The Incredible Predictive Power of String Theory

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If you pay any attention at all to popular science news, you will certainly have heard of string theory. Many physicists make grandiose claims for it – it will unify gravity and quantum theory, it's a 'theory of everything'. Others view the whole field as a gigantic scam, a mirage that has swallowed generations of graduate students and damaged theoretical physics for decades. The reason for the controversy is simple: huge numbers of theorists have been developing it, for about 30 years, and they still can't predict the result of a single experiment. They're not even close. If you want to know if string theory is correct don't hold your breath for a press release from CERN – it's not going to happen in the foreseeable future.

So is my choice of title just sarcastic? No! (Ok, maybe a bit...). String theory does make predictions, but they're of a very different kind. String theory makes predictions about pure mathematics - interesting, difficult and important predictions. And so far, every one of these predictions has been proved right! So whether it's relevant to physics or not - and personally I'm agnostic on this point – there's no question that string theory is relevant to mathematics.

To understand how string theory can make predictions about maths, let's start by going back 300 years, to Newtonian mechanics. Newton's laws describe how a particle moves around in flat 3-dimensional space, and it's trivial to generalize them to a particle moving around in flat n-dimensional space. But mathematicians are interested in lots of other spaces besides ordinary flat space. A particularly important class of spaces are manifolds, these are spaces that "close up" look like flat space, but when you zoom out they look different. A good example is the surface of a sphere – close up it looks like a flat 2d space, but globally it's very different. It turns out that you can formulate Newton's laws for any manifold (technically you first need to equip the manifold with a Riemannian metric, which is a way of measuring distances), then you get a theory which describes a particle moving around on that particular manifold.

If you study that physical theory then it's clear that you'll start to learn some things about the shape of the manifold – for example a particle moving freely on a sphere will eventually get back to its starting point, but in flat space this will never happen.

Now back to string theory. The basic physical idea in string theory is to replace your particles by tiny little loops, or "strings". These loops can move around, but crucially they can also vibrate, like the string of a musical instrument that's just been plucked. String theorists hope that all the usual particles can be described by strings vibrating in different ways, so an electron is just a string playing a particular note, and a quark is the same string playing a different note.

Like our Newtonian particles, we can think of our strings moving about in ordinary flat space if we want, but its more interesting if we let them move around in a different manifold. In fact the physics demands that we do this, because for a technical reason the theory only works properly if the strings move around in a 10-dimensional space. Since our universe appears to be only 4-dimensional, string theorists speculate that there are six additional dimensions that form a closed-up manifold, perhaps something like a 6-dimensional sphere. This is the kind of speculation that irritates more hard-headed physicists!

Strings prefer to move around on a special kind of manifold called a Kähler manifold. These are manifolds which have complex numbers built into their structure, and also the Riemannian metric is of a special form. Perhaps you remember the Riemann sphere – the complex plane curled up with an extra point at infinity – that's the simplest example of a Kähler manifold.

A good way to get more complicated examples is to take polynomials and look at the set of complex solutions, often the resulting shape will be a Kähler manifold.

So Kähler manifolds are things that lots of pure mathematicians are interested in, geometers certainly, but also algebraists (because of the polynomials) and sometimes even number theorists.

If you could understand how a string moves around in your favourite Kähler manifold, you could learn things about the shape of the manifold. Unfortunately there's a serious problem: mathematicians don't understand string theory. In fact the problem goes deeper, mathematicians don't really understand quantum field theory, and that's a piece of physics that's nearly 100 years old. And when I say we don't understand it, I don't mean that we can't solve the equations, I mean that we don't even understand what the equations mean! Physicists write down symbols, and we can't figure out what mathematical objects they're supposed to be referring to. Of course the physicists don't care about this, for them the symbols refer to physical concepts: fields, particles, and so on. Mathematicians try to interpret them as mathematical concepts: sets, functions, vector spaces etc., but we don't always succeed. We have two different mind sets, and this makes it very difficult for mathematicians to understand physicist's calculations.

The thing that bridges this cultural divide is supersymmetry. This is a rather abstract symmetry that some physical theories have; all particles in physics fall into two classes, bosons and fermions, and supersymmetry makes the two kinds swap places. Theorists love supersymmetry, and the LHC is actively looking for experimental evidence of it, but so far the results are disappointing. Of course string theorists are not dissuaded by that fact! No string theorist would dream of studying a theory that didn't have supersymmetry.

The reason supersymmetry is nice is that it makes some computations much easier. If a theory has supersymmetry then the fine details of the physics will still be hard, but certain fundamental pieces of information will be 'invariants', meaning that they do not change if we make small perturbations. Imagine that your strings are moving around in a Kähler manifold, and you deform the manifold a little bit, perhaps squeezing one part of it, and stretching another. This could make a big difference to the trajectories of individual strings.

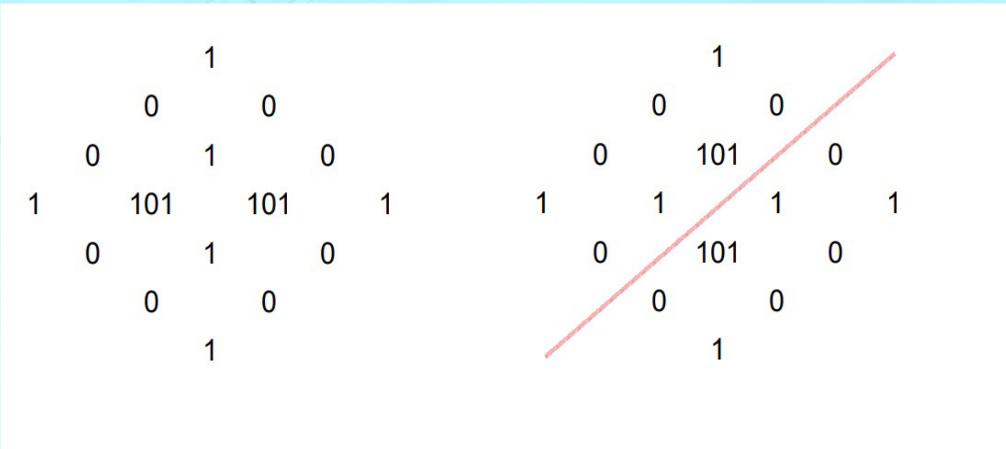

Figure 1: The Hodge diamonds of the cubic threefold (left) and its mirror

However, the invariants provided by supersymmetry are guaranteed to stay the same, and it is this robustness that makes them easy to calculate.

If you're a pure mathematician, your ears prick up at the mention of 'invariants'. We love to calculate invariants of all kinds, and we definitely love to calculate invariants of Kähler manifolds. The invariants that come out of supersymmetric string theory are exactly the kind of thing that mathematicians care about! Some of them are invariants that we already knew about, like Euler characteristics, and homology groups, and some of them are brand new

This leaves us in a rather surprising situation. Physicists' arguments – which we don't understand – can compute geometrical quantities that pure mathematicians are very interested in. Mathematicians now have to listen to the predictions made by string theorists, and their predictions are right!

Most of these predictions, though not all, centre around an astonishing phenomenon called mirror symmetry. Take your favourite Kähler manifold, and write down the physical theory for strings moving around in it. You can now do a really trivial operation on that theory – basically just swap a few plus and minus signs – and get a new physical theory of a similar kind. What does this new theory actually describe? It's definitely doesn't describe strings moving around in your original manifold, because the first theory did that, but perhaps it describes strings moving around in a different Kähler manifold. So perhaps this means that Kähler manifolds come in pairs, with the string theories for each pair being related by this trivial sign change. In this hypothesis, the pairs of manifolds are called 'mirrors' to each other.

Let's assume you believe this idea. If I hand you a Kähler manifold, then it should be possible to produce its partner, the mirror manifold. But how would you know if you'd got the correct mirror? For a start, you could compute some invariants. The most fundamental invariants of a Kähler manifold are called the Hodge numbers, this is a finite set of numbers.

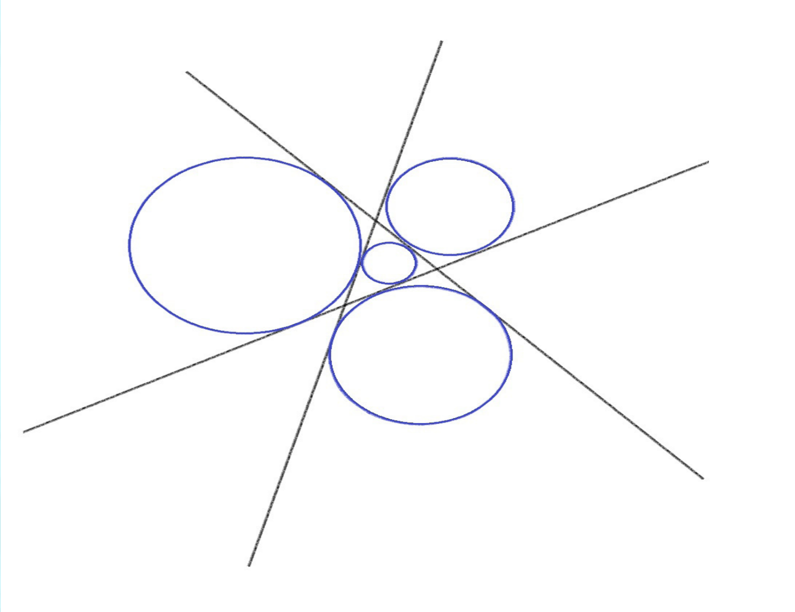

Figure 2: Circles tangent to three lines

In Figure 1 you can see the Hodge numbers of a famous Kähler manifold called the quintic threefold, this manifold is the set of solutions to a quintic polynomial in five variables. Physicists tell us that if two manifolds form a mirror pair then their Hodge diamonds are nearly the same - to get from one to the other you have to do a reflection of the array through a particular diagonal line (marked in red on Figure 1); this is where the name 'mirror' comes from. So now we have a fairly precise prediction: given a Kähler manifold, is there a geometric operation that will produce a new manifold, in a such a way that their Hodge diamonds are mirror images?

At first sight the answer to this question is simply 'no', there is no obvious geometric operation that will do this. But, for reasons I will explain in a moment, mathematicians take this idea extremely seriously. And after 20 years hard work by brilliant people, we can do it, for some examples. I find it absolutely staggering that such a trivial little operation in physics ends up requiring the most monumental effort in geometry and algebra.

So why did mathematicians believe mirror symmetry in the first place? There are now lots of good reasons, but the first really compelling reason was a result by Candelas, de la Ossa, Green and Parkes in 1991. Their result involves things called Gromov-Witten invariants, which are a deeper and more complicated invariant of Kähler manifolds, and exactly the kind of thing that both geometers and string theorists are interested in calculating. To get some kind of flavour of these invariants, imagine drawing three random lines in the plane, and then ask: how many circles can you draw that are tangent to all three lines?

This question is easy - the answer is four (see Figure 2) - but that's because we formulated the question in flat 2-dimensional space. If you ask an analogous question in a Kähler manifold, you may get a much more complicated answer.

For example, if we ask `how many curves of degree three can we draw on the quintic threefold?' then the answer is 317,206,375. This ridiculous number is an example of a Gromov-Witten invariant, and it was first predicted by the four physicists named above. They did it using mirror symmetry.

Mirror symmetry swaps Hodge numbers for Hodge numbers in a simple way, but it turns out that it swaps Gromov-Witten invariants to a completely different kind of invariant, which is sometimes much easier to compute. Candelas et al. had a good guess for the mirror manifold to the quintic threefold, so they simply computed the corresponding invariant for the mirror manifold. The answer was subsequently confirmed by mathematicians, without any mirror symmetry hocus-pocus.

If you can correctly predict a six-digit number then you can win the lottery. Mirror symmetry had correctly predicted a nine-digit number, and mathematicians went crazy for it. If string theory can do that, people asked, then what else might it be able to do? The answer has been 'an awful lot', and there is now a huge body of work on 'pure-maths-inspired-by-string-theory', and several Fields Medals have been won in the process.

The moral of the story is, if you want to learn something new about Kähler manifolds, read a string theory paper.