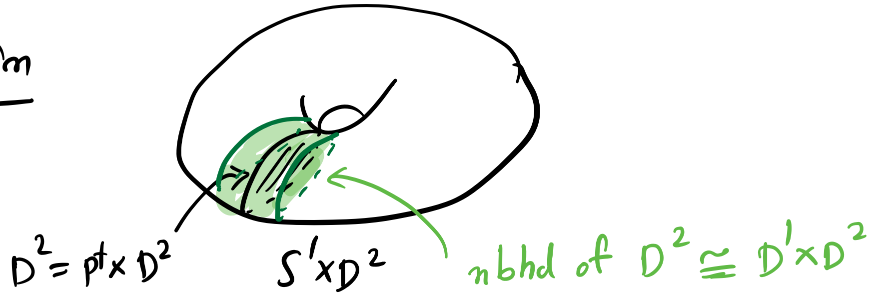
Dehn Filling

Let M be a 3-mfld and T be a torus boundary component of M. For example M could be a knot complement in S^3. Let $S^1 \times D^2$ be the solid torus. Then a <u>Dehn filling of M along T</u> is the operation of gluing $S^1 \times D^2$ to M via a diffeomorphism $\phi: \partial(S^1 \times D^2) \to T$.



<u>Claim</u> The diffeomorphism type of $M \cup_\phi (S^1 \times D^2)$ only depends on the isotopy class of $\phi(\gamma)$ in T.

proof of claim



Attaching $S^1 \times D^2$ can be done in two steps:

1) A nbhd of D^2 diffeomorphic to $D^1 \times D^2$ is attached.

   This step only depends on the isotopy class of $\phi(\mu)$.

2) The complement of $D^1 \times D^2$ in $S^1 \times D^2$ is attached.

   Note that this complement is diffeomorphic to a 3-ball D^3.

We show that if N is a 3-mfld and S is a sphere boundary component of N, and D^3 is a 3-ball, then the diffeomorphism type of $N \cup_\varphi D^3$ does not depend on the choice of attaching diffeomorphism $\partial D^3 \xrightarrow{\varphi} S$.

This is because by Smale's theorem, any self-diffeo of a sphere S^2 is isotopic to either the identity map $id: S^2 \to S^2$ or reflection map $r: S^2 \to S^2$

and both id and r extend to a diffeomorphism of the 3-ball.

Exercise 1

Let N_1, N_2 be n-mflds with boundary and assume that the boundary components $T_1 \subset \partial N_1$ and $T_2 \subset \partial N_2$ are diffeomorphic. Let $\varphi: T_1 \to T_2$ and $\varphi': T_1 \to T_2$ be diffeomorphisms that are isotopic; i.e there is a smooth map $\Phi: T_1 \times [0,1] \to T_2$ such that $\Phi|T_1 \times \{0\} = \varphi$, $\Phi|T_1 \times \{1\} = \varphi'$, and for each $t \in [0,1]$, $\Phi|T_1 \times \{t\}$ is a diffeomorphism. Show that

$$N_1 \underset{\varphi_1}{\cup} N_2 \quad \text{and} \quad N_1 \underset{\varphi_2}{\cup} N_2 \quad \text{are diffeomorphic.}$$

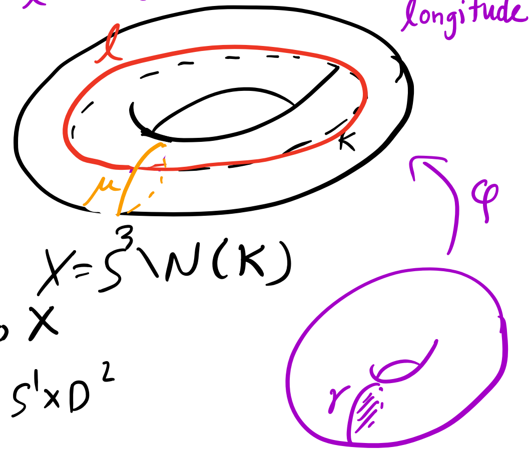
Exercise 2

Let N_1 and N_2 be as above. Let $\varphi: T_1 \to T_2$ be a diffeomorphism, and $\psi: T_2 \to T_2$ be a diffeomorphism that extends to N_2; i.e. there is $\Psi: N_2 \to N_2$ s.t. $\Psi|T_2 = \psi$. Show that $N_1 \underset{\varphi}{\cup} N_2$ and $N_1 \underset{\psi\circ\varphi}{\cup} N_2$ are diffeomorphic.

Example (Lens spaces)

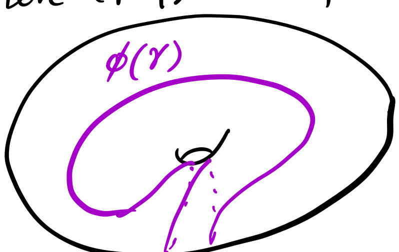
$\mu = \partial(\text{disc})$ in $N(K)$ meridian
$\ell = \partial(\text{disc})$ in $S^3 \setminus \mathring{N}(K)$ longitude

Let $K \subset S^3$ be the unknot and $N(K)$ be a tubular nbhd of K. Let $X = S^3 \setminus \mathring{N}(K)$.



Attach a solid torus $S^1 \times D^2$ to X such that the meridian γ of $S^1 \times D^2$ is sent to the curve $p\mu + q\ell$ on ∂X. Here (p,q) are coprime.

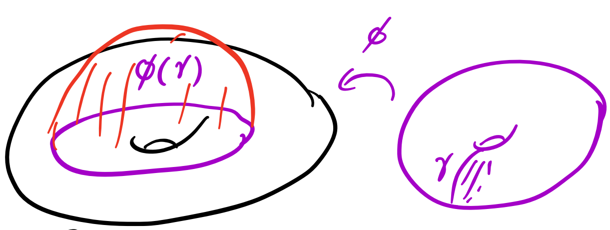
e.g.



$p = 2 \quad q = 1$

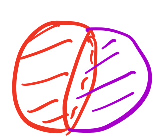
The resulting 3-mfld is called $L(p,q)$ Lens space.

For example if $(p,q) = (1,0)$ we get S^3 back.
If $(p,q) = (0,1)$ then we get $S^2 \times S^1$.



$X = S^3 \setminus N(K)$

you can see a copy of S^2 in the resulting mfld



Exercise 3 Show that $\pi_1(L(p,q)) = \mathbb{Z}/_{p}\mathbb{Z}$.
Deduce that if $L(p,q)$ and $L(p',q')$ are diffeomorphic then $p = p'$.

Exercise 4 Show that lens spaces $L(p,q)$ and $L(p,q')$ are diffeomorphic if $q' \equiv \pm q^{\pm 1} \pmod{p}$.

Note that a lens space is a union of two solid tori glued together along their boundaries.

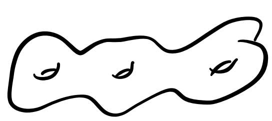
More generally:

Heegaard Decomposition

A Heegaard decomposition of a closed (i.e. compact and without boundary) 3-mfld is a decomposition of the form

$$M = H_g \underset{\phi}{\cup} H_g'$$

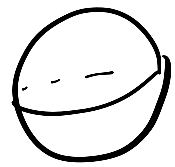
where H_g, H_g' are handle bodies of genus g

(i.e. solid  = diffeo to a regular nbhd of a connected finite graph in $\mathbb{R}^3$)

H_g

and $\phi : \underset{\substack{\cong\\ S_g}}{\partial H_g} \to \underset{\substack{\cong\\ S_g}}{\partial H_g'}$ is a diffeomorphism.

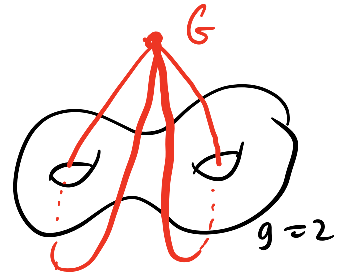
Example The 3-sphere can be written as a union of two solid 3-balls attached along their boundary.



B^3

$S^3 \setminus B^3 \cong B^3$

Similarly for every g, the complement of the standard embedding of a handlebody of genus g in S^3 is another handlebody of genus g. Hence S^3 has a Heegaard decomposition of genus g for every $g \geq 0$.

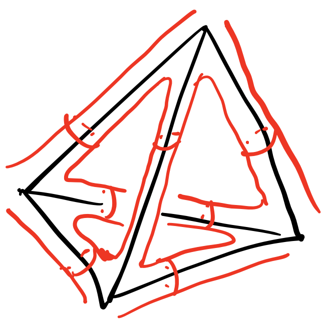


The complement deformation retracts to G.

Caution It is not true that if $f : H_g \hookrightarrow S^3$ is an arbitrary embedding of the genus g handlebody, then $S^3 \setminus f(H_g)$ is another handlebody. For example if K is any non-trivial knot (i.e. not unknot) in S^3 then $S^3 \setminus \mathring{N}(K)$ is not a genus 1 handlebody (i.e. solid torus).

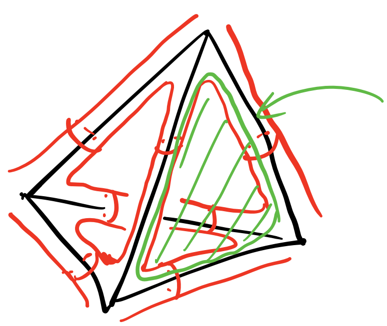
Thm Every closed orientable 3-mfld has a Heegaard decomposition.

proof We use that any such 3-mfld has a triangulation.



Define H = a regular nbhd of the 1-skeleton

H' = closure of the complement of H



one of the discs in H'

Exercise 5 prove the following characterisation of handlebodies between connected orientable 3-mflds: There is a collection of disjoint properly embedded discs in the manifold such that cutting along them produces a union of 3-balls.

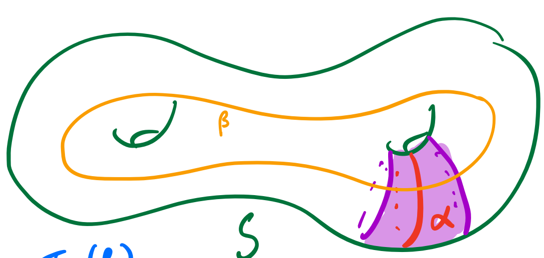
The <u>Heegaard genus</u> of a closed orientable 3-mfld M is the minimum genus of a Heegaard decomposition of M. For example the Heegaard genus of S^3 is 0, and it is the only 3-mfld with Heegaard genus 0. Moreover, the Heegaard genus of a 3-mfld is 1 if and only if it is a nontrivial (i.e. $\neq S^3$) lens space.

<u>Dehn surgery</u> Let L be a link in M^3, and $N(L)$ be a regular nbhd of L. Then Dehn surgery on L is the operation of Dehn filling $X := M \setminus \mathring{N}(L)$ along torus boundary components $\partial N(L)$.

<u>Thm</u> (Lickorish - Wallace)
Every closed orientable 3-mfld can be obtained by Dehn surgery on a link in S^3.

proof Let $H_1 \cup_\psi H_2$ be a Heegaard decomposition of M, and $\psi : \partial H_1 \to \partial H_2$ be the attaching map. By fixing an identification of H_1 and H_2 with a model handlebody H of genus g, we can think of ψ as an orientation-preserving homeomorphism $\psi : S \to S$, where $S = \partial H$ is a surface of genus g. Note that the diffeo type of M only depends on the isotopy class of ψ.

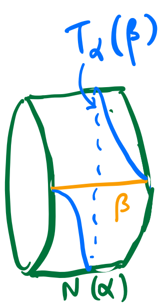
By a thm of Lickorish, every orientation-preserving diffeo of S, up to isotopy, can be written as a product of Dehn twists and their inverses.



α : simple closed curve

T_α : (right handed) Dehn twist about α

$N(\alpha)$ = regular nbhd of α $\cong \alpha \times [0,1]$



$N(\alpha) = S^1 \times [0,1]$

$T_\alpha(x,y) = (x+y, y)$

$x \in S^1 = \mathbb{R}/\mathbb{Z}$, $y \in [0,1]$

$$\begin{cases} T_\alpha | (S \setminus N(\alpha)) = id \\ T_\alpha | N(\alpha) = \text{twisting to the right} \end{cases}$$

We know that the 3-sphere also has a decomposition $H_1 \cup_{\varphi} H_2$ for some $\varphi : S \longrightarrow S$ as well.

By Lickorish thm, we can write $\psi \circ \varphi^{-1}$ as a product of Dehn twists

$$\psi \circ \varphi^{-1} = T_{\alpha_k}^{\pm 1} \circ \cdots \circ T_{\alpha_1}^{\pm 1}$$

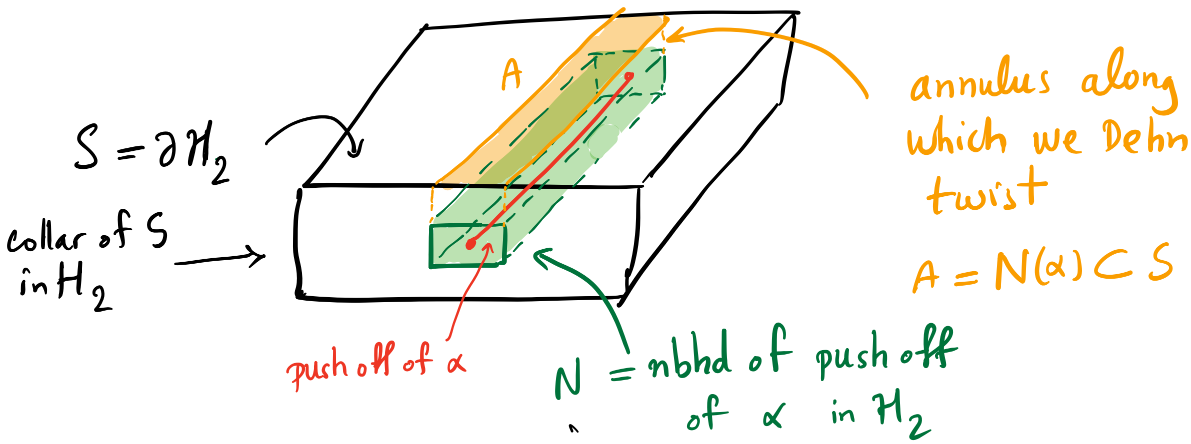
along some simple closed curves $\alpha_1, \ldots, \alpha_k \subset S$.

For simplicity assume that $k = 1$. The general case can be proved by induction on k. Hence

$$\psi \circ \varphi^{-1} = T_{\alpha_1}^{\pm 1} \quad \left(\Rightarrow \psi = T_{\alpha_1}^{\pm 1} \circ \varphi \right)$$

is a single Dehn twist or its inverse. In this special case, we show that M can be obtained by surgery on a knot in S^3 (The general case would be surgery on a k-component link)

push α inside a collar nbhd of ∂H_2 in H_2, and drill a nbhd of the push off of α.



Let $H_2^{drill} = H_2 \setminus \mathring{N}$.

The Dehn twist supported on A extends product-wise to the solid torus $A \times [0,1]$ lying between A and the drilled parallelepiped N, and extends trivially (i.e. by identity) to a self-diffeomorphism

$$T : H_2^{drill} \longrightarrow H_2^{drill}$$

such that $T|S = T_{\alpha_i}^{\pm 1}$. Therefore, T extends to a diffeomorphism

$$\begin{array}{ccccc} T : & S^3 \setminus \mathring{N} & \longrightarrow & M \setminus \mathring{N} \\ & \parallel & & \parallel \\ & H_1 \cup_{\varphi} H_2^{drill} & & H_1 \cup_{T_\alpha^{\pm 1} \circ \varphi} H_2^{drill} \end{array}$$

Hence M is obtained from S^3 by Dehn surgery along α_i. $\square$

Exercise 6 1) check the last step of the proof when T extends to $T: S^3 \setminus \mathring{N} \to M \setminus \mathring{N}$.

2) prove the theorem in the general case by induction on k. In this case define

$$M_i = H_1 \cup_{\varphi_i} H_2 \quad \text{where } \varphi_i = T_{\alpha_i}^{\pm 1} \circ \cdots \circ T_{\alpha}^{\pm 1} \circ \varphi.$$

Then $M_0 = S^3$, $M_k = M$. Show that M_i can be obtained by surgery on an i-component link in S^3 by induction on i. To do this show that M_{i+1} is obtained from M_i by surgery on a knot.

Remark While every 3-mfld can be presented via a Heegaard decomposition, or Dehn surgery on a link in S^3, these presentations are far from unique.

__Example__ (Seifert fibered 3-mfld)

Let S be a compact connected surface with $\partial S \neq \emptyset$ and M be the (unique) oriented S^1-bundle over S. (Hence if S is orientable then $M = S \times S^1$.)

Let S be the zero-section (i.e. $S \times \{0\}$ in $S \times S^1$ when S is orientable).

Let $T_1, \cdots, T_k$ be the torus boundary components of M.

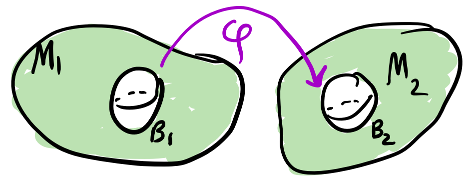
On each T_i, let $m_i = \partial S \cap T_i$ be the meridian and l_i be the fiber of the bundle (both m_i and l_i oriented)

A (p_i, q_i)–Dehn filling on T_i kills the slope $p_i m_i + q_i l_i$. We say that the Dehn-filling is fiber-parallel if $p_i = 0$; i.e. if it kills a fiber.

A __Seifert fibered manifold__ is any 3-mfld N obtained from M by Dehn filling some $n \leq k$ boundary tori in a non-fiber-prallel way.

__Example__ An S^1-bundle over a compact surface is a Seifert fibered space.

Connected sum



$M_1 \# M_2$

M_1, M_2 connected oriented 3-mflds

Pick embedded 3-balls $B_1 \hookrightarrow M_1$, $B_2 \hookrightarrow M_2$

and an orientation-reversing diffeo

$\varphi : \partial B_1 \longrightarrow \partial B_2$. Then glue $M_1 \setminus \mathrm{int}(B_1)$

to $M_2 \setminus \mathrm{int}(B_2)$ via φ to obtain $M_1 \# M_2$.

Exercise 7 Show that $M_1 \# M_2$ does not depend on the choice of embeddings $B_1 \hookrightarrow M_1$, $B_2 \hookrightarrow M_2$ and and diffeomorphism φ. You can use the following two results:

1) Palais' thm: Given an n-mfld M^n, any two embeddings $i_1 : B^n \hookrightarrow M^n$, $i_2 : B^n \hookrightarrow M^n$ are isotopic, where B^n is the unit n-ball in $\mathbb{R}^n$.

2) Smale's thm: any two orientation-reversing (respectively orientation-preserving) diffeo of the 2-sphere are isotopic.

Example $M \# S^3 = M$ for every oriented 3-mfld M

(Because $S^3 \setminus B^3 \cong B^3$).

prime 3-mfld A compact oriented 3-mfld M is prime if it can not be written as a connected sum $M_1 \# M_2$ with $M_i \neq S^3$.

prime decomposition (Kneser, Milnor)

Every compact oriented 3-mfld has a decomposition

$$M = M_1 \# M_2 \# \cdots \# M_k$$

with M_i prime. This decomposition is unique up to reordering of factors.

Irreducible 3-mfld A 3-mfld M is irreducible if every smoothly embedded 2-sphere in M bounds a 3-ball.

Alexander's thm S^3 is irreducible.

Remark This is a smooth version of Jordan curve thm in dimension 3 (The top. version is false due to Alexander horned sphere)

Other examples of irreducible 3-mflds

- Knot complements in S^3
- product 3-mflds $S_g \times S^1$, $S_g \times [0,1]$ (without proof)

Irreducible $\Rightarrow$ prime

Remark Kneser developed normal surface theory in order to show existence of prime decomposition. Alexander used the foliation of $\mathbb{R}^3$ by horizontal planes to show that S^3 is prime. Both of these techniques have proved to be very useful since.

Non-example

Let $L = L_1 \cup L_2$ be a split link in S^3; i.e. there is a 2-sphere S in S^3 separating L_1 from L_2. Then $S^3 \setminus \mathring{N}(L)$ is not prime, since

$$S^3 \setminus \mathring{N}(L) \cong M_1 \# M_2$$

$$M_i = S^3 \setminus \mathring{N}(L_i)$$

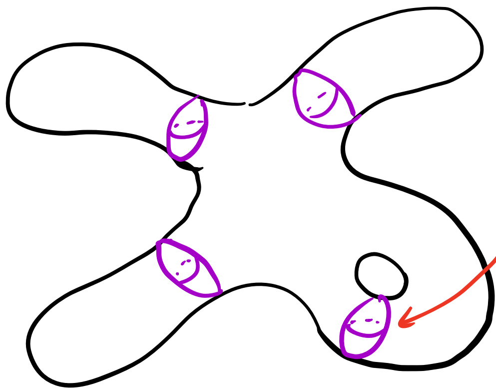
Hence $S^3 \setminus \mathring{N}(L)$ is not irreducible either.

Exercise 8 1) Use Alexander's thm to show that every smoothly embedded 2-sphere in S^3 bounds a 3-ball on each side.

2) Use Alexander's thm to show that a knot complement in S^3 is irreducible.

prime $\Rightarrow$ irreducible or $S^2 \times S^1$

Exercise 9 Let M be a compact oriented 3-mfld that is prime but not irreducible. Show that $M = S^2 \times S^1$. To do this, let S be a sphere in M that does not bound a 3-ball. Then S is necessarily non-separating (i.e. M cut along S has one component). Let α be a simple closed curve in M intersecting S transversely and exactly once. Let N^3 be a regular nbhd of $\alpha \sqcup S$. Show that ∂N^3 is a 2-sphere, and that N^3 is diffeomorphic to $S^2 \times S^1 \setminus \text{int}(3\text{-ball})$. Use M being prime to deduce that $M = S^2 \times S^1$.

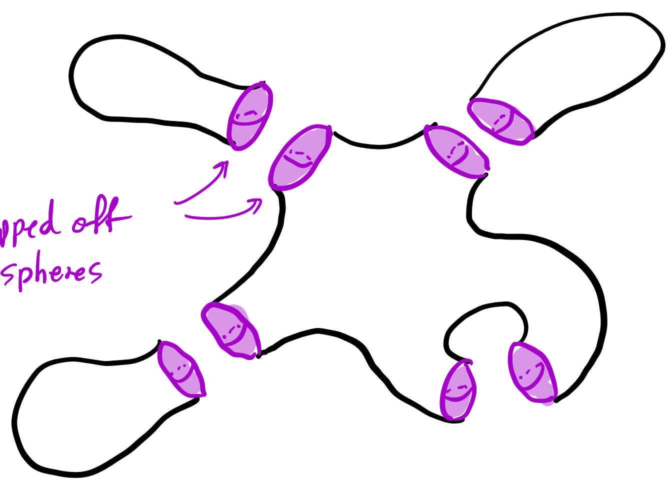


Schematic picture for a 3-mfld

: 'suitable' collection of spheres.

non-separating spheres give rise to $S^2 \times S^1$ prime factors.

cut M along and cap off the sphere boundary components with 3-balls



How to choose the spheres?

this step detects all $S^2 \times S^1$ factors.

1) cut along a non-separating sphere and cap off boundary spheres. This reduces the first Betti number by one. continue until there are no separating spheres.

existence proved by Kneser

2) sphere system: maximal collection of disjoint separating spheres such that no two are parallel (i.e. there is no embedded copy of $S^2 \times [0,1]$ co-bounding them) and no component

of the complement is a 3-ball with holes (i.e. a 3-mfld obtained by removing $k \geq 0$ disjoint open balls from a ball).

cut along a sphere system (not unique!) and cap off sphere boundary components.

Thurston's Geometrisation conjecture (proved by perelman)

In dimension 2, we know by uniformisation thm that every surface admits a metric of constant sectional curvature.

$\chi > 0$	$\chi = 0$	$\chi < 0$
$K > 0$	$K = 0$	$K < 0$

Here we have 3 model geometries (denoted by X)

S^2	$\mathbb{E}^2$	$\mathbb{H}^2$
$K > 0$	$K = 0$	$K < 0$
spherical	Euclidean	hyperbolic

and our surface is equal to the quotient X/G where G is a discrete group of isometries of X. The model space X is <u>homogeneous</u> in the sense that any two point in X look the same (i.e. for every $p, q \in X$ there is an isometry of X taking p to q) and at every point of X any two directions look the same (i.e. for every $p \in X$ and two directions d_1 and d_2 at p there is an isometry of X fixing p and sending d_1 to d_2).

Thurston envisioned a similar geometrisation for 3-mflds. However in the case of 3-mflds

1) homogeneous spaces are too restrictive, and one has to consider a more general notion of <u>locally homogenous</u>: For every two point p, q in X there is an isometry sending p to q.

However, different directions at a point $p \in X$ need not be the same. Of course, the three homogeneous spaces S^3, $\mathbb{E}^3$, $\mathbb{H}^3$ are locally homogeneous as well, as are the product spaces $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$. Thurston showed that there are exactly 3 more locally homogeneous spaces in dimension 3: These are called Nil, Sol, $\widetilde{SL_2}$.

2) One can show that a 3-mfld admitting one of the above 8 geometric structure cannot be a connected sum, except $\mathbb{P}^3 \# \mathbb{P}^3$. So we restrict to irreducible 3-mflds. In fact, one needs to cut along another canonical collection of surfaces (tori in fact) in order to get pieces that admit geometric structures.

We say that a 3-mfld with boundary a union of tori admits a <u>geometric structure</u> if its interior can be written as X/G where X is locally homogeneous and G is a discrete group of isometries of X with $\mathrm{vol}(X/G) < \infty$.

Thurston's Geometrisation conjecture (proved by Perelman)

let M be an irreducible orientable 3-mfld with boundary a (possibly empty) union of tori. Then M has a canonical decomposition (along certain tori) into pieces such that each piece admits a unique geometric structure.

Remark - out of the eight geometries, the least understood is the hyperbolic geometry.

- six out of eight geometries (namely $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{E}^3$, S^3, Nil, $\widetilde{SL_2}$) occur precisely for seifert fibered 3-mflds. The sol geometry only occurs for certain torus bundles over S^1.

References Martelli: An introduction to geometric topology
Scott: The geometries of 3-manifolds