Algebraic Geometry

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$\square$ = *end of proof.* $\triangle$ = *end of example.*

Contents

What is in this course:

- The basics of affine and projective algebraic geometry, over $\mathbb{C}$.

What is not in this course:

- Algebraic geometry over fields which aren't algebraically-closed or have finite characteristic.

- Proofs of the commutative algebra theorems (Nullstellensatz etc.).
- Intersection theory/Bezout's theorem.
- Sheaves.
- Schemes.

I Affine varieties

1 Introduction

Algebraic geometry is about studying spaces which are the solution sets to polynomial equations, we call these spaces *algebraic varieties*. We'll see lots of interplay between the algebraic properties of polynomials, and the geometric properties of varieties. Here's a very simple example of an algebraic variety:

$$V = \left\{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 - 1 = 0\right\}$$

Obviously V is a circle. The equation defining V is a quadratic polynomial in two variables, and we've looked at its real solutions. But we could instead look at its complex solutions, and we'd get a different variety:

$$V_{\mathbb{C}} = \left\{(x, y) \in \mathbb{C}^2,\ x^2 + y^2 - 1 = 0\right\}$$

It's not so obvious what $V_{\mathbb{C}}$ looks like, because $\mathbb{C}^2$ is 4-dimensional (over $\mathbb{R}$) and $V_{\mathbb{C}}$ is a 2-dimensional surface sitting inside it. But things become clearer if we make the co-ordinate change

$$\hat{x} = x + iy, \qquad \hat{y} = x - iy$$

because then:

$$\begin{aligned} V_{\mathbb{C}} &= \left\{(\hat{x}, \hat{y}) \in \mathbb{C}^2,\ \hat{x}\hat{y} - 1 = 0\right\} \\ &= \{(z, z^{-1}),\ z \in \mathbb{C} \setminus 0\} \end{aligned}$$

We see that $V_{\mathbb{C}}$ is a copy of the complex numbers with zero deleted. Our real solutions V sit inside as the unit circle in $\mathbb{C}$.

We could also look at the real solutions to this new polynomial:

$$W = \left\{(\hat{x}, \hat{y}) \in \mathbb{R}^2,\ \hat{x}\hat{y} - 1 = 0\right\}$$

This is a hyperbola. It looks very different from V, but they are just two different 'slices' through the complex variety $V_{\mathbb{C}}$.

This example demonstrates that it's generally easier to work over the complex numbers, and we're going to do so for the whole course. If you want to get a picture of an algebraic variety then it's sometimes helpful to think about the set of real solutions, but it can also be misleading!

We've looked at the solutions to $x^2 + y^2 - 1 = 0$ over $\mathbb{R}$ and $\mathbb{C}$, but in fact (since this polynomial has integer coefficients) it would make sense to look at the solutions over *any* field $\mathbb{K}$, for example a finite field. This is part of the point of algebraic geometry, and although we're going to stick to $\mathbb{C}$ we're only going to do things which are 'algebraic', which means they could be done over any field if we wanted to.

2 First examples

Definition 2.1. An *affine variety* is a subset of $\mathbb{C}^n$ of the form

$$V = \{f_1(\mathbf{x}) = f_2(\mathbf{x}) = ... = f_k(\mathbf{x}) = 0\} \ \subset \mathbb{C}^n$$

where $f_1, ..., f_k \in \mathbb{C}[x_1, ..., x_n]$ is a finite list of polynomials in n variables.

The usual notation for a set of this form is:

$$V(f_1, ..., f_k)$$

The 'V' stands for 'vanishing locus' (or maybe 'variety').

A nice feature of this definition is that it's very easy to write down examples.

Example 2.2. Set $n = 1$.

1. The polynomial equation $x = 0$ has a single solution, so $V(x) \subset \mathbb{C}$ is a single point, the origin.
2. If we fix a complex number $\lambda \in \mathbb{C}$, then $V(x - \lambda)$ is the single point $\{\lambda\} \subset \mathbb{C}$.
3. The polynomial $x^2 - x$ has two roots, so $V(x^2 - x) = \{0, 1\} \subset \mathbb{C}$ consists of two points.

In general, a degree d polynomial $f \in \mathbb{C}[x]$ has at most d roots, so $V(f) \subset \mathbb{C}$ is a finite set of size at most d. It follows that an affine variety in $\mathbb{C}$ must be a finite set (since adding more polynomials will only make the solution set smaller). Moreover every finite set in $\mathbb{C}$ is an affine variety, because $\{\lambda_1, ..., \lambda_d\} \subset \mathbb{C}$ is exactly the vanishing locus of the polynomial:

$$f(x) = (x - \lambda_1)(x - \lambda_2)...(x - \lambda_d)$$

$\triangle$

Example 2.3. Set $n = 2$.

1. The equations $x = y = 0$ cut out a single point, so $V(x, y) = \{(0, 0)\} \subset \mathbb{C}^2$.
2. $V(y - 3x + \frac{1}{2}) \subset \mathbb{C}^2$ is a straight line.
3. We studied $V(xy - 1) \subset \mathbb{C}^2$ in the introduction, this in an example of a *conic*.
4. Fix $a, b \in \mathbb{C}$. Subsets of the form

$$V(y^2 - x^3 - ax - b) \subset \mathbb{C}^2$$

   are called (affine) *elliptic curves* and are very interesting!

If $f \in \mathbb{C}[x, y]$ is a polynomial in two variables then the affine variety $V(f) \subset \mathbb{C}^2$ is called a *plane curve*. As you can see from the previous three examples, the shape of $V(f)$ gets more complicated as the degree of f increases. $\triangle$

Perhaps surprisingly, linear algebra can produce some interesting examples of affine varieties.

Example 2.4. Let $\mathrm{Mat}_{3\times 3}(\mathbb{C})$ be the set of 3×3 complex matrices. It's a 9-dimensional complex vector space, so we can identify it with $\mathbb{C}^9$. Now let

$$V = \{M; \det(M) = 0\} \subset \mathbb{C}^9$$

be the subset of singular matrices. The determinant function here is a (cubic) polynomial, so V is an affine variety. △

Note that $\mathbb{C}^n$ itself is an affine variety, it's the vanishing locus of no polynomials - or if you prefer, it's the vanishing locus of the zero polynomial $0 \in \mathbb{C}[x_1, ..., x_n]$.

An affine variety $V(f) \subset \mathbb{C}^n$ defined by a single polynomial is called a *hypersurface*. Many of our examples will be hypersurfaces because they're simpler to work with.

Before we do any general theory, let's develop a little intuition about what affine varieties can look like. In the introduction we considered two affine varieties over $\mathbb{R}$:

$$V(x^2 + y^2 - 1) \subset \mathbb{R}^2 \qquad \text{and} \qquad V(xy - 1) \subset \mathbb{R}^2$$

The first one is bounded (it's a circle) and the second is not (it's a hyperbola). Over $\mathbb{C}$ these two equations are just related by a co-ordinate change, and the complex affine variety $V(xy - 1) \subset \mathbb{C}^2$ is not bounded. Are there any complex affine varieties that are bounded? The answer is basically no.

Lemma 2.5. *Assuming $n \geq 2$, a hypersurface $V(f) \subset \mathbb{C}^n$ cannot be bounded.*

So the points in $V(f)$ always 'go out to infinity'.

Proof. Fix $x_2, ..., x_n$ to be any complex numbers (as large as you like). Then $f(x_1, x_2, ..., x_n)$ becomes a polynomial in one variable x_1, and by the Fundamental Theorem of Algebra it has a root. □

Of course for $n = 1$ the hypersurface $V(f)$ is a finite set, which is bounded. The general result is that an affine variety $V \subset \mathbb{C}^n$ can only be bounded if it is a finite set, but this is surprisingly tricky to prove (and we won't). This result is not true over $\mathbb{R}$, and this is one reason why looking at real solutions can sometimes be misleading.

Notation: From now on, we're going to write $\mathbb{A}^n$ instead of $\mathbb{C}^n$ when talking about affine varieties.

This '$\mathbb{A}$' stands for *affine*. This is standard notation, and there are two reasons for it:

(1) It doesn't force you to specify which field you're using. Really $\mathbb{C}^n = \mathbb{A}^n(\mathbb{C})$, and $\mathbb{R}^n = \mathbb{A}^n(\mathbb{R})$, etc. But since we're always going to work over $\mathbb{C}$ we'll just write $\mathbb{A}^n$ instead of $\mathbb{A}^n(\mathbb{C})$.

(2) We want to emphasize that we are not interested in the vector space structure on $\mathbb{C}^n$. For example, we don't consider there to be anything special about the origin in $\mathbb{A}^n$, we can apply translation maps to move it to any other point, *c.f.* Example 2.2(ii). More generally we are 'allowed' non-linear (polynomial) changes of co-ordinates on $\mathbb{A}^n$; if we cared about the vector space structure then we would only be interested in linear changes of co-ordinates.

Example 2.6. Let $V = V(y - x^2) \subset \mathbb{A}^2$, this is the graph of the function $x \mapsto x^2$. Now change co-ordinates to:

$$\hat{x} = x, \qquad \hat{y} = y - x^2$$

Note that this transformation is invertible, and sends polynomials to polynomials. In the new co-ordinates V becomes

$$W = V(\hat{y}) \subset \mathbb{A}^2$$

which is just the $\hat{x}$-axis. We view V and W as being essentially 'the same' affine variety. We'll learn how to say this precisely later. △

Now one easy general result.

Lemma 2.7. *Let $V = V(f_1, ..., f_k)$ and $W = V(g_1, ..., g_m)$ be two affine varities in $\mathbb{A}^n$. Then:*

(i) $V \cap W \subset \mathbb{A}^n$ is an affine variety.

(ii) $V \cup W \subset \mathbb{A}^n$ is an affine variety.

Proof. (i) $V \cap W$ is exactly $V(f_1, ..., f_k, g_1, ..., g_m) \subset \mathbb{A}^n$.

(ii) Exercise.

□

Observe that any single point $p = (p_1, ..., p_n) \in \mathbb{A}^n$ is an affine variety, since it's the vanishing locus of the polynomials $x_1 - p_1, ..., x_n - p_n$. It follows immediately (by part (ii) of the Lemma) that any finite set in $\mathbb{A}^n$ is an affine variety.

3 Ideals

Obviously, the polynomials defining an affine variety are not unique.

Example 3.1. The origin $\{(0, 0\} \in \mathbb{A}^2$ is $V(x, y)$, but it's also $V(2x, x+y)$. △

Suppose we pick polynomials $f_1, ..., f_k$, and consider the variety $V = V(f_1, ..., f_k)$. Each polynomial f_i vanishes identically on V. What other polynomials vanish on V? This is a difficult question, but it is obvious that any polynomial of the form

$$f = g_1 f_1 + ... + g_k f_k$$

vanishes on V, where $g_1, ..., g_k$ are any polynomials at all. In other words, if f lies in the *ideal*

$$(f_1, ..., f_k) \subset \mathbb{C}[x_1, ..., x_n]$$

generated by $f_1, .., f_k$, then f vanishes on V.

Lemma 3.2. *Suppose $f_1, ..., f_k$ and $g_1, ..., g_m$ generate the same ideal:*

$$(f_1, ..., f_k) = (g_1, ..., g_m) \subset \mathbb{C}[x_1, ..., x_n]$$

Then $V(f_1, ..., f_k) = V(g_1, ..., g_m) \subset \mathbb{A}^n$ are the same affine variety.

Proof. Each g_i lies in $(f_1, ..., f_k)$, so each g_i vanishes along $V(f_1, .., f_k)$. This means that the common vanishing locus of the g_i includes the subset $V(f_1, ..., f_k)$, *i.e.*

$$V(f_1, ..., f_k) \subset V(g_1, ..., g_m)$$

By symmetry the reverse inclusion also holds. □

So an affine variety doesn't depend on the specific polynomials you chose, but only on the ideal they generate. So we'd like to be able to say that affine variety in $\mathbb{A}^n$ is something that comes from an ideal $I \subset \mathbb{C}[x_1, ..., x_n]$, by defining:

$$V(I) = \{\mathbf{x} \in \mathbb{A}^n,\ f(\mathbf{x}) = 0\ \ \forall f \in I\}$$

If I is the ideal generated by $f_1, .., f_k$ then this definition gives exactly $V(f_1, .., f_k)$.

However, what if we find an ideal I which cannot be generated by a finite list of polynomials? Then $V(I)$ cannot be described by a finite list of polynomial equations, so it isn't an affine variety (at least under our definition). Fortunately, this will never happen.

Theorem 3.3 (Hilbert Basis Theorem)**.** *Every ideal in* $\mathbb{C}[x_1, ..., x_n]$ *is finitely generated.*

This is very important theorem in algebra, and not too difficult, but we're not going to give the proof.

Remark 3.4.

- A ring R with the property that all ideals in R are finitely-generated is called *Noetherian* (after Emmy Noether). It is one of the most important conditions that can be put on a ring, it's a bit like assuming your vector space is finite-dimensional.
- The Hilbert Basis theorem is actually true over any field, not just $\mathbb{C}$.

So there is no problem with the definition $V(I)$ written above, and any ideal I gives us an affine variety. We can always pick a finite list of polynomials that generate the ideal, then we get a finite list of equations defining $V(I)$.

Notice that this procedure $I \mapsto V(I)$ is *order-reversing*: if $J \subset I$ are two ideals then $V(I) \subset V(J)$. If we impose fewer polynomial equations then we get a larger set of solutions.

By Lemma 2.7, if we have a finite collection of affine varieties in $\mathbb{A}^n$ then the intersection of all of them is an affine variety, and the union of all of them is an affine variety. The Hilbert Basis Theorem let's us extend the first statement to infinite collections.

Corollary 3.5. *Let* S *be any set, and let*

$$V_s = V(f_{s,1}, ..., f_{s,k_s}) \subset \mathbb{A}^n, \ \ s \in S$$

be a collection of affine varieties indexed by S*. Then the intersection*

$$V = \bigcap_{s \in S} V_s \subset \mathbb{A}^n$$

is an affine variety.

Proof. V is the vanishing locus of the set of polynomials $\bigcup_{s \in S}\{f_{s,1}, ..., f_{s,k_s}\}$. This set generates some ideal I, and then $V = V(I)$. □

However, the union of an infinite collection of affine varieties may not be an affine variety (exercise).

Now let us return to the question raised earlier: if $V = V(f_1, .., f_k)$, which polynomials vanish on V? We denote this set of polynomials by $I(V)$:

$$I(V) = \{f \in \mathbb{C}[x_1, ..., x_n],\ f|_V \equiv 0\}$$

It is trivial to check that $I(V)$ is an ideal in $\mathbb{C}[x_1, ..., x_n]$. Also notice that the proceduce $V \mapsto I(V)$ is again order-reversing: if $V \subset W$ are two affine varieties then $I(W) \subset I(V)$, since any polynomial that vanishes on W must in particular vanish along V.

We know that $(f_1, ..., f_k) \subset I(V)$, but is the whole of $I(V)$? The answer is no.

Example 3.6. Let $V = V(x^2) \subset \mathbb{A}^1$. Then V consists of the single point $0 \in \mathbb{A}^1$, and the polynomials that vanish on V are exactly the polynomials divisible by x, *i.e.*

$$I(V) = (x)$$

But x does not lie in the ideal (x^2). △

Of course the above example looks stupid, we should have defined V as $V(x)$ instead of $V(x^2)$. But we must accept the possibility of stupidity!

Definition 3.7. Let R be any ring, and $I \subset R$ an ideal. The **radical** of I is the subset:

$$\mathrm{rad}(I) = \{r \in R,\ r^k \in I \text{ for some } k \in \mathbb{N}\}$$

The subset $\mathrm{rad}(I)$ is always an ideal of R (exercise). If $I = \mathrm{rad}(I)$ then we call it a **radical ideal**.

If $V = (f_1, ..., f_k)$ is an affine variety, it is clear that the radical of $(f_1, .., f_k)$ is contained in $I(V)$. The next theorem says that in fact it is the whole of $I(V)$.

Theorem 3.8 (Hilbert's Nullstellensatz)**.** *Let $V = V(f_1, ..., f_k) \subset \mathbb{A}^n$ be an affine variety. Then:*

$$I(V) = \mathrm{rad}(f_1, ..., f_k)$$

This theorem is one of the foundation stones of algebraic geometry. It means that there is an (order-reversing) bijection:

$$\begin{array}{rcl} \text{Affine varieties in } \mathbb{A}^n & \longleftrightarrow & \text{Radical ideals in } \mathbb{C}[x_1, ..., x_n] \\ V & \mapsto & I(V) \\ V(I) & \leftarrow\!\shortmid & I \end{array}$$

We can freely translate between geometry (varieties) and algebra (ideals).

The Nullstellensatz is a more subtle result than Hilbert's Basis Theorem, it does work when $\mathbb{C}$ is replaced by another field $\mathbb{K}$, but only if we assume that $\mathbb{K}$ is algebraically closed. It is not too hard to prove if you assume that your field is uncountable (like $\mathbb{C}$), but the proof for countable algebraically-closed fields (like $\overline{\mathbb{Q}}$) is quite long.

We are not going to discuss the proof, but we will quickly look at two examples that demonstrate why the result is less obvious than it might appear at first.

Example 3.9. For this example we work with the field $\mathbb{R}$ instead of $\mathbb{C}$, so now $\mathbb{A}^1$ means $\mathbb{R}$ not $\mathbb{C}$. Since $\mathbb{R}$ is not algebraically closed we can't expect the Nullstellensatz to hold, and it doesn't. Consider:

$$V = V\big(x(x^2+1)\big) \subset \mathbb{A}^1$$

The only real root of this polynomial is at zero, so V is a single point $\{0\}$, and

$$I(V) = (x) \subset \mathbb{R}[x]$$

But x is not in the radical of $\big(x(x^2+1)\big)$.

For an even worse example, consider $W = V(x^2+1) \subset \mathbb{A}^1$. This variety is empty! So $I(W) = \mathbb{R}[x]$, since asking for a polynomial to vanish along W is a vacuous condition. But the radical of (x^2+1) is not the whole of $\mathbb{R}[x]$. △

Example 3.10. In this example we work over the finite field $\mathbb{F}_p$ of order p, so $\mathbb{A}^1$ means $\mathbb{F}_p$, which is a finite set. Again $\mathbb{F}_p$ is not algebraically closed, and Nullstellensatz does not hold.

Let $V = \mathbb{A}^1$ itself, so $V = V(0)$ for the zero polynomial $0 \in \mathbb{F}_p[x]$. The ideal generated by 0 is the zero ideal $\{0\} \subset \mathbb{F}_p[x]$, and this radical.

However, there are some non-zero polynomials which vanish on every point in V. For example

$$f(x) = x(x-1)(x-2)...(x-p+1)$$

(in fact this f generates $I(V)$). △

Over $\mathbb{C}$ this kind of nonsense does not happen, and this is one of the main reasons we are working with $\mathbb{C}$ in this course.

Corollary 3.11. *Let V be a hypersurface $V = V(f) \subset \mathbb{A}^n$ where f is an irreducible polynomial. Then $I(V) = (f)$.*

Proof. Nullstellensatz says that $I(V) = \operatorname{rad}(f)$, which is the set of polynomials g such that $f|g^k$ for some k. But recall that $\mathbb{C}[x_1, ..., x_n]$ is a UFD, so if f is irreducible then $f|g^k$ implies $f|g$. □

Remark 3.12. More generally, suppose $V = V(f)$ is a hypersurface defined by reducible polynomial. Factorize f as $f = g_1^{m_1}...g_k^{m_k}$ where the g_i are irreducible and distinct, and let $g = g_1...g_k$. It's easy to check that $\operatorname{rad}(f) = (g)$, so $I(V) = (g)$.

Here is another easy corollary of Nullstellensatz which is sometimes useful:

Corollary 3.13 (Weak Nullstellensatz)**.** *If $V(f_1, ..., f_k)$ is empty then there exist polynomials $g_1, .., g_k$ such that:*

$$g_1 f_1 + ... + g_k f_k = 1$$

This may remind you of Bezout's Lemma.

Proof. Nullstellensatz says that $\operatorname{rad}(f_1, .., f_k)$ is the whole polynomial ring $\mathbb{C}[x_1, .., x_n]$. So in particular $1 \in \operatorname{rad}(f_1, ..., f_k)$, which means that $1 \in (f_1, .., f_k)$. □

In fact the 'strong' Nullstellensatz can also be proved from the 'weak' version fairly easily, but we won't give the argument for that either.

4 Rings

Definition 4.1. Let $V \subset \mathbb{A}^n$ be an affine variety. A function $f : V \to \mathbb{C}$ is called a **regular function** if there exists a polynomial $\hat{f} \in \mathbb{C}[x_1, ..., x_n]$ such that

$$f = \hat{f}|_V$$

Regular functions are the most obvious 'algebraic' functions we can consider on an affine variety. The word 'regular' is to distinguish them from rational functions which we'll meet later.

Given a regular function f, the polynomial $\hat{f}$ defining it is not unique. In fact, two polynomials $\hat{f}, \hat{g}$ define the same regular function iff $\hat{f}|_V = \hat{g}|_V$ (by definition), which holds iff:

$$(\hat{f} - \hat{g})|_V \equiv 0$$

In other words, they define the same regular function iff $\hat{f} - \hat{g}$ lies in the ideal $I(V)$. So the set of all regular functions on V, which we denote by $\mathbb{C}[V]$, is exactly the quotient ring:

$$\mathbb{C}[V] \; = \; \mathbb{C}[x_1, ..., x_n]/I(V)$$

This called the **ring of regular functions** or the **co-ordinate ring** of V.

We've now associated two possible algebraic objects to an affine variety V: the radical ideal $I(V)$, and the co-ordinate ring $\mathbb{C}[V]$. The ring is the more useful one.

We shall see later that the co-ordinate ring completely determines the variety. This is a very important observation, it means that algebraic geometry has a dual interpretation as a kind of ring theory. Everything we do in algebraic geometry can be viewed either geometrically, or algebraically.

Unfortunately not every ring can occur as the co-ordinate ring an affine variety (exercise: which rings can occur?). The solution to this is to enlarge the theory of varieties to *schemes*, but that is beyond the scope of this course.

Example 4.2. Let $V = V(x) \subset \mathbb{A}^1$, which is a single point at the origin. Then $I(V) = (x)$, and:

$$\mathbb{C}[V] = \mathbb{C}[x]/(x) \cong \mathbb{C}$$

The latter isomorphism takes a polynomial f to its constant term $f|_0 \in \mathbb{C}$, this is a ring homomorphism with kernel (x). Of course a function on V is just the data of a single complex number, and every function on V is regular. △

Example 4.3. Let $V = V(y - x^2) \subset \mathbb{A}^2$. The polynomial $y - x^2$ is obviously irreducible, so Corollary 3.11 says that $I(V) = (y - x^2)$. Then:

$$\mathbb{C}[V] = \frac{\mathbb{C}[x, y]}{(y - x^2)} \cong \mathbb{C}[x]$$

To see the isomorphism consider the ring homomorphism from $\mathbb{C}[x, y]$ which sends $x \mapsto x$ and $y \mapsto x^2$. This homomorphism is surjective and the kernel is $(y - x^2)$. △

Example 4.4. Let $V = V(x^2 - x) \subset \mathbb{A}^1$, which is two points $\{0, 1\}$. Then $I(V) = (x^2 - x)$, because a polynomial vanishes on V iff it has both x and $(x - 1)$ as a factor. So:

$$\mathbb{C}[V] = \mathbb{C}[x]/(x^2 - x)$$

This is a two dimensional vector space, it's spanned by the equivalence classes of 1 and x, and the ring structure is determined by the rule '$x^2 = x$'.

A better way to understand this ring is to observe that a function on V is just the data of two complex numbers (α, β). The ring structure on the set of functions is just point-wise multiplication, so the product of (α, β) and (γ, δ) is $(\alpha\gamma, \beta\delta)$. So we have a ring homomorphism

$$\begin{aligned} \mathbb{C}[V] &\to \mathbb{C} \oplus \mathbb{C} \\ f &\mapsto (f|_0, f|_1) \end{aligned}$$

given by evaluating the polynomial at both points. This map is actually a ring isomorphism; it's injective by definition, and it's surjective since:

$$(\beta - \alpha)x + \alpha \;\mapsto\; (\alpha, \beta)$$

In this example every function on V is regular. △

We shall prove later that the ring $\mathbb{C}[V]$ knows everything about the variety V. Here is a first example of this phenomenon.

Let $p \in V$ be a point of V. Then given a regular function f on V we can evaluate it at the point p, and this defines a map:

$$\begin{aligned} \mathrm{ev}_p : \mathbb{C}[V] &\to \mathbb{C} \\ f &\mapsto f(p) \end{aligned}$$

Note that ev_p is a ring homomorphism, and it is also linear over $\mathbb{C}$.

Lemma 4.5. *Let V be an affine variety. There is a bijection:*

$$\begin{aligned} \{ \text{ points of } V\} &\xrightarrow{\sim} \{\mathbb{C}\text{-linear ring homomorphisms } \mathbb{C}[V] \to \mathbb{C}\} \\ p &\mapsto \mathrm{ev}_p \end{aligned}$$

Note that not every ring homomorphism $\mathbb{C}[V] \to \mathbb{C}$ is $\mathbb{C}$-linear; for example sending f to $\overline{f|_p}$ (the complex conjugate) is a homomorphism but is not $\mathbb{C}$-linear.

Proof. We just need to construct the inverse map. Pick generators $f_1, ..., f_k$ for $I(V)$, so $V = V(f_1, ..., f_k)$, and:

$$\mathbb{C}[V] = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$$

Suppose we have a $\mathbb{C}$-linear homomorphism $\epsilon : \mathbb{C}[V] \to \mathbb{C}$. We can think of this a ($\mathbb{C}$-linear) homomorphism $\epsilon : \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}$ which sends each f_j to zero.

Now each co-ordinate function x_i is a polynomial, so using ϵ we can get n complex numbers

$$p_i = \epsilon(x_i) \in \mathbb{C}$$

and this defines a point $p = (p_1, ..., p_n) \in \mathbb{A}^n$. These numbers completely determine ϵ; if g is any polynomial then applying ϵ to g must give

$$\epsilon(g) = g(p_1, ..., p_n)$$

because ϵ is a homomorphism and also $\mathbb{C}$-linear. So ϵ is exactly the evaluation homomorphism ev_p.

Finally, the fact that $\epsilon(f_j) = 0$ says that $f_j(p) = 0$ for all j, so p is in V. □

5 Irreducibility

Recall that a ring R is an **integral domain** (or just a **domain**) if there do not exist non-zero elements $f, g \in R$ such that $fg = 0$.

Example 5.1. Let $V = V(xy) \subset \mathbb{A}^2$. The solutions to $xy = 0$ are $\{y = 0\}$ and $\{x = 0\}$, so V is the union of the x-axis and the y-axis. This is a basic but useful example, it's called a *node*.

We can write V as a union

$$V = V(x) \cup V(y)$$

of two smaller affine varieties. How is this fact reflected in the ring $\mathbb{C}[V]$? The answer is that

$$\mathbb{C}[V] = \mathbb{C}[x, y]/(xy)$$

fails to be an integral domain: we have two non-zero elements x, y whose product xy is zero in $\mathbb{C}[V]$. Geometrically, $x|_V$ is a regular function which vanishes along the y-axis, and $y|_V$ is a rgular function which vanishes along the x-axis, and their product $xy|_V$ vanishes at all points. △

Definition 5.2. An affine variety $V \subset \mathbb{A}^n$ is called **reducible** if

$$V = V_1 \cup V_2$$

for two affine varieties $V_1, V_2 \subset \mathbb{A}^n$, where $V_1 \neq V$ and $V_2 \neq V$. If V is not reducible we call it **irreducible**.

Definition 5.3. An ideal I (in some ring R) is called **prime** if there do not exist elements $f, g \in R$ such that $f \notin I$ and $g \notin I$ but $fg \in I$.

Proposition 5.4. *Let V be an affine variety. Then*

$$V \text{ is irreducible} \iff I(V) \text{ is prime} \iff \mathbb{C}[V] \text{ is a domain}$$

Proof. The claim that $I(V)$ is prime iff $\mathbb{C}[V]$ is a domain is trivial, since the class of f is zero in $\mathbb{C}[V]$ exactly when $f \in I(V)$. So let's prove V is irreducible iff $I(V)$ is prime.

Say V is reducible, so $V = V_1 \cup V_2$ in a non-trivial way. Then $V_1 \neq V$ so $I(V) \subsetneq I(V_1)$. So there is a polynomial $f \in I(V_1)$ with $f \notin I(V)$. Similarly there is a $g \in I(V_2)$ with $g \notin I(V)$. But fg vanishes on $V_1 \cup V_2$ so $fg \in I(V)$.

Conversely, suppose we have f, g such that neither lies in $I(V)$ but $fg \in I(V)$. Set $V_1 = V \cap V(f)$ and $V_2 = V \cap V(g)$. Then $V \subset V(fg) = V(f) \cup V(g)$ so $V = V_1 \cup V_2$. But $V_1 \neq V$ since $f \notin I(V)$, and $V_2 \neq V$ since $g \notin I(V)$. □

Example 5.5.

1. If V is a single point then it's obviously irreducible, and indeed $\mathbb{C}[V] = \mathbb{C}$ is a field, in particular a domain.

2. If V is two points then it's obviously reducible, and $\mathbb{C}[V] = \mathbb{C} \oplus \mathbb{C}$ is not a domain since $(1, 0)(0, 1) = (0, 0)$.

3. A node $V(xy) \subset \mathbb{A}^2$ is reducible, it splits as $V = V(x) \cup V(y)$. But the two pieces $V_1 = V(x)$ and $V_2 = V(y)$ themselves are irreducible, since:

$$\mathbb{C}[V_1] = \mathbb{C}[x,y]/(x) \cong \mathbb{C}[y] \qquad \text{and} \qquad \mathbb{C}[V_2] = \mathbb{C}[x,y]/(y) \cong \mathbb{C}[x]$$

So we can split V as a union of two irreducible pieces.

$\triangle$

Suppose V is a reducible affine variety, so we can split it as $V = V_1 \cup V_2$. If V_2 (say) is still reducible, we can split V up further as $V = V_1 \cup V_3 \cup V_4$. Continuing in this way, we should end up with a decomposition

$$V = V_1 \cup ... \cup V_r$$

of V into irreducible pieces (this is analogous to prime factorization). But: does the process always terminate after a finite number of steps? The answer is yes.

Proposition 5.6. *Any affine variety V can be decomposed as $V = V_1 \cup ... \cup V_r$ where each V_i is irreducible (and $V_i \not\subseteq V_j$ for $i \neq j$). The decomposition is unique up to ordering.*

The V_i are called the **irreducible components** of V.

Proof. We leave uniqueness as an exercise, the harder part is finiteness (for prime factorization it's the other way around!).

Suppose V is reducible, and we begin splitting it into pieces as described above. If the procedure never terminates, we will obtain an infinite chain

$$V \supsetneq V_1 \supsetneq V_2 \supsetneq ...$$

of subvarieties, hence an infinite chain of ideals:

$$I(V) \subsetneq I(V_1) \subsetneq I(V_2) \subsetneq ...$$

Let's write $I_i = I(V_i)$ and $I_0 = I(V)$, and set:

$$I = \bigcup_{i=0}^{\infty} I_i$$

By Hilbert's Basis Theorem (Theorem 3.3) I must have a finite generating set $f_1, ..., f_k$. By definition, each f_t is an element of some I_{i_t}, and if we set $m = \max\{i_1, ..., i_t\}$ then each f_t is an element of I_m. But this implies that $I \subset I_m$, which means $I_{m+1} \subset I_m$, and this is a contradiction. $\square$

Remark 5.7. We've just proved a purely algebraic statement: if every ideal in R is finitely-genererated then we cannot have an infinite strictly-increasing chain of ideals in R. This is called the *ascending chain condition.* The converse result also holds (exercise), so either conditions can be used as the definition of a Noetherian ring.

For a hypersurface $V(f)$ it is straight-forward to find the irreducible components. Recall from Remark 3.12 that f generates $I(V)$ iff the irreducible factors of f all occur with multiplicity one.

Lemma 5.8. *Let $V = V(f)$ be a hypersurface, and assume that f generates $I(V)$. Then:*

(i) V is irreducible iff f is irreducible.

(ii) If V is reducible then the irreducible components of V are the hypersurfaces defined by the irreducible factors of f.

Proof. Since $\mathbb{C}[x_1, ..., x_n]$ is a UFD the ideal (f) is prime iff f is irreducible, so part (i) follows from Proposition 5.4. If f has irreducible factors $f = g_1...g_k$ then $V = V(g_1) \cup ... \cup V(g_k)$, and part (i) says that each $V(g_i)$ is irreducible. □

A note on terminology: In many algebraic geometry texts you will see the words *algebraic set*. We are not going to use these words, for two reasons.

- Some people insist that a 'variety' must be irreducible, and use the words 'algebraic set' for what we would call a reducible variety.
- Over other fields ($\mathbb{R}$, $\mathbb{F}_p$,...) the set of points of a variety V may not tell you much information about the polynomials defining V; we saw examples of this when we discussed the Nullstellensatz. So it's helpful to distinguish between the 'algebraic set' which is the set of points of V, and a more abstract notion of a 'variety' which remembers more information. Since we work over $\mathbb{C}$ we don't need this distinction.

6 Regular functions and isomorphisms

The affine variety $V(y) \subset \mathbb{A}^2$ is just the x-axis, so intuitively it's 'the same' as the affine line $\mathbb{A}^1$. To say this precisely we need a definition of when two affine varieties are *isomorphic*. But before we do that, we need think about functions between two affine varieties.

Definition 6.1. Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ be affine varieties. A function $F : V \to W$ is called a **regular function** if there exist polynomials $\hat{f}_1, .., \hat{f}_k \in \mathbb{C}[x_1, .., x_n]$ such that:

$$F = (\hat{f}_1, ..., \hat{f}_k)|_V$$

So a regular function from $\mathbb{A}^n$ to $\mathbb{A}^k$ is just a function whose components are polynomials, and a regular function from V to W is a the restriction of some regular function $\widehat{F} : \mathbb{A}^n \to \mathbb{A}^k$ such that $\widehat{F}(V) \subset W$. If we set $W = \mathbb{A}^1$ then we exactly recover Definition 4.1.

Note that different k-tuples of polynomials can define the same regular function $V \to W$, because if $h \in I(V)$ then we can add h to any component f_i and the restriction of this function to V does not change. So more precisely, a regular function from V to W is a k-tuple

$$F = (f_1, ..., f_k)$$

of elements $f_i \in \mathbb{C}[V]$, satisfying $F(V) \subset W$.

Definition 6.2. A regular function $F : V \to W$ is an **isomorphism** if there exists a regular function $G : W \to V$ such that $F \circ G = 1_W$ and $G \circ F = 1_V$. If there exists an isomorphism between V and W we say they are **isomorphic**, and write $V \cong W$.

A regular function F which is a bijection is *not* automatically an isomorphism, because the inverse function might not be regular. We'll see a counterexample in Example 6.12 below.

Example 6.3. Let $V = V(y) \subset \mathbb{A}^2$. Define

$$\widehat{F} : \mathbb{A}^1 \to \mathbb{A}^2 \qquad \widehat{G} : \mathbb{A}^2 \to \mathbb{A}^1$$
$$x \mapsto (x, 0) \qquad (x, y) \mapsto x$$

These are both regular, and the image of $\widehat{F}$ lies in V, so we get regular functions $F : \mathbb{A}^1 \to V$ and $G = \widehat{G}|_V : V \to \mathbb{A}^1$. Then $F \circ G$ and $G \circ F$ are both the identity, so V is isomorphic to $\mathbb{A}^1$. This is reassuring!

Notice that $\widehat{F} \circ \widehat{G}$ is not the identity function on $\mathbb{A}^2$, but this doesn't matter. △

Example 6.4. Let $V = V(y - x^2,\, z - x^3) \subset \mathbb{A}^3$. Then $V \cong \mathbb{A}^1$, using:

$$F : \mathbb{A}^1 \to V \qquad G : V \to \mathbb{A}^1$$
$$x \mapsto (x, x^2, x^3) \qquad (x, y, z) \mapsto x$$

△

We have made an important conceptual leap. Up until now an affine variety only made sense as a *subset* of some ambient $\mathbb{A}^n$. Now we have freed ourselves of this: we can say that $V(y) \subset \mathbb{A}^2$ and $V(y - x^2,\, z - x^3) \subset \mathbb{A}^3$ are *the same mathematical object*, they've just been embedded into different ambient spaces.

An isomorphism $F : \mathbb{A}^n \xrightarrow{\sim} \mathbb{A}^n$ is just an algebraic change of co-ordinates. If $V \subset \mathbb{A}^n$ is an affine variety then it's obvious that $F(V)$ is also an affine variety; just take the polynomials defining V and write them in the new co-ordinates. And it's obvious that

$$F : V \to F(V)$$

is an isomorphism.

Example 6.5. The hypersurfaces $V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ and $V(xy - 1) \subset \mathbb{A}^2$ are isomorphic, since they're related by the co-ordinate change:

$$(x, y) \mapsto (x + iy, x - iy)$$

We saw this in the introduction. △

Example 6.6. The hypersurface $V(y - x^2)$ is isomorphic to $\mathbb{A}^1$, see Example 2.6. △

We have claimed that everything in algebraic geometry can be expressed either geometrically or algebraically, so a regular function from V to W should have an algebraic interpretion. It does: it corresponds to a ring homomorphism from $\mathbb{C}[W]$ to $\mathbb{C}[V]$. Let's explain why.

Suppose we have a regular function:

$$F : \mathbb{A}^n \longrightarrow \mathbb{A}^k$$
$$\mathbf{x} \mapsto (f_1(\mathbf{x}), ..., f_k(\mathbf{x}))$$

If $g \in \mathbb{C}[y_1, ..., y_k]$ is a polynomial on $\mathbb{A}^k$, then we can get a polynomial on $\mathbb{A}^n$ by defining:

$$F^*(g) = g \circ F : \mathbf{x} \mapsto g(f_1(\mathbf{x}), ..., f_k(\mathbf{x}))$$

This is called the **pull-back** of g along F. So we have a function:

$$F^* : \mathbb{C}[y_1, ..., y_k] \longrightarrow \mathbb{C}[x_1, ..., x_n]$$

It's obvious that this is a ring homomorphism, and also linear over $\mathbb{C}$.

Example 6.7. Let $F : \mathbb{A}^2 \to \mathbb{A}^3$ be the regular function:

$$F(x_1, x_2) = (x_1 + x_2,\, x_1^2,\, x_2^2)$$

Then

$$F^* : \mathbb{C}[y_1, y_2, y_3] \longrightarrow \mathbb{C}[x_1, x_2]$$

is the ring homomorphism such that:

$$y_1 \mapsto x_1 + x_2, \qquad y_2 \mapsto x_1^2, \qquad y_3 \mapsto x_2^2$$

So for example $F^*(y_1 + y_2^2) = x_1 + x_2 + x_1^4$. △

Every $\mathbb{C}$-linear ring homomorphism from $\mathbb{C}[y_1, ..., y_k]$ to $\mathbb{C}[x_1, ..., x_n]$ arises in this way. If Φ is such a homomorphism then we can get k polynomials (in n-variables) by setting

$$f_1 = \Phi(y_1), \quad ..., \quad f_k = \Phi(y_k)$$

and this is exactly the data of a regular map:

$$F = (f_1, ..., f_k) : \mathbb{A}^n \to \mathbb{A}^k$$

The f_i's determine Φ completely. Because Φ is a $\mathbb{C}$-linear homomorphism, if $g \in \mathbb{C}[y_1, ..., y_k]$ is an arbitrary polynomial we must have:

$$\Phi(g) = g(f_1, ..., f_k) = F^*(g)$$

Thus Φ is exactly the pull-back homomorphism associated to the regular map F. So we have a bijection:

$$\left\{ \begin{matrix} \text{Regular} \\ \text{functions} \end{matrix} \;\; \mathbb{A}^n \to \mathbb{A}^k \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \mathbb{C}\text{-linear} \\ \text{homomorphisms} \end{matrix} \;\; \mathbb{C}[y_1, .., y_k] \to \mathbb{C}[x_1, ..., x_n] \right\}$$

Now suppose we have a regular function $F : V \to W$ between two general affine varieties. Just as before, if we have regular function $g : W \to \mathbb{C}$ then we can define the *pull-back* of g along F to be the function:

$$F^*(g) = g \circ F : \; V \to \mathbb{C}$$

It's clear that $F^*(g)$ will also be regular; F is the restriction of some polynomial map $\widehat{F} : \mathbb{A}^n \to \mathbb{A}^k$, and g is the restriction of some polynomial $\hat{g} \in \mathbb{C}[y_1, ..., y_k]$, and then $F^*(g)$ is the restriction of $\hat{g} \circ \widehat{F}$. So we have a map:

$$F^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$$

It's easy to see that F^* is a $\mathbb{C}$-linear ring homomorphism.

Proposition 6.8. *Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ be two affine varieties. Then we have a bijection*

$$\{\text{Regular functions } V \to W\} \xrightarrow{\sim} \{\mathbb{C}\text{-linear homomorphisms } \mathbb{C}[W] \to \mathbb{C}[V]\}$$

sending F to F^*.

The proof of this result is similar to the proof of Lemma 4.5. In fact that lemma is a special case of this proposition: if V is a point then $\mathbb{C}[V] = \mathbb{C}$, and a regular function $V \to W$ is just the choice of a point of W.

Proof. Let $\Phi : \mathbb{C}[W] \to \mathbb{C}[V]$ be a $\mathbb{C}$-linear homomorphism. We can think of this as a ($\mathbb{C}$-linear) homomorphism from $\mathbb{C}[y_1, ..., y_k]$ to $\mathbb{C}[V]$ which sends the ideal $I(W)$ to zero. If we evaluate Φ on the co-ordinate functions y_i we get

$$f_1 = \Phi(y_1), \quad ..., \quad f_k = \Phi(y_k) \quad \in \mathbb{C}[V]$$

which is the data of a regular map:

$$F = (f_1, ..., f_k) : V \to \mathbb{A}^k$$

The fact that Φ is a $\mathbb{C}$-linear homomorphism means that for an arbitrary polynomial $g \in \mathbb{C}[y_1, .., y_k]$ we must have

$$\Phi(g) = g(f_1, ..., f_k) = F^*(g) \in \mathbb{C}[V]$$

so Φ is the pull-back homomorphism F^*. Finally, we must show that $F(V) \subset W$. If $p \in V$ is any point, and g is any function vanishing on W, then by assumption $\Phi(g) = g(F(p)) = 0$. So $F(p)$ lies in $V(g)$, and this is true for all $g \in I(W)$, which means $F(p) \in W$. □

This bijection between regular functions and ring homomorphisms behaves nicely when we compose regular functions. Suppose

$$V \xrightarrow{F} W \xrightarrow{G} X$$

are two regular functions. It's clear that $G \circ F$ is regular: just choose polynomial maps $\widehat{F} : \mathbb{A}^n \to \mathbb{A}^k$ and $\widehat{G} : \mathbb{A}^k \to \mathbb{A}^m$ that restrict to give F and G, then $G \circ F$ is the restriction of $\widehat{G} \circ \widehat{F}$. So we have ($\mathbb{C}$-linear) ring homomorphisms:

$$F^* : \mathbb{C}[W] \to \mathbb{C}[V], \qquad G^* : \mathbb{C}[X] \to \mathbb{C}[W], \qquad \text{and} \qquad (G \circ F)^* : \mathbb{C}[X] \to \mathbb{C}[V]$$

But it's easy to see that $(G \circ F)^*$ is the composition $F^* \circ G^*$, because:

$$(G \circ F)^*(h) = h \circ G \circ F = (G^*(h)) \circ F = F^*(G^*(h))$$

So the bijection between regular functions and ring homomorphisms respects compositions. Just remember that the order of compositions gets reversed!

Remark 6.9. If you know what a *functor* is, we've just shown that the procedure sending V to $\mathbb{C}[V]$ and F to F^* is a contravariant functor.

Corollary 6.10. *Let $F : V \to W$ be a regular map between two affine varieties. Then F is an isomorphism iff*

$$F^* : \mathbb{C}[W] \to \mathbb{C}[V]$$

is an isomorphism of rings.

Proof. If F has an inverse G then F^* and G^* are inverse homomorphisms. Conversely, suppose F^* is a ring isomorphism. Then $(F^*)^{-1}$ is automatically $\mathbb{C}$-linear, so there is a regular function $G : W \to V$ such that $G^* = (F^*)^{-1}$, which means that $(G \circ F)^*$ is the identity function on $\mathbb{C}[V]$ and $(F \circ G)^*$ is the identity function on $\mathbb{C}[W]$. Then (using Proposition 6.8 again) $G \circ F$ must be the identity on V, and $F \circ G$ must be the identity on W. □

So the ring $\mathbb{C}[V]$ really does know everything about the variety V.

Example 6.11. Let $V = V(y - x^2) \subset \mathbb{A}^2$. Then we saw in Example 4.3 that $\mathbb{C}[V] \cong \mathbb{C}[x]$ using the ($\mathbb{C}$-linear) homomorphism defined by $x \mapsto x$ and $y \mapsto x^2$. The associated regular function

$$\begin{aligned} F : \mathbb{A}^1 &\to V \\ x &\mapsto (x, x^2) \end{aligned}$$

is an isomorphism. △

Example 6.12. Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$. This is a very nice example that we'll come back to many times, it's called a *cusp singularity*. It's actually a very degenerate example of an elliptic curve.

Consider the regular map:

$$\begin{aligned} F : \mathbb{A}^1 &\to V \\ t &\mapsto (t^2, t^3) \end{aligned}$$

This function is evidently an injection, and it's easy to check that it's also a surjection. However, F is *not* an isomorphism.

To see this, observe that $I(V) = (y^2 - x^3)$ since this polynomial is irreducible (Corollary 3.11). Now consider the ring homomorphism associated to F:

$$\begin{aligned} F^* : \mathbb{C}[V] = \frac{\mathbb{C}[x,y]}{(y^2 - x^3)} &\longrightarrow \mathbb{C}[t] \\ f &\mapsto f(t^2, t^3) \end{aligned}$$

Since $x \mapsto t^2$ and $y \mapsto t^3$ it's clear that F^* is not a surjection, its image does not contain the polynomial t. So F^* is not an isomorphism and hence neither is F.

In fact we can make the stronger statement that $\mathbb{C}[V]$ is not isomorphic to $\mathbb{C}[t]$; the ring $\mathbb{C}[t]$ is a UFD but $\mathbb{C}[V]$ is not (since $y^2 = x^3$). So there is no possible isomorphism between V and $\mathbb{A}^1$. △

Here's a lemma which is often useful for proving that a variety is irreducible:

Lemma 6.13. *Let $F : V \to W$ be a surjective regular map between two affine varieties. If V is irreducible then W must be irreducible.*

Proof. Exercise. □

Example 6.14. The regular function F from Example 6.12 is surjective, and $\mathbb{A}^1$ is irreducible, so the cusp must be irreducible. This also follows from Lemma 5.8 since $y^2 - x^3$ is an irreducible polynomial. △

Of course saying that $F : V \to W$ is surjective is the same as saying that W is the image of F. But it's important to realize that the image of an affine variety under a regular map might not be an affine variety.

Example 6.15. Consider the regular map:

$$\begin{aligned} F : \mathbb{A}^2 &\to \mathbb{A}^2 \\ (x, y) &\mapsto (x, xy) \end{aligned}$$

The image of F is the subset

$$\begin{aligned} \{(x, xy),\ x \neq 0\} \cup \{(0, 0\} &= \big(\mathbb{A}^2 \setminus \{(0, y)\}\big) \cup \{(0, 0)\} \\ &= \mathbb{A}^2 \setminus \{(0, y),\ y \neq 0\} \end{aligned}$$

This cannot be an affine variety in $\mathbb{A}^2$, because it is not a closed subset. △

7 Zariski open subsets

An affine variety $V \subset \mathbb{A}^n$ is always a closed subset of $\mathbb{A}^n (= \mathbb{C}^n)$, since it is the zero locus of some polynomials. But most closed subsets of $\mathbb{A}^n$ are not affine varieties.

Example 7.1. The subsets $\{|z| \leq 1\}$ and $\{\mathrm{Re}(z) \geq 0\}$ in $\mathbb{A}^1$ are both closed, but neither is an affine variety because an affine variety in $\mathbb{A}^1$ must be a finite set (see Example 2.2). △

Here is some new terminology:

Definition 7.2. A **Zariski-closed** subset of $\mathbb{A}^n$ is an affine variety. A **Zariski open** subset of $\mathbb{A}^n$ is the complement $\mathbb{A}^n \setminus V$ of some affine variety V.

The simplest kind of Zariski open subset in $\mathbb{A}^n$ is the complement of a hyperplane, *i.e.* a subset of the form:

$$U = \mathbb{A}^n \setminus V(f)$$

Every affine variety is (by definition) the intersection of a finite set of hyperplanes, so every Zariski open subset is the union of a finite number of subsets of the form above.

Remark 7.3. If you know what a *topology* is, you can observe (using Lemma 2.7 and Corollary 3.5) that the Zariski open subsets define a topology on $\mathbb{A}^n$. It's called the *Zariski topology.*

Despite the previous remark Zariski open subsets are a poor substitute for more general open subsets. In complex analysis we might talk about an 'open neighbourhood' of a point $x \in \mathbb{C}$, and have in mind something like an open ball $B(x, \epsilon)$ for a small real number ϵ. But this is not a Zariski open neighbourhood; a Zariski open neighbourhood must always be much larger.

Example 7.4. In $\mathbb{A}^1$, a Zariski-closed subset is exactly a finite set. So a Zariski open neighbourhood must be the complement of a finite set.

In particular if we take two points $x, y \in \mathbb{A}^1$, then if we choose ϵ_1 and ϵ_2 small enough then the open balls $B(x, \epsilon_1)$ and $B(y, \epsilon_2)$ will be disjoint. But we cannot achieve this with Zariski open neighbourhoods, because if $U_1, U_2 \subset \mathbb{A}^1$ are any two Zariski open subsets then U_1 and U_2 must intersect. (This says that the Zariski topology on $\mathbb{A}^1$ is not *Hausdorff*). △

Definition 7.5. Let $V \subset \mathbb{A}^n$ be an affine variety. A **Zariski-closed** subset of V is the intersection $V \cap W$ for some other affine variety $W \subset \mathbb{A}^n$.

A **Zariski open** subset of V is the complement $V \setminus V \cap W$ of some Zariski-closed subset.

Of course $V \cap W$ is an affine variety, and any affine variety contained in V looks like this, so another name for a Zariski-closed subset is an **affine subvariety**. Also note that a Zariski open subset of V is exactly a subset of the form $V \cap U$ where U is a Zariski open subset of $\mathbb{A}^n$ (so we're just talking about the *induced topology* on the subset $V \subset \mathbb{A}^n$).

We observed in Example 7.4 that any two Zariski open subsets in $\mathbb{A}^1$ must intersect. The following easy lemma generalizes this observation:

Lemma 7.6. *Let V be an irreducible affine variety, and $U_1, U_2 \subset V$ be two non-empty Zariski open subsets. Then $U_1 \cap U_2$ is non-empty.*

Conversely if V is reducible then there exist two disjoint non-empty Zariski open subsets $U_1, U_2 \subset V$.

Proof. Exercise. □

Example 7.7. The node $V = V(xy) \subset \mathbb{A}^2$ is reducible. The Zariski open subsets $U_1 = V \setminus V(x)$ and $U_2 = V \setminus V(y)$ are non-empty and disjoint. △

You may recall the definition of the *closure* of a subset $S \subset \mathbb{C}^n$, it's the intersection of all closed subsets containing S. We can do the same thing with Zariski-closed subsets (actually we can do the same thing in any topology).

Definition 7.8. Let $S \subset \mathbb{A}^n$ be any subset. The **Zariski closure** of S is the intersection of all Zariski closed subsets containing S.

By Corollary 3.5 the Zariski closure of S is an affine variety, it's the smallest affine variety containing S. For another way to understand this variety, let

$$I(S) \subset \mathbb{C}[x_1, ..., x_n]$$

denote the set of polynomials which vanish at every point of S. It's trivial to check that this is an ideal.

Lemma 7.9. *Let $S \subset \mathbb{A}^n$ be any subset. Then $V(I(S))$ is the Zariski closure of S.*

Proof. Exercise. □

Since Zariski closed subsets are also closed in the usual sense it's immediate the closure of S (in the usual sense) is contained in the Zariski closure. But the Zariski closure of S may be much bigger.

Example 7.10. Let $S = \{x; x \in \mathbb{Z}\} \subset \mathbb{A}^1$. This is a closed subset, but not Zariski closed. If a polynomial f vanishes at all points of S then it must be the zero polynomial, so the Zariski closure of S is the whole of $\mathbb{A}^1$.

The same argument applies to any infinite subset S. △

In particular if U is a Zariski open subset of $\mathbb{A}^1$ then the Zariski closure of U is the whole of $\mathbb{A}^1$. The same fact is true in $\mathbb{A}^n$. But if we have a more general affine variety V, and a Zariski open subset $U \subset V$, it may not be true that the Zariski closure of U is the whole of V.

Lemma 7.11. *Let $V \subset \mathbb{A}^n$ be an irreducible affine variety. If $U \subset V$ is a non-empty Zariski open subset then the Zariski closure of U is V.*

Conversly if V is reducible then there exists a non-empty Zariski open subset $U \subset V$ whose Zariski closure is smaller than V.

Proof. Exercise. □

Example 7.12. Let $V = V(xy) \subset \mathbb{A}^2$ and let $U = V \setminus V(x)$. The Zariski closure of U is $V(y)$. △

Now we have three equivalent definitions of 'irreducible'.

8 Quasi-affine varieties

Yet more redundant terminology:

Definition 8.1. A **quasi-affine variety** $U \subset \mathbb{A}^n$ is a Zariski open subset of an affine variety.

This a larger class of spaces that we can study in algebraic geometry. Note that affine varieties are a special case of quasi-affine varieties.

Example 8.2. $\mathbb{A}^1 \setminus \{(0,0)\}$ is a quasi-affine variety, and so is $\mathbb{A}^n \setminus \{(0,0)\}$. △

More generally any Zariski open subset of $\mathbb{A}^n$ is a quasi-affine variety. These are the simplest kind of quasi-affine variety, but not the only kind.

Example 8.3. Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$, the cusp singularity. Then

$$U \;=\; V \cap \{y \neq 0\} \;=\; V \setminus \{(0,0)\}$$

is a Zariski open subset of V, so it's a quasi-affine variety. Notice that the function $F : t \mapsto (t^2, t^3)$ from Example 6.12 is a bijection between $\mathbb{A}^1 \setminus (0,0)$ and this U. Once we've defined *isomorphisms* between quasi-affine varieties it will be a straight-forward exercise to show that this is an isomorphism. △

Example 8.4. Recall the function $F : \mathbb{A}^2 \to \mathbb{A}^2$, $(x,y) \mapsto (x, xy)$ from Example 6.15, whose image is the subset:

$$S = \mathbb{A}^2 \setminus \{(0,y), y \neq 0\}$$

This is not a quasi-affine variety. The Zariski-closure of S is the whole of $\mathbb{A}^2$; in fact the closure of S in the usual sense is the whole of $\mathbb{A}^2$. So the only affine variety containing S is $\mathbb{A}^2$ itself, but S is not Zariski open in $\mathbb{A}^2$. △

Before we can define isomorphisms between quasi-affine varieties we must first define regular functions between quasi-affine varieties, and to do this we need to generalize from polynomial functions to *rational functions.*

In general rational functions are subtle, but we'll start with rational functions on $\mathbb{A}^n$ which are pretty straight-forward.

Definition 8.5. A **rational function** on $\mathbb{A}^n$ is a ratio

$$f/g$$

where $f, g \in \mathbb{C}[x_1, ..., x_n]$ are two polynomials and $g \neq 0$.

There's an obvious equivalence relation we should use here, if h is any non-zero polynomial we should regard

$$f/g \qquad \text{and} \qquad fh/gh$$

as the same thing. Any rational function can be written uniquely in lowest terms by cancelling all common factors between f and g. (This only works because the polynomial ring is a UFD! We'll return to this point later.) Also note that f_1/g_1 and f_2/g_2 are equivalent rational functions iff:

$$f_1 g_2 = f_2 g_1$$

The set of all rational functions on $\mathbb{A}^n$, up to equivalence, is denoted:

$$\mathbb{C}(x_1, .., x_n)$$

This set is evidentally a field, rational functions can be added, subtracted, multiplied and divided in the obvious way. In particular the inverse of f/g is g/f.

Try not to confuse the field $\mathbb{C}(x_1, ..., x_n)$ with the ring $\mathbb{C}[x_1, ..., x_n]$. It's unfortunate that history has left us with such similar notation.

Despite the name, a rational function is *not* a function on $\mathbb{A}^n$. Approximately, it's the function

$$x \mapsto f(x)/g(x)$$

but this only makes sense if $g(x) \neq 0$, so f/g only defines a function on the Zariski open subset $\mathbb{A}^n \setminus V(g)$. This subset is called the set of **regular points** of the rational function f/g.

Remark 8.6. Perhaps you'd like to say that if $g(x) = 0$ then the value of the function is infinity. Once we've learnt about projective geometry we'll see that it is possible to make sense of this, as long as we don't also have $f(x) = 0$.

Rational functions give us more examples of functions which are 'algebraic', *i.e.* they could be defined with coefficients in any field. On quasi-affine varieties this extra flexibility is essential, and we're going to use them to formulate the definition of a *regular function.* The full definition is a bit complicated so we'll start with the easiest class of quasi-affine varieties: the Zariski open subsets in $\mathbb{A}^n$.

Definition 8.7. Let $U \subset \mathbb{A}^n$ be a Zariski open subset. A **regular function** on U is a rational function $f/g \in \mathbb{C}(x_1, ..., x_n)$ such that g never vanishes on U.

In other words, U is contained in the set of regular points of f/g. In this definition we may as well assume that f/g is written in lowest terms, since $fh/gh = f/g$ defines a regular function on $\mathbb{A}^n \setminus V(g)$ not just on the smaller open set $\mathbb{A}^n \setminus V(gh)$.

It's clear that the set of regular functions on U forms a ring, which is a subring of the field $\mathbb{C}(x_1, ..., x_n)$. This ring can be interesting but it's not as important as it was in the affine case, as we shall see.

Example 8.8.

1. Let $U = \mathbb{A}^1 \setminus 0$. A rational function $f/g \in \mathbb{C}(x)$ is regular in U iff g has no zeroes away from the origin, which means g must be a scalar multiple of x^k for some $k \in \mathbb{N}$. So the set of regular functions on U is:

$$\left\{ f(x)/x^k,\ f(x) \in \mathbb{C}[x],\ k \in \mathbb{N} \right\} \ \subset \mathbb{C}(x)$$

It's easy to see that this is a subring of $\mathbb{C}(x)$. In this ring the element x has a multiplicative inverse $1/x$, which it didn't have in the ring $\mathbb{C}[x]$; in fact it is the smallest subring of $\mathbb{C}(x)$ containing $\mathbb{C}[x]$ with this property. It's called the *localization* of $\mathbb{C}[x]$ at the element x.

2. Let $U = \mathbb{A}^1 \setminus \{0, 1\}$. Then the set of regular functions on U is the subring

$$\left\{ \frac{f(x)}{x^k(x-1)^m},\ f(x) \in \mathbb{C}[x],\ k, m \in \mathbb{N} \right\} \subset \mathbb{C}(x)$$

of rational functions where the denominators only have zeroes at 0 or 1. In this ring both x and $x - 1$ have multiplicative inverses.

3. Generalizing the above, let $U \subset \mathbb{A}^1$ be any Zariski open subset. Then U is the complement of some finite set $\{p_1, ..., p_t\} \subset \mathbb{A}^1$, and the ring of regular functions on U is:

$$\left\{ \frac{f(x)}{(x-p_1)^{k_1}...(x-p_t)^{k_t}}\ f(x) \in \mathbb{C}[x],\ k_1, ..., k_t \in \mathbb{N} \right\} \subset \mathbb{C}(x)$$

$\triangle$

The more points we delete from $\mathbb{A}^1$, the more regular functions we get. In some sense, the field $\mathbb{C}(x)$ is the set of regular functions we get when we delete *all* the points of $\mathbb{A}^1$! But maybe we shouldn't take this too seriously.

Example 8.9. Let $U = \mathbb{A}^2 \setminus (0, 0)$. It is not possible for a polynomial $g(x, y)$ to vanish only at the origin, unless g is constant (see Lemma 2.5), hence f/g only defines a regular function on U if it is actually a polynomial. So the ring of regular functions on U is the polynomial ring $\mathbb{C}[x, y]$. $\triangle$

The previous example shows that looking at the ring of regular functions is not usually enough to distinguish quasi-affine varieties, since $\mathbb{A}^2$ and $\mathbb{A}^2 \setminus (0, 0)$ produce the same ring.

We now have enough to define regular maps between quasi-affine varieties, as long as the source is either affine or a Zariski open subset of $\mathbb{A}^n$.

Definition 8.10. Let $U \subset \mathbb{A}^n$ be a Zariski open subset and let $V \subset \mathbb{A}^k$ be any quasi-affine variety. A **regular function** $F : U \to V$ is a k-tuple

$$F = (f_1/g_1,\ ...,\ f_k/g_k)$$

of regular functions on U, such that $F(U) \subset V$.

As always, an **isomorphism** is a regular function with a regular inverse.

Example 8.11. Let $U = \mathbb{A}^1 \setminus 0$, and let V be the affine variety:

$$V = V(xy - 1) \subset \mathbb{A}^2$$

Consider the functions:

$$\begin{aligned} F : U &\to \mathbb{A}^2 \\ t &\mapsto (t, 1/t) \end{aligned} \qquad\qquad \begin{aligned} G : V &\to \mathbb{A}^1 \\ (x, y) &\mapsto x \end{aligned}$$

Observe that F is indeed a regular function on U, and $F(U) \subset V$. Also G is a regular function on V, and $G(V) \subset U$. These two functions are inverse to each other, hence U and V are isomorphic.

So although U looks like it's just a quasi-affine variety, it's actually isomorphic to an affine variety. This isomorphism is reflected in the ring of regular functions, since

$$\mathbb{C}[V] = \mathbb{C}[x, y]/(xy - 1)$$

is isomorphic to the ring of regular functions on U that we computed in Example 8.8, using the map $y \mapsto 1/x$. △

This example generalizes easily to show that any quasi-affine variety of the form

$$U = \mathbb{A}^n \setminus V(f)$$

is isomorphic to an affine variety (exercise). However these quasi-affine varieties are quite special; for example it's possible to prove that $\mathbb{A}^2 \setminus (0, 0)$ is not isomorphic to any affine variety. The key point is that the ring of regular functions cannot tell that this space is different from $\mathbb{A}^2$.

Now let's try to understand regular functions on more complicated quasi-affine varieties. To see why the definition is subtle we have to look at something higher dimensional.

Example 8.12. Let V be the affine variety:

$$V = V(xy - zw) \subset \mathbb{A}^4$$

V contains the whole plane $y = z = 0$ and if we delete this plane we get a quasi-affine variety:

$$U = V \setminus V(z, y)$$

Consider the rational functions:

$$x/z \qquad \text{and} \qquad w/y \quad \in \mathbb{C}(x, y, z, w)$$

The first one is regular whenever $z \neq 0$, so it defines a function on the Zariski open subset $U_1 = U \setminus V(z)$, which is not the whole of U. The second one is regular when $y \neq 0$, so it defines a function on $U_2 = U \setminus V(y)$, which again is not the whole of U. But on the intersection $U_1 \cap U_2$ these are actually the same function, because

$$(x, y, z, w) \in V \implies \frac{x}{z} = \frac{w}{y}$$

provided neither z nor y are zero. Between them U_1 and U_2 cover the whole of U so we can define a function on U by:

$$F : U \to \mathbb{C}$$
$$(x, y, z, w) \mapsto \begin{cases} x/z & \text{if } z \neq 0 \\ w/y & \text{if } y \neq 0 \end{cases}$$

But there is no single rational function in $\mathbb{C}(x, y, z, w)$ which is regular on the whole of U and restricts to give this function. △

We want our definition of 'regular function' to allow functions such as F from the above example.

Definition 8.13. Let $U \subset \mathbb{A}^n$ be any quasi-affine variety. We say a function $F : U \to \mathbb{C}$ is **regular** if there exists a finite cover

$$U = U_1 \cup ... \cup U_k$$

of U by Zariski open subsets, and rational functions

$$f_1/g_1,\ ...,\ f_k/g_k\ \in \mathbb{C}(x_1, ..., x_n)$$

such for each i the polynomial g_i does not vanish in U_i, and:

$$\left.\frac{f_i}{g_i}\right|_{U_i} \equiv F|_{U_i}$$

So a regular function can be described by a ratio of polynomials locally (in the Zariski topology), but perhaps not globally. Given this definition it's easy to write down the definition of a **regular function** between any two quasi-affine varieties, it's a function from $F : U \to U'$ whose components are regular functions.

Lemma 8.14. *If U, U', U'' are quasi-affine varieties and $F : U \to U'$ and $G : U' \to U''$ are regular functions then the composition $G \circ F$ is regular.*

Proof. Exercise. ☐

And now it's clear what it means for two quasi-affine varieties to be **isomorphic**.

However, there is a technical issue here that we must address. There are two kinds of varieties for which we have already defined the notion of a 'regular function', namely affine varieties, and Zariski open subsets of $\mathbb{A}^n$. So we must check that our new definition agrees with our previous definitions in these two cases. The precise statements we need are:

(1) Let $U \subset \mathbb{A}^n$ be a Zariski open subset. Suppose $F : U \to \mathbb{C}$ is a regular function in the sense of Definition 8.13. Then there is a single rational function $f/g \in \mathbb{C}(x_1, ..., x_n)$ such that g never vanishes inside U and $F = (f/g)|_U$.

(2) Let $V \subset \mathbb{A}^n$ be an affine variety, and let $F : V \to \mathbb{C}$ be a regular function in the sense of Definition 8.13. Then there is a single polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ such that $F = f|_V$.

Since the proofs are a little fiddly (particularly for (2)) we banish them to Appendix A.

Writing down a regular function on a quasi-affine variety U looks like a lot of work: we must first specify an open cover $U = U_1 \cup ... \cup U_k$, then write down a rational function f_i/g_i for each U_i (and make sure they agree on the overlaps). But if we assume that U is an open subset of an *irreducible* affine variety V then the process is much simpler.

Proposition 8.15. *Let U be a Zariski open subset of an irreducible affine variety $V \subset \mathbb{A}^n$, and let*

$$F, G : U \to \mathbb{C}$$

be two regular functions. Suppose $U' \subset U$ is a non-empty Zariski open subset such that $F|_{U'} = G|_{U'}$. Then F and G agree on the whole of U.

Proof. Pick any point $p \in U \setminus U'$, we want to show that $F(p) = G(p)$.

By assumption, we can find some Zariski open set $U_1 \ni p$ and some rational function $f_1/g_1 \in \mathbb{C}(x_1, ..., x_n)$ which is regular on U_1, such that $F|_{U_1} = (f_1/g_1)|_{U_1}$. We can also find a $U_2 \ni p$ and f_2/g_2 such that $G|_{U_2} = (f_2/g_2)|_{U_2}$. Consider the Zariski open subset:

$$W = U' \cap U_1 \cap U_2$$

Since V is irreducible Lemma 7.6 says that W is non-empty. Inside W the functions F and G agree, so the polynomial

$$f_1 g_2 - f_2 g_1$$

vanishes on W. But by Lemma 7.11 the Zariski closure of W is the whole of V, hence $f_1 g_2 - f_2 g_1$ vanishes on the whole of V. This means that $F(p) = G(p)$. □

This means that on a variety of this form the first bit of data $(U_1, f_1/g_1)$ determines the whole regular function F. Any regular function which agrees with f_1/g_1 on the subset U_1 must agree with F everywhere.

However, if you just write down U_1 and f_1/g_1 then there is no guarantee that you can extend it to a regular function on the whole of U. We will think about this issue soon when we study *rational functions* on irreducible affine varieties. The proposition we've just proved will be a key technical result.

If we drop the assumption that V is irreducible then the proposition fails.

Example 8.16. Let U be the quasi-affine variety:

$$U = V(xy) \setminus \{(0, 0)\} \subset \mathbb{A}^2$$

We can cover U by the two Zariski opens subsets $U_1 = U \setminus V(x)$ and $U_2 = U \setminus V(y)$, and they don't intersect. A rational function f_1/g_1 defines a regular function on U_1 provided that g_1 is never zero in U_1, and f_2/g_2 defines a regular function on U_2 provided that g_2 is never zero in U_2. If we patch the two together we get a function F on U, and F is regular by definition. But knowing the values of F on U_1 tells you nothing about its values on U_2. △

9 Rational functions: function fields

Rational functions can be important even when they're not regular everywhere. However if we allow this possibility then they stop being actual functions, so what are they?

There are (at least) two answers to this question, and we'll start with a more algebraic approach. If you've ever seen the formal construction of $\mathbb{Q}$ from $\mathbb{Z}$, then the following definition should look familiar.

Definition 9.1. Let R be any integral domain. The **fraction field** of R is the set of expressions

$$\frac{f}{g}, \quad f, g \in R, \quad g \neq 0$$

up to the equivalence relation generated by:

$$\frac{f}{g} \sim \frac{hf}{hg} \quad \text{for any } h \neq 0 \in R$$

This definition is not as simple as it first appears.

- This definition goes badly wrong if R is not an integral domain. If we have zero divisors g, h, then $1/g$ is a sensible fraction, but multiplying top and bottom by h gives $h/hg = h/0$ which is not allowed.
- As the name suggests the fraction field of R is indeed a field, using the obvious operations. The ring R sits inside this field as the set of elements of the form $f/1$.
- The binary relation $\sim$ occuring in this definition is not itself an equivalence relation; for example it is not usually symmetric. But (like any relation) it does *generate* an equivalence relation, and that is the equivalence relation we use. It's easy to prove that this equivalence relation is exactly:

$$f_1/g_1 \sim f_2/g_2 \quad \text{iff} \quad f_1g_2 = f_2g_1 \in R$$

Of course if $R = \mathbb{C}[x_1, ..., x_n]$ then the fraction field is $\mathbb{C}(x_1, ..., x_n)$. If you felt like Definition 8.5 was a bit sloppy you were correct, and this is one way to make it precise.

If R is a UFD, for example $\mathbb{Z}$ or a polynomial ring, then it is straight-forward to tell if two elements of the fraction field are equivalent. Any fraction f/g can be written in *lowest terms* by cancelling all common factors, and then f_1/g_1 is equivalent to f_2/g_2 iff when we write them in lowest terms we get the same expression. This means that the fields $\mathbb{Q}$ or $\mathbb{C}(x_1, .., x_n)$ are quite easy to work with. However, if the ring is not a UFD then we must be more careful.

Example 9.2. Let $V = V(xy - zw) \subset \mathbb{A}^4$, as in Example 8.12. It's easy enough to convince yourself that $xy - zw$ is an irreducible polynomial, so V is irreducible, $I(V) = (xy - zw)$ and

$$\mathbb{C}[V] = \frac{\mathbb{C}[x, y, z, w]}{(xy - zw)}$$

is a domain (Proposition 5.4). Hence this ring has an associated fraction field.

Now neither y nor z are zero in $\mathbb{C}[V]$, so

$$x/z \qquad \text{and} \qquad w/y$$

both define elements of the fraction field. What's less obvious is that they define the *same* element. But this is true, because $xy = zw \in \mathbb{C}[V]$ by definition. If you want to think in terms of 'multiplying by common factors', we have to use the chain of relations

$$\frac{x}{z} \sim \frac{xw}{zw} = \frac{xw}{xy} \sim \frac{w}{y}$$

(it can't be done in a single step). In this field, when we write a fraction in 'lowest terms' the answer is not unique. △

Now we can give a precise definition of a rational function. We'll only consider rational functions on affine varieties, not on quasi-affines, and we must assume that our varieties are irreducible (or $\mathbb{C}[V]$ will not be a domain).

Definition 9.3. Let V be an irreducible affine variety. The **function field** of V is the field of fractions of $\mathbb{C}[V]$, and we denote it by $\mathbb{C}(V)$. A **rational function** on V is an element of $\mathbb{C}(V)$.

For our purposes, the terms 'function field' and 'fraction field' are interchangable. Again, try not to get confused between the ring $\mathbb{C}[V]$ and the field $\mathbb{C}(V)$.

Example 9.4. If $V = \mathbb{A}^n$ then $\mathbb{C}(V) = \mathbb{C}(x_1, ..., x_n)$. △

Example 9.5. Let $V = V(xy) \subset \mathbb{A}^2$ (the node). This variety is not irreducible, and $\mathbb{C}(V)$ is not defined. The problem is that $1/x$ and $1/y$ make sense, but $1/xy$ does not since xy vanishes identically. We should not attempt to work with rational functions on reducible varieties! △

Example 9.6. Let $V = V(xy - z^2) \subset \mathbb{A}^3$. This is called an *ordinary double point* (ODP) singularity, also known as an A_1 singularity. The defining polynomial is irreducible, so V is irreducible and

$$\mathbb{C}[V] = \frac{\mathbb{C}[x, y, z]}{(xy - z^2)}$$

is a domain. Let's examine the function field $\mathbb{C}(V)$.

In this field the variable y is redundant, because:

$$y \sim \frac{z^2}{x} \tag{9.7}$$

So any rational function on V is equivalent to an expression of the form:

$$\frac{f(x, z)}{g(x, z)}$$

Since (9.7) is the 'only' relation, this suggests that $\mathbb{C}(V)$ is actually isomorphic to the field $\mathbb{C}(x, z)$.

We can say this more precisely. A ring homomorphism $\Phi : R \to S$ between two domains will induce a homomorphism between their fraction fields, provided that Φ has no kernel. We can define a ring homomorphism

$$\Phi : \mathbb{C}[x, z] \to \mathbb{C}[V]$$

by sending x and z to their equivalence classes in $\mathbb{C}[V]$, and this homomorphism has no kernel because no polynomial of the form $f(x, z)$ can lie in the ideal $(xy - z^2)$. So we get a field homomorphism from $\mathbb{C}(x, z)$ to $\mathbb{C}(V)$, and this is surjective because of (9.7). So it must be an isomorphism of fields.

So if we look at function fields, we don't see a difference between V and $\mathbb{A}^2$. But V is certainly not isomorphic to $\mathbb{A}^2$, because $\mathbb{C}[V]$ is not a UFD. This shows that function fields only know some information about a variety, not everything. We will understand this more precisely in due course. △

10 Rational functions: partially-defined regular functions

Let $V \subset \mathbb{A}^n$ be an irreducible affine variety. Roughly speaking, a rational function on V is what we get when we take a rational function f/g on $\mathbb{A}^n$ and restrict it to V, since

$$\left.\frac{f}{g}\right|_V = \frac{f|_V}{g|_V}$$

is a ratio of regular functions on V. But f/g is not actually a function on $\mathbb{A}^n$, only on the open set $\mathbb{A}^n \setminus V(g)$, so when we 'restrict to V' we only get a function on the open subset $V \setminus V(g)$. We're going to show that we can interpret the function field $\mathbb{C}(V)$ as regular functions, but which are only defined on some open subset of V.

Example 10.1. Let $V = V(y - x^2) \subset \mathbb{A}^2$. The rational function $1/x$ defines a function on the Zariski open set

$$U = V \setminus (0,0) \subset V$$

by $F : (x,y) \to 1/x$. This F is a regular function on the quasi-affine variety U.

Now consider the rational function $1/(y - x^2) \in \mathbb{C}(x,y)$. The denominator vanishes everywhere on V, so this doesn't define an actual function on any subset on V. Algebrically, we can observe that the class of $y - x^2$ is zero in $\mathbb{C}[V]$, so $1/(y - x^2)$ doesn't define an element of the fraction field $\mathbb{C}(V)$. △

Example 10.2. Let $V = V(xy - zw) \subset \mathbb{A}^4$. As we observed in Example 8.12, the rational function $x/z \in \mathbb{C}(x,y,z,w)$ defines a regular function:

$$F_1 : U_1 = V \setminus V(z) \longrightarrow \mathbb{C}$$

Similarly,the rational function $w/y \in \mathbb{C}(x,y,z,w)$ defines a regular function:

$$F_2 : U_2 = V \setminus V(y) \longrightarrow \mathbb{C}$$

But x/z and w/y define the same element of $\mathbb{C}(V)$ (Example 9.2), and indeed F_1 and F_2 agree on the open set $U_1 \cap U_2$. As we saw, we can patch them together to define a function

$$F : U = U_1 \cup U_2 \longrightarrow \mathbb{C}$$

on this larger open set, and (by definition) F is a regular function. △

In the preceding example the information of (U_1, F_1) gives a 'partially-defined regular function' on V, and so does (U_2, F_2), or (U, F). We want to view all three as being equivalent.

Definition 10.3. Let V be an irreducible affine variety. A **rational function** on V is an equivalence class of pairs (U, F), where $U \subset V$ is a non-empty Zariski open subset, and

$$F : U \to \mathbb{C}$$

is a regular function. Two such pairs (U_1, F_1) and (U_2, F_2) are defined to be equivalent if:

$$F_1|_{U_1 \cap U_2} \equiv F_2|_{U_1 \cap U_2}$$

We must verify that this definition makes sense, and agrees with our previous definition that a *rational function* is an element of the function field $\mathbb{C}(V)$.

Proposition 10.4.

(1) *The relation in Definition 10.3 is an equivalence relation.*

(2) *A rational function on V is the same thing as an element of $\mathbb{C}(V)$.*

We've done all the real work for the proof of this lemma in Section 8, now it's just a matter of tediously unwinding all the definitions.

Proof.

(1) The relation is symmetric and reflexive by definition so we just have to check transitivity. Suppose $(U_1, F_1), (U_2, F_2), (U_3, F_3)$ are three such pairs such that F_1 and F_2 agree on $U_1 \cap U_2$ and F_2 and F_3 agree on $U_2 \cap U_3$. Since V is irreducible $U_1 \cap U_2 \cap U_3$ must be non-empty, and F_1 and F_3 agree on this set. Then by Proposition 8.15 they must agree on the whole of $U_1 \cap U_3$.

(2) First we show that elements of $\mathbb{C}(V)$ define rational functions in the sense of Definition 10.3. Let $\psi \in \mathbb{C}(V)$. Pick a representative expression f_1/g_1 for ψ with $f_1, g_1 \in \mathbb{C}[V]$. Now pick polynomials $\hat{f}_1$ and $\hat{g}_1$ representing f_1 and g_1, then we have an open set $U_1 = V \setminus V(\hat{g}_1)$ and a regular function

$$F_1 : U_1 \to \mathbb{C}$$

by restricting $\hat{f}_1/\hat{g}_1$ to U. Obviously U_1 and F_1 only depend on f_1 and g_1 and not on our choice of polynomials $\hat{f}_1$ and $\hat{g}_1$. Now suppose f_2/g_2 is another representative of ψ, where $f_2, g_2 \in \mathbb{C}[V]$. Picking polynomials again, we get a regular function

$$F_2 : U_2 \to \mathbb{C}$$

where $U_2 = V \setminus V(\hat{g}_2)$ and F_2 is the restriction of $\hat{f}_2/\hat{g}_2$. By definition $f_1 g_2 - f_2 g_1 = 0 \in \mathbb{C}[V]$, so $\hat{f}_1 \hat{g}_2 - \hat{f}_2 \hat{g}_1$ vanishes on V, hence F_1 and F_2 agree on the intersection $U_1 \cap U_2$. So (U_1, F_1) and (U_2, F_2) are equivalent.

Now we go in the other direction. Let $U_1 \subset V$ be a Zariski open subset, and $F_1 : U_1 \to \mathbb{C}$ a regular function. By assumption we can find a non-empty Zariski open $U_1' \subset U_1$ and a rational function $f_1/g_1 \in \mathbb{C}(x_1, ..., x_n)$, regular on U_1', such that $F|_{U_1'} = (f_1/g_1)|_{U_1'}$. Then g_1 doesn't vanish on V so

$$\frac{f_1|_V}{g_1|_V}$$

is an element of $\mathbb{C}(V)$.

Now let $U_2 \subset V$ be another Zariski open, and $F_2 : U_2 \to \mathbb{C}$ another regular function, such that F_1 and F_2 agree on the intersection $U_1 \cap U_2$. Again we can pick a rational function $f_2/g_2 \in \mathbb{C}(x_1, ..., x_n)$ which restricts to give F_2 on some Zariski open subset $U_2' \subset U_2$, and again f_2/g_2 defines a class in $\mathbb{C}(V)$. Since F_1 and F_2 agree where they are both defined, the polynomial $f_1 g_2 - f_2 g_1$ vanishes on $U_1' \cap U_2'$, so it vanishes on the whole of V. This says that

$$\frac{f_2|_V}{g_2|_V} \quad \text{and} \quad \frac{f_1|_V}{g_1|_V}$$

are equivalent in $\mathbb{C}(V)$.

□

If $\psi \in \mathbb{C}(V)$ is a rational function we can represent it as a regular function $F : U \to \mathbb{C}$ defined on some Zariski open subset of V. All the points in U are 'regular points' of the rational function ψ. But our chosen (U, F) might not be optimal, it might be possible to extend F to a regular function $F' : U' \to \mathbb{C}$ on some larger open subset. So to see all the 'regular points' of ψ we must consider all possible representatives.

Definition 10.5. Let $\psi \in \mathbb{C}(V)$ be a rational function on V. A point $p \in V$ is called a **regular point** of V if there is a representative f/g for ψ such that $g(p) \neq 0$.

If we want to think of a rational function as a partially-defined regular function (U, F) there is an equivalent way to say this: a point $p \in V$ is a **regular point** if there is some (U', F'), equivalent to (U, F), with $p \in U'$. The set of regular points is the union of all such open sets, hence it's an open subset of V.

Example 10.6. Revisiting Examples 8.12 or 10.2 again, let

$$V = V(xy - zw) \subset \mathbb{A}^4$$

and let $\psi \in \mathbb{C}(V)$ be the rational function defined by x/z. Every point in $U_1 = V \backslash V(z)$ is a regular point of ψ, but since ψ has an equivalent representative z/y every point in $U_2 = V \setminus V(y)$ is also regular. So the set of regular points of ψ is at least

$$U = U_1 \cup U_2 = V \setminus V(y, z)$$

and ψ defines a regular function F on U.

It's not hard to show that U is exactly the set of regular of points of ψ. Observe that the complement $V \setminus U$ is just the plane $V(y, z) \subset \mathbb{A}^4$, which is isomorphic to $\mathbb{A}^2$, and has co-ordinate ring $\mathbb{C}[x, w]$. Now suppose f/g is any representative of ψ. By definition we have

$$zf = xg$$

in the ring $\mathbb{C}[V]$. If we restrict these regular functions to $V(y, z)$ the function fz vanishes, and we get:

$$xg(x, 0, 0, w) = 0 \ \in \mathbb{C}[x, w]$$

But $\mathbb{C}[x, w]$ is a domain, so g must vanish on $V(y, z)$. Hence no point in $V(y, z)$ is regular. △

11 Rational maps

Now that we understand a rational function is a 'partially-defined regular function' from V to $\mathbb{C}$, we can start to think about rational functions between V and some other variety W.

Definition 11.1. Let $V \subset \mathbb{A}^n$ be an irreducible affine variety. A **rational map** from V to $\mathbb{A}^k$ is a k-tuple

$$\Psi = (\psi_1, ..., \psi_k), \quad \psi_i \in \mathbb{C}(V)$$

of rational functions on V.

If we let $U_i \subset V$ be the set of regular points of ψ_i then we see that Ψ only defines an actual function on the intersection:

$$U = \bigcap_{i=1}^{k} U_i$$

Each ψ_i is a regular function on U, so by definition Ψ defines a regular function from U to $\mathbb{A}^k$. Also note that U is not empty (since V is irreducible) so Ψ really is giving us a regular function on 'most' of V.

As before there is an alternative version of this definition:

Definition 11.2. Let $V \subset \mathbb{A}^n$ be an irreducible affine variety. A **rational map** from V to $\mathbb{A}^k$ is a pair (U, F) where $U \subset V$ is a non-empty Zariski open subset and $F : U \to \mathbb{A}^k$ is a regular map. Two such pairs are equivalent if the two functions agree on the intersection of the open subsets.

We can apply Proposition 10.4 to the individual components of Ψ or F and it follows immediately that Definitions 11.1 and 11.2 are equivalent (and that the relation in the second definition is an equivalence relation). The set of regular points of Ψ is the largest possible U.

We define the **image** of a rational map Ψ to be the image of the set of regular points points, *i.e.* the set $\{\Psi(x)\}$ for all x where this actually makes sense.

Definition 11.3. Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ be two irreducible affine varieties. A **rational map** from V to W is a rational map Ψ from V to $\mathbb{A}^k$ whose image lies in W.

We write rational maps as

$$\Psi : V \dashrightarrow W$$

to remind us that they're not really functions on V.

Example 11.4. Let:

$$\Psi = (t, 1/t) : \mathbb{A}^1 \dashrightarrow \mathbb{A}^2$$

This is a rational map, and it's regular on $U = \mathbb{A}^1 \setminus \{0\}$. If we let $V = V(xy - 1) \subset \mathbb{A}^2$ then $\Psi(U) \subset V$, so Ψ defines a rational map:

$$\Psi : \mathbb{A}^1 \dashrightarrow V$$

This map is 'nearly' an isomorphism, since $\Psi : U \to V$ is an isomorphism (see Example 8.11). △

Here's a small technical lemma which is helpful.

Lemma 11.5. *Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ be two irreducible affine varieties, let $U \subset V$ be a non-empty Zariski open subset, and let $F : U \to W$ be a regular function. Then (U, F) defines a rational map $\Psi : V \dashrightarrow W$.*

Proof. Using Definition 11.2 it's immediate that (U, F) defines a rational map $\Psi : V \dashrightarrow \mathbb{A}^k$. By assumption $F(U) \subset W$, but the set of regular points of Ψ might be larger than U so we must check that any additional regular points still get mapped into W.

Let $U' \subset V$ be the set of all regular points of Ψ, and let h be any polynomial that vanishes on W. Then $h \circ \Psi$ is a regular map from U' to $\mathbb{C}$ and it vanishes on the subset $U \subset U'$. By Proposition 8.15 $h \circ \Psi$ must be the zero function on the whole of U'. This holds for any $h \in I(W)$ so $\Psi(U') \subset W$. □

Next we want to talk about 'pulling-back' rational functions along rational maps.

Suppose $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ are irreducible. Let $\Psi : V \dashrightarrow W$ be a rational map, and let ϕ be a rational function on W. If we write down Ψ explicitly it's just a k-tuple

$$(f_1/g_1, ..., f_k/g_k)$$

of rational functions on $\mathbb{A}^n$, and ϕ is just a rational function on $\mathbb{A}^k$. Then naively, the composition of ϕ and Ψ is just:

$$\Psi^*(\phi) = \phi \circ \Psi = \phi\left(\frac{f_1}{g_1}, ..., \frac{f_k}{g_k}\right) \in \mathbb{C}(x_1, ..., x_n) \tag{11.6}$$

If $p \in \mathbb{A}^n$ is a regular point of Ψ, and $\Psi(p)$ is a regular point of ϕ, then certainly $\phi(\Psi(p))$ is given by evaluating the expression above at the point p.

The problem is that rational functions are not regular everywhere, so we must be slightly careful.

Example 11.7. Let

$$\Psi = (t, 1/t) : \mathbb{A}^1 \dashrightarrow \mathbb{A}^2$$

as in the previous example, and let:

$$\phi = \frac{1}{xy - 1} \in \mathbb{C}(x, y)$$

If try naively to compose ϕ and Ψ we get

$$\phi \circ \Psi : t \mapsto \frac{1}{t(\frac{1}{t}) - 1} = \frac{1}{0}$$

which is nonsense. The problem is of course that ϕ is not regular anywhere on the image of Ψ, so we can't define the value of $\phi \circ \Psi$ at any point.

We've seen this before in Example 10.1: the rational function $1/(xy - 1)$ doesn't define a rational function on $V(xy - 1)$. △

To avoid this problem we introduce the following definition:

Definition 11.8. A rational map $\Psi : V \dashrightarrow W$ is called **dominant** if the image of Ψ is not contained in any proper affine subvariety of W.

In other words, Ψ is dominant if the Zariski closure of the image of Ψ is the whole of W.

Lemma 11.9. *If $\Psi : V \dashrightarrow W$ is dominant, and $\phi \in \mathbb{C}(W)$ is any rational function on W, then there is a rational function $\Psi^*(\phi) \in \mathbb{C}(V)$ which agrees with the composition $\phi \circ \Psi$ wherever this is defined.*

Proof. Let $U \subset V$ be the regular points of Ψ, and $U' \subset W$ be the regular points of ϕ. Then Ψ and ϕ define regular functions:

$$F : U \to W \qquad \text{and} \qquad G : U' \to \mathbb{C}$$

The complement of U' is an affine subvariety, so Φ being dominant ensures that $F^{-1}(U')$ is non-empty. Then

$$G \circ F : F^{-1}(U') \to \mathbb{C}$$

is a regular function, and (the equivalence class of) this data is a rational function on V. □

In practice this procedure is very simple, if we write Ψ and ϕ down explicitly then the composition $\Psi^*(\phi)$ is just given by plugging the components of Ψ into ϕ, as in (11.6). On the the open subset $F^{-1}(U') \subset V$ we really are just composing functions, and nothing clever is happening. The point of the above lemma is that even if this subset is not the whole of V then we can still make sense of $\Psi^*(\phi)$ as a rational function on V.

Obviously every point in $F^{-1}(U')$ is a regular point of $\Psi^*(\phi)$. However it's important to realise that $\Psi^*(\phi)$ might have more regular points than this.

Example 11.10. Let $\Psi = (t, 1/t) : \mathbb{A}^1 \dashrightarrow \mathbb{A}^2$ again, and let $\phi = x^2 y \in \mathbb{C}(x, y)$. Then

$$\Psi^*(\phi) = t^2 \frac{1}{t} = t \quad \in \mathbb{C}(t)$$

which is regular everywhere. So even though Ψ is not regular at the point $t = 0$, the composition $\Psi^*(\phi)$ *is* regular at this point.

This is another good demonstration of why 'partially-defined regular functions' are subtle; at first sight we can only define $\phi \circ \Phi$ on the open set $\{t \neq 0\}$, but it actually extends to a regular function on the whole of $\mathbb{A}^1$. △

If we have a dominant rational map $\Psi : V \dashrightarrow W$ between two irreducible affine varieties we get a function

$$\Psi^* : \mathbb{C}(W) \longrightarrow \mathbb{C}(V)$$

by sending ϕ to $\Psi^*(\phi)$. It's easy to see that this function is a homomorphism of fields, and also $\mathbb{C}$-linear.

Proposition 11.11. *Let V and W be irreducible affine varieties. We have a bijection:*

$$\left\{ \begin{matrix} \textit{Dominant rational} \\ \textit{maps} \end{matrix} \quad V \dashrightarrow W \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \mathbb{C}\textit{-linear field} \\ \textit{homomorphisms} \end{matrix} \quad \mathbb{C}(W) \to \mathbb{C}(V) \right\}$$

$$\Psi \mapsto \Psi^*$$

Proof. This is very similar to Proposition 6.8 so we won't fill in all the details. Say $W \subset \mathbb{A}^k$, and let $y_1, ..., y_k$ be the co-ordinates on $\mathbb{A}^k$. If we have a field homomorphism $\alpha : \mathbb{C}(W) \to \mathbb{C}(V)$ then each $\alpha(y_i)$ is a rational function on V, and these form the components of a rational map $\Psi : V \dashrightarrow \mathbb{A}^k$. The fact that α lands in W, and is a dominant map to W, follows from the fact that α is well-defined on $\mathbb{C}(W)$. □

Example 11.12. Let $\Psi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ be the rational map:

$$\Psi(x, y) = \left(\frac{x}{y}, \frac{1}{x}\right)$$

Let's use (s, t) for the co-ordinates on the target $\mathbb{A}^2$. Then

$$\Psi^* : \mathbb{C}(s, t) \to \mathbb{C}(x, y)$$

is the field homorphism defined by $\Psi^*(s) = x/y$ and $\Psi^*(t) = 1/x$.

Now let $\Phi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ be the rational map:

$$\Phi(x, y) = \left(\frac{x}{y}, \frac{y}{x}\right)$$

This is not dominant, its image is contained in the subvariety $V = V(st - 1)$. Correspondingly if we try to define a field homomorphism $\Phi^* : \mathbb{C}(s, t) \to \mathbb{C}(x, y)$ by declaring $\Phi^* : s \mapsto x/y$ and $t \mapsto y/x$ then we fail, because $\Phi^*(1/(st - 1))$ doesn't make any sense.

However, Φ does define a dominant rational map $\Phi : \mathbb{A}^2 \dashrightarrow V$. It's easy to see that $\mathbb{C}(V) \cong \mathbb{C}(s)$ (since $t = 1/s \in \mathbb{C}(V)$), and we have a field homomorphism:

$$\Phi^* : \mathbb{C}(V) \to \mathbb{C}(x, y)$$
$$s \mapsto x/y$$

△

Now we can talk about composing rational maps in general. Suppose we have two rational maps $\Psi : V \dashrightarrow W$ and $\Phi : W \dashrightarrow X$. We'd like to define the 'composition' as a rational map:

$$\Phi \circ \Psi : V \dashrightarrow X$$

But we need to make the following assumption:

> There is at least one point $x \in V$ such that x is regular for Ψ and $\Psi(x)$ is regular for W.

Let's write $U \subset V$ for the set of regular points of Ψ and $U' \subset W$ for the set of regular points of Φ, and let $U'' = \Psi^{-1}(U') \cap U$. This subset is Zariski open, and our assumption is that it is non-empty. Then $\Phi \circ \Psi$ defines a regular function from U'' to W and hence (by Lemma 11.5) it defines a rational map from V to W. Note:

- If U'' is non-empty then it must be 'almost all' of V. On this subset we are just composing two functions, but outside this subset the composition doesn't make sense. If U'' was empty then the composition wouldn't make any sense, even as a rational function.

- In practice composing rational maps is easy, we just compose their components in the obvious way (11.6). But if you haven't checked the hypotheses this might produce nonsensical answers!

- If we assume Ψ is dominant then $\Phi \circ \Psi$ makes sense for any Ψ. If we assume that Φ is regular then $\Phi \circ \Psi$ makes sense for any Ψ.

Example 11.13. Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$, the cusp singularity. Since $x \notin I(V)$ the expression y/x is a rational function on V, *i.e.* it's a rational map:

$$\Psi = \frac{y}{x} : V \dashrightarrow \mathbb{A}^1$$

In Example 6.12 we considered the regular map:

$$\begin{aligned} F : \mathbb{A}^1 &\to V \\ t &\mapsto (t^2, t^3) \end{aligned}$$

Since Ψ is certainly regular on $V \setminus (0,0)$, and F is surjective and regular everywhere, both compositions

$$\Psi \circ F : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1 \qquad \text{and} \qquad F \circ \Psi : V \dashrightarrow V$$

are defined. The first composition is easy to compute, we have:

$$\Psi \circ F : t \mapsto t^3/t^2 = t$$

This is the identity map on $\mathbb{A}^1$, and it's regular everywhere. This is slightly surprising: $F(0) = (0,0)$, and $\Psi(0,0)$ isn't defined, but still the composed function $\Psi \circ F$ can be extended to $t = 0$ with no problems.

For the second composition we calculate

$$F \circ \Psi : (x, y) \mapsto \left(\left(\frac{y}{x}\right)^2, \left(\frac{y}{x}\right)^3 \right)$$

and we claim this is the identity map on V. To see this, let's restrict to the open subset $V \setminus (0,0)$. Here we are just composing regular functions, and $(x,y) \in V$ means that $y^2 = x^3$, hence $(y/x)^2 = x$ and $(y/x)^3 = y$. So the rational function $F \circ \Psi$ is given by the identity map on this open subset, and it's equivalent to the identity map on V. Again this might look surprising, Ψ is not regular at $(0,0)$ but the composition $F \circ \Psi$ is regular everywhere.

So in some sense F and Ψ are each other's inverses. But Ψ is not really a function on V! What's really happening is that we have regular functions

$$\Psi : V \setminus (0,0) \longrightarrow \mathbb{A}^1 \setminus 0 \qquad \text{and} \qquad F : \mathbb{A}^1 \setminus 0 \longrightarrow V \setminus (0,0)$$

which are mutually inverse, so this is an isomorphism of quasi-affine varieties. But the strange language of rational functions lets us view this as a 'partially-defined isomorphism' between V and $\mathbb{A}^1$. △

Let's capture the phenomenon of the previous example in a definition.

Definition 11.14. A rational map $\Psi : V \dashrightarrow W$ between two irreducible affine varieties is a **birational equivalence** if there exists a rational map $\Phi : W \dashrightarrow V$ such that both $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are defined, and $\Psi \circ \Phi = 1_W$ and $\Phi \circ \Psi = 1_V$.

If a birational equivalence exists we say V and W are **birationally equivalent**, or just **birational**.

Lemma 11.15. *Two irreducible affine varieties V and W are birational if and only if we can find Zariski open subsets $U \subset V$ and $U' \subset W$ such that U is isomorphic to U'.*

Proof. Exercise. □

An immediate corollary of Proposition 11.11 is:

Corollary 11.16. *V and W are birational iff we can find a $\mathbb{C}$-linear isomorphism of fields:*

$$\mathbb{C}(W) \xrightarrow{\sim} \mathbb{C}(V)$$

Example 11.17.

1. From Example 11.13 the cusp $V = V(y^2 - x^3)$ is birational to $\mathbb{A}^1$. So the function field $\mathbb{C}(V)$ must be isomorphic to $\mathbb{C}(t)$. What's the isomorphism? (Exercise.)

2. In Example 9.6 we considered the ODP singularity $V = V(xy - z^2)$, and we showed that $\mathbb{C}(V)$ is the field $\mathbb{C}(x, z)$. It follows that V is birational to $\mathbb{A}^2$. What's the birational equivalence? (Exercise.)

△

We now have a precise understanding of what information is carried in the function field $\mathbb{C}(V)$. Varieties V and W have the same function field iff their birational, *i.e.* iff they become isomorphic once we cut out a subvariety from both sides.

This means that field theory can tell us useful things about algebraic geometry. For example:

Theorem 11.18. *Any finitely-generated field extension of $\mathbb{C}$ is the fraction field of $\mathbb{C}[x_1, ..., x_n]/(f)$ for some $n \in \mathbb{N}$ and some irreducible polynomial f.*

This theorem is not particularly difficult to prove if you know some things about field extensions, but we won't do it. However, applying it to function fields tells us:

Corollary 11.19. *Any irreducible affine variety V is birational to a hypersurface.*

Proof. Apply the previous theorem to the field $\mathbb{C}(V)$, then by Corollary 11.16 V is birational to the hypersurface $V(f) \subset \mathbb{A}^n$. □

12 Tangent spaces, singularities and dimension

It's intuitively obvious that (for example) the cusp singularity $V(y^2 - x^3) \subset \mathbb{A}^2$ is a 'one-dimensional' variety, and the ODP singularity $V(xy - z^2)$ is 'two-dimensional'. But giving a precise defintion of 'dimension' for a variety is not so easy. There are at least three possible approaches, we will start with the approach that uses *tangent spaces*. As a bonus, this will also lead us to a precise definition of 'singularity'.

If we draw a curve in $\mathbb{R}^2$, and pick a point on the curve, it's clear what we mean by the 'tangent line' to the curve through our chosen point. If the curve is defined by an equation $f(x,y)=0$, you may remember that the tangent line to p is the set of vectors that are orthogonal to the vector:

$$\nabla f_p = (\partial_x f, \partial_y f)|_p$$

Similarly if we have a surface in $\mathbb{R}^3$ defined by $f(x,y,z)=0$, and we choose a point p on the surface, then there is a whole plane tangent to the surface at p, and again it's the set of vectors orthogonal to ∇f_p. Based on these observations, we're going to write down a definition of the *tangent space* T_pV to a point p on an affine variety V. The tangent space T_pV will be a vector space, so it has a dimension, and this will lead us to the definition of $\dim(V)$.

Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a polynomial, so it's a function $f : \mathbb{A}^n \to \mathbb{C}$. Fix $p \in \mathbb{A}^n$. Recall that the *derivative* of f at p is the linear map

$$Df_p = (\partial_1 f, ..., \partial_n f)|_p : \ \mathbb{C}^n \longrightarrow \mathbb{C}$$

where $\partial_i f$ is the ith partial derivative of f. Note:

- We're writing $\mathbb{C}^n$ not $\mathbb{A}^n$, because here we *do* care about the vector space structure. This function Df_p is a linear map.
- Even though derivatives are appearing, what we're doing here is algebraic. Of course in general differentiation needs analysis to make sense of it, but if f is just a polynomial then the procedure which sends f to $\partial_i f$ is a purely algebraic operation on polynomials, *i.e.* it makes sense over any field.

Definition 12.1. Let $V \subset \mathbb{A}^n$ be an affine variety and let $f_1, .., f_k$ generate $I(V)$. Fix $p \in V$. The **tangent space** to V at p is:

$$T_pV = \{v \in \mathbb{C}^n, \ (Df_j)_p(v) = 0, \ \forall j \in [1,k]\}$$

T_pV is a linear subspace of $\mathbb{C}^n$, it's the intersection of the kernels $\mathrm{Ker}(Df_j)_p$. Another way to say this is that T_pV is the kernel of the *Jacobian matrix*:

$$J_p = \left.\begin{pmatrix} \partial_1 f_1 & ... & \partial_n f_1 \\ \vdots & & \vdots \\ \partial_1 f_k & ... & \partial_n f_k \end{pmatrix}\right|_p \ : \ \mathbb{C}^n \to \mathbb{C}^k$$

This definition doesn't depend on your choice of generators for the ideal $I(V)$, because:

Lemma 12.2. *A vector v lies in T_pV if and only if $Df_p(v) = 0$ for every $f \in I(V)$.*

Proof. Exercise. □

This lemma gives an alternative definition of T_pV which is theoretically nicer since it doesn't involve any choices. But in practice we use the first definition.

Example 12.3. Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$ (the cusp). Then $I(V)$ is generated by $f = y^2 - x^3$ and for any $p \in \mathbb{A}^2$ we have:

$$Df_p = (-3x^2, \ 2y)|_p$$

If we choose $p = (1,1)$ this lies on V, and $Df_p = (-3, 2)$, so:

$$T_pV = \{(v_1, v_2) \in \mathbb{C}^2,\ -3v_1 + 2v_2 = 0\}$$

This is a 1-dimensional subspace.

In fact, suppose p is any point of V apart from $(0,0)$. Then Df_p is a non-zero vector, hence the linear map $Df_p : \mathbb{C}^2 \to \mathbb{C}$ has rank 1, so its kernel T_pV has dimension 1. However, at the point $p = (0,0) \in V$ we have $Df_p = (0,0)$, and $T_{(0,0)}V$ is the whole of $\mathbb{C}^2$. △

Example 12.4. Let $V = V(z - xy, z) \subset \mathbb{A}^3$ (this is actually isomorphic to the node $V(xy) \subset \mathbb{A}^2$). Then $I(V)$ is generated by $z - xy$ and z, and the Jacobian matrix at a point p is:

$$J_p = \begin{pmatrix} -y & -x & 1 \\ 0 & 0 & 1 \end{pmatrix}\bigg|_p : \mathbb{C}^3 \to \mathbb{C}^2$$

The points of V fall into three types, and we look at each in turn:

1. $p = (x, 0, 0)$ for $x \neq 0$. Then the rank of J_p is 2, so $\dim T_pV = 1$. In fact T_pV is the subspace $\langle(1,0,0)\rangle \subset \mathbb{C}^2$, regardless of the value of x.
2. $p = (0, y, 0)$ for $y \neq 0$. Again the rank of J_p is 2 and $\dim T_pV = 1$. Also $T_pV = \langle(0,1,0)\rangle$ for any y.
3. $p = (0,0,0)$. Here the rank of J_p drops to 1, so the dimension of T_pV jumps up to 2. It's the subspace spanned by $(1,0,0)$ and $(0,1,0)$.

△

When you're computing tangent spaces it's important to make sure that your chosen polynomials really do generate $I(V)$.

Example 12.5. Let $V = V(x^2) \subset \mathbb{A}^2$, which is the y-axis. If we set $f = x^2$ then:

$$Df|_{(x,y)} = (2x, 0) : \mathbb{C}^2 \to \mathbb{C}$$

At any point $(0, y) \in V$ this linear map is zero, so you might conclude that T_pV is 2-dimensional at every point. But this is obviously wrong since V is a line! Of course the mistake is that that x^2 doesn't generate $I(V)$, and if we set $f = x$ instead then we get the correct answers. △

Examples 12.3 and 12.4 show an important feature: the dimension of T_pV is 'generally' constant, but at certain special points it jumps up to higher values.

Definition 12.6. Let V be an irreducible affine variety. The **dimension** of V is:

$$\min_{p \in V} \dim T_pV$$

If $p \in V$ is such that $\dim T_pV > \dim V$ we call p a **singular point** of V

Points that are not singular are called either **non-singular points** or **regular points**. We'll try to use the former to avoid confusion with 'regular points' of rational functions. Unfortunately the latter is more common.

Example 12.7. From Example 12.3 we see that the cusp $V(y^2 - x^3)$ has dimesion 1, and $(0,0)$ is the only singular point. △

Why do we assume V is irreducible in this definition?

Example 12.8. Let

$$\begin{aligned} V &= V(xz, yz) \subset \mathbb{A}^3 \\ &= V(z) \cup V(x,y) \end{aligned}$$

which is the union of the x, y-plane and the z-axis. If we take a point $p = (x, y, 0) \in V$, where $(x, y) \neq (0, 0)$, then

$$T_pV = \big\langle (1,0,0), (0,1,0) \big\rangle \subset \mathbb{C}^3$$

which is 2-dimensional. But if we take a point $p = (0, 0, z)$ on the other irreducible component (with $z \neq 0$) then

$$T_pV = \big\langle (0,0,1) \big\rangle \subset \mathbb{C}^3$$

which is 1-dimensional. At the origin the tangent space is the whole of $\mathbb{C}^3$.

So the minimum value of T_pV is 1, but this misses the geometric fact that one of the irreducible components of V is 2-dimensional. Also it doesn't seem right to declare that every point in $V(z)$ is 'singular', only the origin should be a singular point. △

Some authors define the dimension of a reducible variety to be the maximum of the dimension of each irreducible component, so the previous example would have dimension 2. But in my opinion it's more helpful to say that a reducible variety might be a union of varieties of different dimensions, and not try to assign it a single dimension.

However, we can generalise the definition of a *singular point* to reducible varieties. Before we do so, observe that in the examples we've seen so far the set of singular points in V is always a subvariety, so the set of non-singular points is Zariski open. In fact this is always true.

Lemma 12.9. *Let $V \subset \mathbb{A}^n$ be an irreducible affine variety. Then the set of singular points of V is an affine subvariety of V.*

Proof. Suppose $V \subset \mathbb{A}^n$, and $I(V) = (f_1, ..., f_k)$, and $\dim V = d$. Then the set of singular points in V is the locus where the Jacobian matrix

$$J = \begin{pmatrix} \partial_1 f_1 & \dots & \partial_n f_1 \\ \vdots & & \vdots \\ \partial_1 f_k & \dots & \partial_n f_k \end{pmatrix}$$

has rank $< n - d$. Now recall two facts from linear algebra:

- If M is an $k \times n$ matrix then an $(r \times r)$ **minor** of M is a square matrix obtained by deleting $k - r$ rows and $n - r$ columns from M.
- The rank of M is $< r$ if and only if the determinant of every $r \times r$ minor of M is zero.

(This is elementary to prove, just put M in echelon form.) Since J is a matrix of polynomials the $(n-d)\times(n-d)$ minors of J are polynomials. These polynomials cut out the singular locus, so it's a subvariety of V. □

In fact we could prove a bit more here. Let's define a function

$$\begin{aligned} d : V &\longrightarrow \mathbb{N} \\ p &\mapsto \dim T_pV \end{aligned}$$

and set $V_{\geq k} = \{p \in V, d(p) \geq k\}$ for each $k \in \mathbb{N}$. The proof of the previous lemma shows that each $V_{\geq k}$ is an affine subvariety. By definition $V = V_{\geq \dim V}$ and $V_{\geq (\dim V+1)}$ is the singular locus. For $k > \dim V + 1$ the subvarieties $V_{\geq k}$ are a chain of subvarieties of the singular locus, consisting of 'worse and worse' singular points.

The set of non-singular points is the Zariski open set:

$$V \setminus V_{\geq (\dim V+1)}$$

The function d is constant (and equal to $\dim V$) on this open set and it jumps up in value outside this set. This suggests how we should define a singular point on a reducible variety.

Definition 12.10. Let V be any affine variety. We say a point $p \in V$ is a **non-singular point** (or **regular point**) if there exists a Zariski open neighbourhood $U \subset V$ of p such that:

$$\dim T_qV = \dim T_pV \text{ for all } q \in U$$

If not we say p is a **singular point**.

So p is non-singular if d is constant in a Zariski neighbourhood of p. We leave it as an exercise to verify that when V is irreducible this definition agrees with our previous one (Definition 12.6).

Example 12.11. If $V = V(xz, yz)$ then the origin is the only singular point (exercise). △

Why is our definition of dimension so complicated? If $V = V(f_1, ..., f_k) \subset \mathbb{A}^n$ then V is cut out of an n-dimensional space by k equations, so isn't it obvious that $\dim V = n - k$? Actually no, this is clearly false in general.

Example 12.12. Let $V = V(x, y, x - y) \subset \mathbb{A}^3$, which is the z-axis. This is 1-dimensional not 0-dimensional. △

OK, but in this example clearly we could have set $V = V(x, y)$ instead. So perhaps you'd like to write:

(Incorrect) Definition. Let $V \subset \mathbb{A}^n$ be an affine variety and let $f_1, ..., f_k$ be a minimal set of generators for $I(V)$. Then the **dimension** of V is $n - k$.

Example 12.13. Let $V = V(xy, xz, yz) \subset \mathbb{A}^3$, which is the union of the three axes. It's possible to show that this ideal cannot be generated by less than 3 polynomials, so under the above definition we'd have to say that $\dim V = 0$. But clearly the dimension of V should be 1. △

If you object that this example is reducible, consider the next one:

Example 12.14. Define:

$$\begin{aligned} F : \mathbb{A}^2 &\longrightarrow \mathbb{A}^4 \\ (s,t) &\mapsto (s^3, s^2t, st^2, t^3) \end{aligned}$$

If we use x, y, z, w as co-ordinates on the target $\mathbb{A}^4$ then clearly the image of F is contained in:

$$V = V(xz - y^2,\, yw - z^2,\, xw - yz)$$

This variety is called the *twisted cubic*. With a little messing around you can show that F is a surjection onto V, which implies that V is irreducible (Lemma 6.13). Also it's possible to prove that $I(V)$ cannot be generated by fewer than 3 polynomials.

However the dimension of V is 2, not 1. The Jacobian matrix of these polynomials is

$$\begin{pmatrix} z & -2y & x & 0 \\ 0 & w & -2z & y \\ w & -z & -y & x \end{pmatrix}$$

and you can check that this matrix has rank 2 at every point in V, except for the origin where it has rank zero. So $\dim V = 2$ and the origin is the only singular point. △

However, for hypersurfaces the obvious guess for the dimension is always correct.

Lemma 12.15. *Let $V = V(f) \subset \mathbb{A}^n$ be an irreducible hypersurface. Then $\dim V = n - 1$.*

Proof. Exercise. □

Suppose $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ are isomorphic affine varieties, via some isomorphism $F : V \xrightarrow{\sim} W$. Since V and W are 'the same', if we pick a point $p \in V$ then we should expect T_pV and $T_{F(p)}W$ to be 'the same' vector space. But T_pV is (by definition) a subspace of $\mathbb{C}^n$, whereas $T_{F(p)}W$ is a subspace of $\mathbb{C}^k$, so they are not literally the same vector space. The correct statement is that they are isomorphic.

Recall that if $F = (f_1, ..., f_k) : \mathbb{A}^n \to \mathbb{A}^k$ is a regular map (or in fact any smooth function) then for $p \in \mathbb{A}^n$ the *derivative* of F at p is the linear map

$$DF_p : \mathbb{C}^n \to \mathbb{C}^k$$

given by the Jacobian matrix with entries $(\partial_j f_i|_p)$.

Lemma 12.16. *Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ be affine varieties and let $F : V \to W$ be a regular map. Pick $p \in V$ and set $q = F(p)$.*

(i) We have an induced linear map:

$$DF_p : T_pV \longrightarrow T_qW$$

(ii) If $G : W \to X$ is a second regular map then:

$$D(G \circ F)_p = DG_q \circ DF_p$$

Proof. *(i)* Pick polynomials representing each component of F, so we get a regular map $\widehat{F} : \mathbb{A}^n \to \mathbb{A}^k$. For $p \in V$ we get a linear map

$$D\widehat{F}_p : \mathbb{C}^n \to \mathbb{C}^k$$

and our first claim is that this maps the subspace T_pV to the subspace T_qW. To see this, pick any $h \in I(W)$. Then $h \circ \widehat{F} \in I(V)$ (because $\widehat{F}(V) \subset W$) and differentiating gives

$$Dh|_q \circ D\widehat{F}_p = D(h \circ \widehat{F})_p$$

by the chain rule. If $v \in \mathbb{C}^n$ lies in T_pV then $D(h \circ \widehat{F})_p(v) = 0$ by definition, so $D\widehat{F}_p(v)$ lies in the kernel of $Dh|_q$. This holds for any $h \in I(W)$, so $D\widehat{F}_p(v)$ lies in T_qW. Hence we can define

$$DF_p : T_pV \to T_qW$$

to be the restriction of $D\widehat{F}_p$ to T_pV. However, we need to show that this linear map doesn't depend on our choice of $\widehat{F}$. We can change $\widehat{F}$ by taking a polynomial $g \in I(V)$ and adding it to (say) the first component, then this is another representative of F. This changes the first row of the matrix $D\widehat{F}_p$, by adding the row vector Dg_p. However, if v is a vector in T_pV then $Dg_p(v) = 0$, so $D\widehat{F}_p(v)$ does not change.

(ii) Just pick representatives $\widehat{F}$ and $\widehat{G}$ for F and G and apply the usual chain rule. □

Corollary 12.17. *If $F : V \xrightarrow{\sim} W$ is an isomorphism of affine varieties then for any $p \in V$ we get a linear isomorphism:*

$$DF_p : T_pV \xrightarrow{\sim} T_{F(p)}W$$

This shows that the tangent space T_pV is in some sense 'intrinsic' to the variety V, it doesn't depend on the embedding $V \subset \mathbb{A}^n$. In particular $\dim T_pV = \dim T_{F(p)}W$. It follows immediately that:

1. $\dim V = \dim W$ (assuming V and W are irreducible).
2. A point $p \in V$ is a singular point iff $F(p)$ is a singular point of W.

This is all reassuring.

We end this section with a bit of a digression. We've have repeatedly claimed that the ring $\mathbb{C}[V]$ knows everything about the variety V, for example we saw in Lemma 4.5 that a point of p corresponds to a $\mathbb{C}$-linear ring homomorphism:

$$\mathrm{ev}_p : \mathbb{C}[V] \to \mathbb{C}$$

If our claim is true, the ring $\mathbb{C}[V]$ must also know what the tangent space T_pV is. How can we extract this information?

The trick is to consider the ring

$$R = \mathbb{C}[t]/(t^2)$$

(sometimes called *the dual numbers*). This is not the co-ordinate ring of an affine variety, since the ideal (t^2) is not radical. Intuitively, you could think that it describes a point which has been 'thickened up' to order 1. The obvious quotient homomorphism $\mathbb{C}[t] \to R$ remembers the constant and linear terms of a polynomial f, equivalently it remembers both the value $f(0)$ at the origin, and also the value of the first derivative $f'(0)$.

Let's write

$$q : R \to \mathbb{C}$$

for the ($\mathbb{C}$-linear) homomorphism which sends t to zero.

Proposition 12.18. *Fix a point $p \in V$. There is a bijection*

$$T_pV \xrightarrow{\sim} \left\{ \begin{array}{c} \mathbb{C}\text{-linear homomorphisms } \alpha : \mathbb{C}[V] \to R \\ \text{such that } q \circ \alpha = \mathrm{ev}_p \end{array} \right\}$$

Proof. Say $V \subset \mathbb{A}^n$, so $\mathbb{C}[V]$ is generated by the co-ordinates $x_1, ..., x_n$. A ($\mathbb{C}$-linear) homomorphism $\alpha : \mathbb{C}[V] \to R$ is determined by the elements

$$\alpha(x_i) = a_i + b_i t \quad \in R, \qquad \text{for } i = 1, ..., n$$

where each $a_i, b_i \in \mathbb{C}$. Requiring that $q \circ \alpha = \mathrm{ev}_p$ says exactly that

$$(q \circ \alpha)(x_i) = a_i = \mathrm{ev}_p(x_i)$$

so $\mathbf{a} = (a_1, ..., a_n)$ must be the co-ordinates of the point p. Now pick generators $f_1, ..., f_k$ for $I(V)$. By definition we must have $\alpha(f_j) = 0$ for all j (since α is defined on the quotient ring $\mathbb{C}[V]$), but:

$$\begin{aligned} \alpha(f_j) &= f_j(a_1 + b_1 t, \ ..., \ a_n + b_n t) \\ &= f_j(\mathbf{a}) + t \sum_{i=1}^{n} \partial_i f_j(\mathbf{a}) b_i \quad \in R \end{aligned}$$

The second equality here comes from Taylor expanding around the point $\mathbf{a}$, but this a genuine equality not an approximation, because $t^2 = 0$ in the ring R. Now $f_j(\mathbf{a}) = 0$ automatically because $p \in V$, so our possible homomorphisms α are given by vectors $(b_1, ..., b_n)$ which are orthogonal to $Df_j(\mathbf{a})$ for each j. This is exactly the tangent space T_pV. □

It's possible to improve this result by defining a vector space structure on the set of α's, then the bijection becomes an isomorphism of vector spaces.

13 Other approaches to dimension

We saw above (Corollary 12.17) that if two irreducible affine varieties V and W are isomorphic then they have the same dimension. This is hardly surprising, but we can say something stronger:

Proposition 13.1. *If V and W are birational then* $\dim V = \dim W$.

This also shouldn't be surprising, since birational means 'almost isomorphic'.

Proof. If V and W are birational then we can find Zariski open subsets $U \subset V$ and $U' \subset W$ and an isomorphism $F : U \xrightarrow{\sim} U'$ (Lemma 11.15). The set of non-singular points in V is Zariski open, so since V is irreducible it must intersect with U, *i.e.* there is a point $p \in U$ such that $\dim T_pV = \dim V$. Lemma 12.16 and Corollary 12.17 work for regular maps between quasi-affine varieties (exercise), so we get an isomorphism:

$$DF_p : T_pV \xrightarrow{\sim} T_{F(p)}W$$

Hence $\dim W \leq \dim V$, and the reverse argument shows $\dim V \leq \dim W$. □

But the birational equivalence class of V is detected by the function field $\mathbb{C}(V)$. So it must be possible to detect $\dim V$ just from the field $\mathbb{C}(V)$. Here's how you do it.

Definition 13.2. Let $\mathbb{L}/\mathbb{K}$ be a field extension. A set $\alpha_1, ..., \alpha_n \in \mathbb{L}$ is **algebraically independent** over $\mathbb{K}$ if they satisfy no non-trivial polynomial over $\mathbb{K}$, *i.e.* if the subfield $\mathbb{K}(\alpha_1, ..., \alpha_n) \subset \mathbb{L}$ is isomorphic to $\mathbb{K}(x_1, ..., x_n)$.

The **transcendence degree** of $\mathbb{L}$ over $\mathbb{K}$ is the maximal size of a set of algebraically independent elements. We denote it by:

$$\operatorname{Tr}\deg_{\mathbb{K}} \mathbb{L}$$

Some facts from field theory:

- If $[\mathbb{L} : \mathbb{K}]$ is finite then the extension is algebraic and $\operatorname{Tr}\deg_{\mathbb{K}} \mathbb{L} = 0$.
- $\operatorname{Tr}\deg_{\mathbb{K}} \mathbb{K}(x_1, ..., x_n) = n$. This is not by definition! *A priori* it could be bigger than n.
- If $\mathbb{M}$ is an extension of $\mathbb{L}$ then $\operatorname{Tr}\deg_{\mathbb{K}} \mathbb{M} = \operatorname{Tr}\deg_{\mathbb{L}} \mathbb{M} + \operatorname{Tr}\deg_{\mathbb{K}} \mathbb{L}$.

Now suppose $\mathbb{K}$ is a finitely-generated extension of $\mathbb{C}$. We claimed in Theorem 11.18 that $\mathbb{K}$ must be the fraction field of $\mathbb{C}[x_1, ..., x_n]/(f)$ for some irreducible polynomial f. We can assume WLOG that x_n appears in f, then the set $\{x_1, ..., x_{n-1}\} \subset \mathbb{K}$ is algebraically independent. Moreover we can think of f as a polynomial in the variable x_n with coefficients in the field $\mathbb{C}(x_1, ..., x_{n-1})$, then it's clear that $\mathbb{K}$ is an algebraic extension of $\mathbb{C}(x_1, ..., x_{n-1})$. Using the above facts it follows that:

$$\operatorname{Tr}\deg_{\mathbb{C}} \mathbb{K} = n - 1$$

Proposition 13.3. *Let V be an irreducible affine variety. Then:*

$$\operatorname{Tr}\deg_{\mathbb{C}} \mathbb{C}(V) = \dim V$$

Proof. Since $\mathbb{C}(V)$ is a finitely-generated extension of $\mathbb{C}$ we can find an irreducible polynomial f in n-variables such that $\mathbb{C}(V) \cong \mathbb{C}(W)$ where W is the hypersurface $W = V(f) \subset \mathbb{A}^n$. By the calculation above, $\operatorname{Tr}\deg_{\mathbb{C}} \mathbb{C}(V) = n-1$. But V is birational to W, so $\dim V = \dim W = n - 1$ by Proposition 13.1 and Lemma 12.15. □

Example 13.4. Let $V = V(xy - z^2)$, the ODP singularity. This is a hypersurface in $\mathbb{A}^3$ so $\dim V = 2$. Also we argued in Example 9.6 that $\mathbb{C}(V) \cong \mathbb{C}(x, z)$, which has transendence degree 2. △

Proposition 13.3 gives us an alternative definition of *dimension*, we could define the dimension of V to be the transcendence degree of $\mathbb{C}(V)$ (as before this only makes sense if V is irreducible). Let's very briefly discuss a third possible definition, based on *chains of irreducible subvarieties*.

What is the smallest possible irreducible subvariety of V? The answer is easy, any single point $Z_0 = \{p\}$ is an irreducible subvariety of V, and we can't get any smaller. Now suppose $Z_1 \subset V$ is an irreducible subvariety containing Z_0, but strictly bigger. Then Z_1 can't be a finite set (it would be reducible), so the minimal possibility is that $\dim Z_1 = 1$. Keep going, making minimal choices at each step, and you'll get a chain of irreducible subvarieties:

$$Z_0 \subsetneq Z_1 \subsetneq ... \subsetneq Z_{k-1} \subset Z_k = V$$

The best case scenario is that $\dim Z_p = p$ at each step, and $k = \dim V$.

Definition 13.5. The **Krull dimension** of V is the maximal length of a strictly-increasing chain of irreducible subvarieties in V.

Theorem 13.6. *The Krull dimension of V equals $\dim V$.*

This is difficult theorem and requires a lot of algebra. Let's just remark that this should remind you of the definition of a *Noetherian ring*, since chains of subvarieties in V correspond to chains of ideals in $\mathbb{C}[V]$. The geometric meaning of 'Noetherian' is that your space is finite-dimensional. This shows why it's such an important condition!

II Projective varieties

There are some ways in which affine varieties are less-than-ideal spaces. For example, an affine variety in $\mathbb{A}^n$ will always go 'off to infinity', unless it's a finite set. This means it is not compact in the usual topology (in the Zariski topology everything is compact, but that's just because the Zariski topology is weird). For comparison, the unit sphere $S^{2n-1} \subset \mathbb{A}^n$ is a bounded, compact space.

Since compact spaces generally have better properties it would be nice if we could find them in the setting of algebraic geometry. The solution is to consider subvarieties not in affine space, but in *projective space*.

14 Projective space

You may previously met the *Riemann sphere*, this is a copy of the complex plane $\mathbb{C}$ with an additional point "infinity". We identify it with the 2-dimensional sphere S^2 by putting "infinity" at the north pole, then using stereopraphic projection from the north pole to give a bijection between the rest of S^2 and the plane $\mathbb{R}^2$, which we identify with $\mathbb{C}$. Under this bijection the south pole maps to the origin $0 \in \mathbb{C}$ and the equator maps to the unit complex numbers.

In algebraic geometry we call this space $\mathbb{P}^1$, which means *1-dimensional projective space*, or sometimes *the projective line*. The correct way to define it is:

Definition 14.1. $\mathbb{P}^1$ is the set of 1-dimensional complex subspaces of $\mathbb{C}^2$.

This definition takes some time to get used to, and it may not be clear yet why this is the same as the Riemann sphere.

To start understanding it, let's think about the analogue over the real numbers:

$$\mathbb{P}^1(\mathbb{R}) = \{1\text{d } \mathbb{R} \text{ subspaces of } \mathbb{R}^2\}$$

Of course there's an analogue $\mathbb{P}^1(\mathbb{K})$ for any field $\mathbb{K}$; really to be consistent we should have written $\mathbb{P}^1(\mathbb{C})$ above, but for us $\mathbb{C}$ is the default field.

A 1-dimensional subspace of $\mathbb{R}^2$ is just a straight line through the origin. Any point $(x, y) \in \mathbb{R}^2$, apart from the origin, lies in a unique such line. We use the notation $x:y$ to denote this line, *i.e.*

$$x:y \; = \; \{(\mu x, \mu y), \; \mu \in \mathbb{R}\} \; \subset \mathbb{R}^2$$

Obviously if λ is any non-zero real number then $\lambda x : \lambda y$ is the same line as $x:y$. So $\mathbb{P}^1(\mathbb{R})$ can also be described as '$\mathbb{R}^2$ without the origin, modulo rescaling':

$$\mathbb{P}^1(\mathbb{R}) = \frac{\mathbb{R}^2 \setminus (0,0)}{(x,y) \sim (\lambda x, \lambda y), \; \forall \lambda \in \mathbb{R}^*}$$

From this point-of-view $x:y$ denotes the equivalence class of (x, y). Note that $(0, 0)$ does not determine a point in $\mathbb{P}^1(\mathbb{R})$ and $0:0$ is meaningless.

How can we parametrize points in $\mathbb{P}^1(\mathbb{R})$? One option is to use gradient, which maps a line $x:y$ to its slope $y/x \in \mathbb{R}$. For every real number $t \in \mathbb{R}$ there is a unique line with that gradient, namely the line through $(1,t)$. So we have an injection:

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathbb{P}^1(\mathbb{R}) \\ t &\mapsto 1\!:\!t \end{aligned}$$

But this map is not a surjection, it misses exactly the line $0:1$ (the y-axis). Another way to say this is to think about the line $\{x=1\} \cong \mathbb{R} \subset \mathbb{R}^2$. A line through the origin $x:y$ intersects this line in the single point $1:(y/x)$, except for the y-axis which doesn't intersect it at all. So:

$$\begin{aligned} \mathbb{P}^1(\mathbb{R}) &= \{x\!:\!y,\ x \neq 0\} \ \sqcup\ \{0\!:\!1\} \\ &\cong \mathbb{R} \sqcup \{\text{one point}\} \end{aligned}$$

The line $0:1$ has "infinite slope", so we can think of $\mathbb{P}^1(\mathbb{R})$ as a copy of $\mathbb{R}$ with an extra point 'at infinity'. But it's important to realise that there is nothing special or different about the point $0:1$ in $\mathbb{P}^1(\mathbb{R})$, this is just an artifact of our parametrization. We could instead have mapped a line $x:y$ to the number $x/y \in \mathbb{R}$. This makes sense for every line except the x-axis $1:0$, and it has an inverse:

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathbb{P}^1(\mathbb{R}) \\ s &\mapsto s\!:\!1 \end{aligned}$$

From this point-of-view, the line $1:0$ is the point 'at infinity'.

We've constructed two bijections:

$$\begin{aligned} \mathbb{P}^1(\mathbb{R}) \setminus 0\!:\!1 &\xrightarrow{\sim} \mathbb{R} \\ x\!:\!y &\mapsto y/x \end{aligned} \qquad\qquad \begin{aligned} \mathbb{P}^1(\mathbb{R}) \setminus 1\!:\!0 &\xrightarrow{\sim} \mathbb{R} \\ x\!:\!y &\mapsto x/y \end{aligned}$$

These are called the *standard charts* on $\mathbb{P}^1(\mathbb{R})$. Observe that if a line maps to $t \in \mathbb{R}$ under the first chart, and $t \neq 0$, then it maps to $s = 1/t$ under the other chart. If $t = 0$ then this line is not in the domain of the second chart. The domains of the charts cover $\mathbb{P}^1(\mathbb{R})$ between them, so another way to think of $\mathbb{P}^1(\mathbb{R})$ is as a union:

$$\mathbb{P}^1(\mathbb{R}) \cong \mathbb{R} \cup \mathbb{R}$$

This is not-at-all a disjoint union, almost all points lie in the overlap of the two charts.

Remark 14.2. Everything we've done so far is algebraic, and will work over $\mathbb{C}$ (or any other field). There are two more ways to think about $\mathbb{P}^1(\mathbb{R})$ which are helpful but involve non-algebraic constructions. Both of them make it clear that $\mathbb{P}^1(\mathbb{R})$ is the circle S^1.

(i) Consider the semi-circle in $\mathbb{R}^2$ with radius 1, lying in the region $\{x \geq 0\}$. This semi-circle intersects every line once, except for the y-axis which it hits at both ends. So we can identify $\mathbb{P}^1(\mathbb{R})$ with a unit interval that has its ends glued together:

$$\mathbb{P}^1(\mathbb{R}) \ \cong\ \frac{[0,1]}{0 \sim 1} \ \cong\ S^1$$

(ii) Consider the circle of radius 1 in $\mathbb{R}^2$. It intersects every line in exactly two points, which are antipodal. So:

$$\mathbb{P}^1(\mathbb{R}) \;\cong\; \frac{S^1}{(x,y)\sim(-x,-y)} \;\cong\; S^1$$

Construction (i) has no analogue over $\mathbb{C}$, construction (ii) does but it's considerably harder to understand (it's called the Hopf fibration).

Now let's get back to $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. A 1d subspace of $\mathbb{C}^2$ (a *line*) is the complex span of some point $(x,y) \neq (0,0) \in \mathbb{C}^2$, and as before we write $x\!:\!y$ for this subspace:

$$x\!:\!y \;=\; \big\{(\mu x, \mu y),\ \mu \in \mathbb{C}\big\} \;\subset \mathbb{C}^2$$

So any $(x,y) \in \mathbb{C}^2 \setminus (0,0)$ defines a point $x\!:\!y \in \mathbb{P}^1$. As before $\mathbb{P}^1$ is the quotient:

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 \setminus (0,0)}{(x,y)\sim(\lambda x, \lambda y),\ \forall \lambda \in \mathbb{C}^*}$$

We can parametrize points in $\mathbb{P}^1$ by their 'complex gradient' $x\!:\!y \mapsto y/x \in \mathbb{C}$, this makes sense for every point except $0\!:\!1$, and has an inverse:

$$\begin{aligned} \mathbb{A}^1 &\xrightarrow{\sim} \mathbb{P}^1 \setminus 0\!:\!1 \\ z &\mapsto 1\!:\!z \end{aligned}$$

As in the real case, this means we can decompose $\mathbb{P}^1$ as:

$$\begin{aligned} \mathbb{P}^1 &= \{x\!:\!y,\ x \neq 0\} \;\sqcup\; \{0\!:\!1\} \\ &= \mathbb{A}^1 \sqcup \{\text{one point}\} \end{aligned}$$

So $\mathbb{P}^1$ is the affine line $\mathbb{A}^1$ plus one extra point 'at infinity'. This is the Riemann sphere.

Once again there is nothing special about the point $0\!:\!1$, it's only 'at infinity' because of our particular choice of chart. We have a second standard chart

$$\begin{aligned} \mathbb{P}^1 \setminus 1\!:\!0 &\xrightarrow{\sim} \mathbb{A}^1 \\ x\!:\!y &\mapsto x/y \\ w\!:\!1 &\leftarrow\!\!\shortmid\ w \end{aligned}$$

and using this chart makes $1\!:\!0$ the point 'at infinity'. The domains of the two standard charts cover $\mathbb{P}^1$, so

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$$

where as before a point $z \neq 0 \in \mathbb{A}^1$ in the first chart corresponds to the point $w = 1/z \in \mathbb{A}^1$ in the second chart. If we identify $\mathbb{P}^1$ with the sphere S^2, then these two charts are exactly the stereographic projection maps from the north and south poles.

Here's two more things to note:

- We've defined two 'standard charts', but this depended on using the standard basis for $\mathbb{C}^2$. We could use another basis (*i.e.* apply a linear coordinate change) and we'll get other charts. For example

$$x\!:\!y \;\mapsto\; \frac{x}{x-y}$$

defines a bijection between $\mathbb{P}^1 \setminus 1:1$ and $\mathbb{A}^1$. So there really is nothing special about $1:0$ and $0:1$.

- Later you may be tempted to think or write something like:

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{C} \\ x:y &\mapsto x \end{aligned}$$

But this is not a well-defined function! The notation $x : y$ means an equivalence class, and $x:y = \lambda x:\lambda y$ for any non-zero λ. Only the function x/y makes sense, and even this only makes sense in the subset $\mathbb{P}^1 \setminus 1:0$.

Now it's time to generalize to higher dimensions.

Definition 14.3. $\mathbb{P}^n$ is the set of 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$. We call it n-dimensional **projective space**.

Any point in $\mathbb{C}^{n+1}$ (except the origin) spans a unique 1-d subspace, so it defines a point in $\mathbb{P}^n$. So $\mathbb{P}^n$ can be viewed as the quotient:

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus 0}{\mathbf{x} \sim \lambda\mathbf{x},\ \forall \lambda \in \mathbb{C}^*}$$

Given a point $(x_0, ..., x_n) \in \mathbb{C}^n \setminus 0$, we write

$$x_0 : ... : x_n \ \in \mathbb{P}^n$$

for the corresponding equivalence class, or 1-d subspace.

Let's focus on the case $n = 2$ so we don't get overwhelmed by notation. A point in $\mathbb{P}^2$ is a class $x:y:z$ for a point $(x, y, z) \in \mathbb{C}^3 \setminus (0, 0, 0)$, and

$$x:y:z = \lambda x:\lambda y:\lambda z$$

for any non-zero $\lambda \in \mathbb{C}$. To parametrize points in $\mathbb{P}^2$ we can start by considering the plane

$$\Pi_1 = \{z = 1\} \subset \mathbb{C}^3$$

Most lines $x:y:z$ intersect the plane Π_1 in a single point, $(x/z,\ y/z,\ 1)$. The only lines that don't are the lines lying in the (x, y)-plane, *i.e.* lines of the form $x:y:0$. So we have a bijection:

$$\begin{aligned} \mathbb{P}^2 \setminus \{z = 0\} &\xrightarrow{\sim} \mathbb{A}^2 \\ x:y:z &\mapsto (x/z,\ y/z) \\ s:t:1 &\leftarrow\!\shortmid (s, t) \end{aligned}$$

This is the first 'standard chart' on $\mathbb{P}^2$. Notice that the set of points missed by this chart is a copy of $\mathbb{P}^1$, so we could decompose $\mathbb{P}^2$ as

$$\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{P}^1$$

where the $\mathbb{A}^2$ is all the lines $x : y : z$ with $z \neq 0$, and the $\mathbb{P}^1$ is the lines with $z = 0$. To form $\mathbb{P}^1$ we took $\mathbb{A}^1$ and added an extra point 'at infinity'; to form $\mathbb{P}^2$ we take $\mathbb{A}^2$ and add a whole $\mathbb{P}^1$ at infinity. When we get out to infinity we remember the line we were travelling along.

You may find the real picture helpful here. The plane $\mathbb{R}^2$ can be shrunk homeomorphically to a disc. Now add the boundary of the disc so we have points at infinity, but antipodal points on the boundary must be glued together because they represent the same line, so the boundary becomes $\mathbb{P}^1(\mathbb{R})$ (see part (ii) of Remark 14.2). This gives us a picture of $\mathbb{P}^2(\mathbb{R})$, which is a 2-dimensional non-orientable manifold.

It's pretty hard to visualize $\mathbb{P}^2$ over the complex numbers because it's a 2-dimensional complex manifold, so a 4-dimensional real manifold. It's not the sphere S^4 or the torus T^4 (the fact that $\mathbb{P}^1 \cong S^2$ is a coincidence that doesn't generalize to higher dimensions). Nevertheless $\mathbb{P}^2$ is one of the easiest 4-dimensional manifolds to work with.

When we say that the lines with $z = 0$ lie 'at infinity' this is only from the point-of-view of the first standard chart. There are two more standard charts:

$$\begin{aligned} \mathbb{P}^2 \setminus \mathbb{P}^1_{x:0:z} &\xrightarrow{\sim} \mathbb{A}^2 \\ x:y:z &\mapsto (x/y,\ z/y) \end{aligned} \qquad\qquad \begin{aligned} \mathbb{P}^2 \setminus \mathbb{P}^1_{0:y:z} &\xrightarrow{\sim} \mathbb{A}^2 \\ x:y:z &\mapsto (y/x,\ z/x) \end{aligned}$$

So we can also decompose $\mathbb{P}^2$ as

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2$$

by taking the domains of the three standard charts (the subsets where either x, y or z are not zero), and these subsets have big overlaps. Just as in the $\mathbb{P}^1$ case these standard charts are not special, they depended on us using the standard basis for $\mathbb{C}^3$. If we pick a different basis we get other charts.

Everything we've done for $\mathbb{P}^2$ generalizes immediately to higher dimensions. $\mathbb{P}^n$ is covered by $n+1$ standard charts

$$\begin{aligned} \mathbb{P}^n \setminus \{x_k = 0\} &\xrightarrow{\sim} \mathbb{A}^n \\ x_0 : x_1 : \ldots : x_n &\mapsto (x_0/x_k,\ x_1/x_k, \ldots\hat{k}\ldots, x_n/x_k) \end{aligned}$$

so $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^n \cup \ldots \cup \mathbb{A}^n$ with big overlaps. The complement of each chart is a copy of $\mathbb{P}^{n-1}$, because it's the set of lines lying in the n-dimensional subspace $\{x_k = 0\} \subset \mathbb{C}^{n+1}$. So each chart gives us a decomposition

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$$

into a copy of $\mathbb{A}^n$ and a $\mathbb{P}^{n-1}$ 'at infinity'.

15 Projective varieties

A projective variety is a certain kind of subset of $\mathbb{P}^n$. First we need the following:

Definition 15.1. A polynomial f is **homogeneous** of degree d if every term in f has degree d.

Example 15.2.

- $x^3 + x^2y + 2xy^2$ is homogeneous of degree 3.
- $x - y + \frac{3\pi}{4}z$ is homogeneous of degree 1.
- $y^2 - x^3$ is not homogeneous.

△

Observe that f is homogeneous of degree d iff it satisfies:

$$f(\lambda \mathbf{x}) = \lambda^d f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{A}^n,\ \lambda \in \mathbb{C}$$

This means that if f is homogeneous (of any degree) and $f(\mathbf{x}) = 0$ for some $\mathbf{x} \in \mathbb{A}^n$, then f vanishes on the whole line spanned by $\mathbf{x}$. If f is not homogeneous then this is not true, *e.g.* $y^2 - x^3$ vanishes at $(1,1)$ but not at $(2,2)$.

Definition 15.3. Let f be a homogeneous polynomial in $n+1$ variables. We define

$$\mathbb{V}(f) \subset \mathbb{P}^n$$

to be the set of lines $l \subset \mathbb{C}^{n+1}$ such that f vanishes on l. A subset of this form is called a **projective hypersurface**.

Warning: A homogeneous polynomial does *not* define a function $f : \mathbb{P}^n \to \mathbb{C}$. The value of $f(\mathbf{x})$ changes when we rescale $\mathbf{x}$, since $f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x})$, so f is not a well-defined function on the set of lines. Only the set $\{f = 0\}$ makes sense.

Example 15.4. Let $f(x, y) = x^2y + 2xy^2 = xy(x + 2y)$, which is homogeneous of degree 3. This f vanishes on exactly 3 lines:

1. $\{x = 0\}$, *i.e.* the line $0:1 \in \mathbb{P}^1$.
2. $\{y = 0\}$, *i.e.* the line $1:0 \in \mathbb{P}^1$.
3. $\{x + 2y = 0\}$, *i.e.* the line $2:-1 \in \mathbb{P}^1$.

So the corresponding projective hypersurface $\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^1$ consists of these 3 points.

Recall we have a standard chart $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ where $x \mapsto x:1$. The intersection of $\mathbb{V}$ with this chart is:

$$\mathbb{V} \cap \mathbb{A}^1_x = \left\{x^2 + 2x = 0\right\} = \{0\} \cup \{-2\}$$

This is an affine variety, cut out by the polynomial $f(x, 1)$ (*i.e.* set $y = 1$ in f). This affine variety only has 2 points, because we've missed the point $1:0$. To get a complete picture we must also consider the second standard chart $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ where $y \mapsto 1:y$, intersecting with this chart gives:

$$\mathbb{V} \cap \mathbb{A}^1_y = \{y + 2y^2 = 0\} = \{0\} \cup \{-\frac{1}{2}\}$$

The point $y = 0$ is the point we missed in the first chart, and in this chart we miss the point $0 : 1$. Also note that the point $x = -2$ in the first chart corresponds to the point $y = -\frac{1}{2}$ in the second chart, since $x \mapsto 1/x$ when we change charts. △

Example 15.5. Let $f(x, y, z) = x$, which is homogeneous of degree 1. It defines a projective hypersurface

$$\mathbb{V} = \mathbb{V}(x) \subset \mathbb{P}^2$$

This $\mathbb{V}$ is the set of lines $\{0 : y : z\} \subset \mathbb{P}^2$, it's just the complement of the third standard chart. We said previously that this is 'a copy of' $\mathbb{P}^1$, the precise statement is that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$, and we'll prove this as soon as we've defined isomorphisms for projective varieties. △

Here is a more interesting example.

Example 15.6. Let $f(x, y, z) = xy - z^2$, which is homogeneous of degree 2. It defines a projective hypersurface:

$$\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^2$$

The first standard chart $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ is given by $(x, y) \mapsto x:y:1$, and intersecting $\mathbb{V}$ with this chart gives the affine hypersurface:

$$V_1 = V(xy - 1) \subset \mathbb{A}^2_{x,y}$$

This was our very first example of an affine variety, and we know it's isomorphic to the quasi-affine variety $\mathbb{A}^1 \setminus 0$ (Example 8.11).

What points have we missed? The complement of the first standard chart is a copy of $\mathbb{P}^1$, it's the set of lines:

$$\mathbb{P}^1_{x:y} = \{x:y:0\} \subset \mathbb{P}^2_{x:y:z}$$

The intersection of $\mathbb{V}$ with this $\mathbb{P}^1$ 'at infinity' is:

$$\mathbb{V} \cap \mathbb{P}^1_{x:y} = \{x:y:0,\ xy = 0\} = \{1:0:0\} \cup \{0:1:0\}$$

This is a projective hypersurface in $\mathbb{P}^1$, it's cut out by the polynomial $f(x, y, 0)$ obtained by setting $z = 0$ in f, and it consists of two points. So $\mathbb{V}$ consists of V_1 plus these two extra points.

The real picture is quite helpful here. Over the real numbers, the set $\{xy - 1 = 0\} \subset \mathbb{R}^2$ is a hyperbola. When we go out to infinity in $\mathbb{R}^2$ the hyberbola has two asympotes, which are the x-axis and the y-axis. In $\mathbb{P}^2(\mathbb{R})$ we add in both of these points, making the hyperbola into the compact space S^1.

What does $\mathbb{V}$ look like over the complex numbers? Since V_1 is $\mathbb{A}^1 \setminus 0$ adding a single point probably gives us $\mathbb{A}^1$, and then adding another point should give us the Riemann sphere. So $\mathbb{V}$ probably looks like $\mathbb{P}^1$.

We can confirm this guess by using the other two standard charts. If we intersect $\mathbb{V}$ with the second standard chart in $\mathbb{P}^2$ we get the affine hypersurface:

$$V_2 = V(y - z^2) \subset \mathbb{A}^2_{y,z}$$

This is the set of points (t^2, t) and is isomorphic to $\mathbb{A}^1$. If we intersect $\mathbb{V}$ with the third standard chart we get

$$V_3 = V(x - z^2) \subset \mathbb{A}^2_{x,z}$$

which is the set $\{(s^2, s)\}$ and is also isomorphic to $\mathbb{A}^1$. These two charts together cover all of $\mathbb{V}$, since the only point they miss is $0:0:1$ which doesn't lie in $\mathbb{V}$. So $\mathbb{V}$ consists of two copies of $\mathbb{A}^1$ glued together, and with a little more work you can check that the gluing is given by $s = 1/t$, so the resulting space is exactly $\mathbb{P}^1$.

This is all a bit heuristic, but later we will prove the precise statement which is that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$. △

A general projective variety is just an interection of projective hypersurfaces.

Definition 15.7. A **projective variety** is a subset of $\mathbb{P}^n$ of the form

$$\mathbb{V} = \mathbb{V}(f_1) \cap \mathbb{V}(f_2) \cap ... \cap \mathbb{V}(f_k) \ \subset \mathbb{P}^n$$

where $f_1, .., f_k$ are homogeneous polynomials in $n+1$ variables.

So $\mathbb{V}$ is the set of lines which are the common vanishing locus of $f_1, ..., f_k$. We could also write

$$\mathbb{V} = \mathbb{V}(f_1, ..., f_k)$$

like we did in the affine case. Note that each f_i must be homogeneous, but they don't all have to have the same degree.

Most of our examples will be hypersurfaces because they're simpler, but it's easy to write down examples which are not hypersurfaces.

Example 15.8. Let $f(x, y, z) = xy - z^2$ as before and let $g(x, y, z) = x - y$. These are two homogeneous polynomials in three variables, and they define a projective variety:

$$\mathbb{V} = \mathbb{V}(f, g) \subset \mathbb{P}^2$$

Note that $\mathbb{V}(g)$ is the set of lines of the form $x : x : z$, so

$$\mathbb{V} = \mathbb{V}(f) \cap \mathbb{V}(g) = \{x : x : z,\ x^2 - z^2 = 0\} = \{1:1:1\} \cup \{1:1:-1\}$$

consists of two points. △

Say $\mathbb{V} = \mathbb{V}(f)$ is a projective hypersurface in $\mathbb{P}^2$, for some homogeneous $f(x, y, z)$. As in some of our examples above, we can intersect $\mathbb{V}$ with the first standard chart and get an affine hypersurface:

$$V_1 = V(f(x, y, 1)) \subset \mathbb{A}^2_{x,y}$$

Perhaps you think we are restricting the function f to the chart $\{z \neq 0\} \subset \mathbb{P}^2$. We are not! Because f is not a function on $\mathbb{P}^2$.

Is it possible to reverse this procedure? Suppose $V \subset \mathbb{A}^2$ is an affine hypersurface, is there some projective hypersurface $\mathbb{V} \subset \mathbb{P}^2$ that gives us V in the first standard chart? The answer is yes, because there is a way to turn any polynomial $g(x, y)$ into a homogeneous polynomial $f(x, y, z)$.

Let's write $\mathbb{C}[x, y, z]_k$ for the set of homogeneous polynomials in three variables of degree k, this is a finite-dimensional vector space and it has a basis $\{x^a y^b z^c,\ a + b + c = k\}$. Let's also write $\mathbb{C}[x, y]_{\leq k}$ for the set of polynomials in two variables with degree at most k, this is again a finite dimensional vector space with a basis $\{x^a y^b, a + b \leq k\}$. The map

$$\mathbb{C}[x, y, z]_k \longrightarrow \mathbb{C}[x, y]_{\leq k}$$
$$f(x, y, z) \mapsto f(x, y, 1)$$

is evidently a linear isomorphism. The inverse map is called *homogenizing* a polynomial, on monomials it is the map:

$$x^a y^b \mapsto x^a y^b z^{k-a-b}$$

Suppose $V(g) \subset \mathbb{A}^2$ is an affine hypersurface, where g has degree k. Homogenize g to get a homogeneous polynomial $f(x, y, z)$ of degree k. Then we get a projective hypersurface $\mathbb{V}(f) \subset \mathbb{P}^2$ such that the intersection of $\mathbb{V}$ with the first standard chart is V. We call $\mathbb{V}$ the **projective completion** of V.

Example 15.9. Let $V = V(g) \subset \mathbb{A}^2$ for $g = y^2 - x^3 + x - 1$ (this is an affine elliptic curve). The corresponding homogeneous degree 3 polynomial is

$$f(x, y, z) = y^2 z - x^3 + xz^2 - z^3$$

and then $\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^2$ is a projective hypersurface (it's a projective elliptic curve), and the intersection of $\mathbb{V}$ with $\{z \neq 0\}$ is V.

What points have we added? The complement of the first standard chart is the $\mathbb{P}^1$ 'at infinity' where $z = 0$. So

$$\mathbb{V} \setminus V = \mathbb{V} \cap \mathbb{P}^1_{x:y:0} = \{x\!:\!y\!:\!0,\ x^3 = 0\} = \{0\!:\!1\!:\!0\}$$

is a single point. The real picture here is not misleading; if we plot $y^2 = x^3 - x + 1$ in $\mathbb{R}^2$ we see it has a single asymptote which is (a translate of) the y-axis. If you have studied elliptic curves you will know that this extra point at infinity is the unit for the group law. △

Example 15.10. If $V = V(xy - 1) \subset \mathbb{A}^2$ then the projective completion is:

$$\mathbb{V} = \mathbb{V}(xy - z^2) \subset \mathbb{P}^2$$

This adds two points to V, $1\!:\!0\!:\!0$ and $0\!:\!1\!:\!0$, as we saw in Example 15.6. △

Obviously this procedure generalizes to any dimension, if $V = V(g) \subset \mathbb{A}^n$ is an affine hypersurface then it has a projective completion $\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^n$ obtained by homogenizing g. The extra points $\mathbb{V} \setminus V$ live in the $\mathbb{P}^{n-1}$ at infinity, where the last co-ordinate is zero. In fact $\mathbb{V} \setminus V$ is a projective hypersurface in $\mathbb{P}^{n-1}$, cut out by the polynomial $f(x_1, ..., x_n, 0)$. This polynomial consists of all the highest-degree terms in g.

If $V \subset \mathbb{A}^n$ is not a hypersurface then it's still possible to define the projective completion $\mathbb{V} \subset \mathbb{P}^n$ but there's a subtlety (see exercises).

There is another way to relate projective varieties and affine varieties. Suppose we have homogeneous polynomials $f_1, ..., f_k \in \mathbb{C}[x_1, ..., x_n]$, so they define a projective variety $\mathbb{V} = \mathbb{V}(f_1, ..., f_k) \subset \mathbb{P}^{n-1}$. Then these same polynomials also define an affine variety:

$$V = V(f_1, ..., f_k) \subset \mathbb{A}^n$$

Because the defining polynomials are homogeneous this V is of a special form. If $\mathbf{x} \in \mathbb{A}^n$ lies in V then so does $\lambda\mathbf{x}$ for any $\lambda \in \mathbb{C}$, *i.e.* the whole line spanned by $\mathbf{x}$ lies in V. An affine variety with this property is called a **cone**. We say V is the *affine cone* associated to the projective variety $\mathbb{V}$.

Example 15.11. Let $f(x, y, z) = xy - z^2$. This defines:

1. An affine cone $V(f) \subset \mathbb{A}^3$, which the 2-dimensional ODP singularity (Example 9.6).
2. A projective hypersurface $\mathbb{V}(f) \subset \mathbb{P}^2$ which we think is isomorphic to $\mathbb{P}^1$ (Example 15.6).

△

The relationship between a projective variety $\mathbb{V} \subset \mathbb{P}^n$ and the affine cone $V \subset \mathbb{A}^{n+1}$ is that:

$$\mathbb{V} = \frac{V \setminus \{0\}}{\mathbf{x} \sim \lambda \mathbf{x}}$$

Hopefully it's intuitively clear that the dimension of $\mathbb{V}$ is one less than the dimension of V; we'll prove this once we've defined dimension for projective varieties.

Now suppose that $W \subset \mathbb{A}^n$ is an affine variety and that it's a cone, *i.e.* if $\mathbf{x} \in W$ then $\lambda \mathbf{x} \in W$ for all λ. There's an associated subset $\mathbb{W} \subset \mathbb{P}^{n-1}$ consisting of all the lines that lie in W, but it's not instantly obvious that $\mathbb{W}$ is a projective variety.

Example 15.12. Let:

$$\begin{aligned} W &= V(xy - z, xy + z) \ \subset \mathbb{A}^3 \\ &= \{(x, 0, 0)\} \cup \{(0, y, 0)\} \end{aligned}$$

This is clearly a cone, but the defining polynomials were not homogeneous. But we could have used the homogenous polynomials xy and z to generate $I(W)$ instead, then it becomes clear that $\mathbb{W} = \mathbb{V}(xy, z) \subset \mathbb{P}^2$ is a projective variety. △

What we just did can be done for any cone, because:

Lemma 15.13. *If $W \subset \mathbb{A}^n$ is a cone then $I(W)$ can be generated by homogeneous polynomials.*

Note that most polynomials in $I(W)$ are not homogenous (look at the previous example).

Proof. Any polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ can be written uniquely as a sum of homogeneous polynomials

$$f = f_d + f_{d-1} + ... + f_1 + f_0$$

simply by grouping together the terms in f which have the same degree.

Claim: If $f \in I(W)$ then each homogeneous summand f_i also lies in $I(W)$.

This claim immediately proves the lemma: take any generating set $f, ..., g$ for $I(W)$, split each generator into its homogeneous summands, then the set of all these homogeneous summands $f_d, ..., g_0$ generates $I(W)$.

Now we prove the claim. Pick an $f \in I(W)$, and fix a point $\mathbf{x} \in W$. Then $f(\lambda \mathbf{x}) = 0$ for all $\lambda \in \mathbb{C}$ since W is a cone. But we can think of $f(\lambda \mathbf{x})$ as a polynomial in the single variable λ, and:

$$f(\lambda \mathbf{x}) = \lambda^d f_d(\mathbf{x}) + \lambda^{d-1} f_{d-1}(\mathbf{x}) + ... + \lambda f_1(\mathbf{x}) + f_0(\mathbf{x})$$

We know this polynomial vanishes for all λ so every coefficient must be zero, so each f_i vanishes at $\mathbf{x}$, and this is true for all $\mathbf{x} \in W$. □

Given this lemma we have a bijection between affine cones in $\mathbb{A}^n$ and projective varieties in $\mathbb{P}^{n-1}$. This is a useful bijection because many properties of projective varieties can be expressed in terms of their affine cones, and we've already developed many results for affine varieties.

16 Irreducibility, quasi-projective varieties, dimension

Some ideas from affine varieties can be easily imported to projective varieties.

Definition 16.1. A projective variety $\mathbb{V} \subset \mathbb{P}^n$ is **irreducible** if it can't be written as a union $\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2$ of two proper projective subvarieties.

Proposition 16.2. *Let $\mathbb{V} \subset \mathbb{P}^n$ be a projective variety and let $V = \mathbb{V} \cap \mathbb{A}^n$ be the intersection of $\mathbb{V}$ with a standard chart. If $\mathbb{V}$ is irreducible then V is irreducible.*

Proof. Exercise. □

But the converse to this result is false: knowing that V is irreducible does not imply that that $\mathbb{V}$ is irreducible.

Example 16.3. Let $\mathbb{V} = \mathbb{V}(yz) = \mathbb{V}(y) \cup \mathbb{V}(z) \subset \mathbb{P}^2_{x:y:z}$. This is the union of two copies of $\mathbb{P}^1$, meeting at a single point $1:0:0$.

If we intersect $\mathbb{V}$ with the chart $\{z \neq 0\}$ we get $V(y) \subset \mathbb{A}^2_{x,y}$, which is a copy of $\mathbb{A}^1$ and is irreducible. But $\mathbb{V}$ itself is reducible. This chart has missed an entire irreducible component of $\mathbb{V}$. △

In the example above we could switch to the chart $\{x \neq 0\}$ and get the reducible affine variety $V(yz) \subset \mathbb{A}^2_{y,z}$, which reveals that $\mathbb{V}$ is reducible (by Proposition 16.2). But it's not hard to find examples of reducible varieties which look irreducible in every standard chart (exercise).

So irreduciblity is not always easy to detect by looking in charts. To get a stronger result we can look at the affine cone.

Lemma 16.4. *If $V \subset \mathbb{A}^n$ is a cone and is reducible then there exist homogeneous polynomials $f, g \notin I(V)$ such that $fg \in I(V)$.*

Proof. If V is reducible then $I(V)$ is not prime (Proposition 5.4) so we can find two polynomials $\hat{f}, \hat{g} \notin I(V)$ such that $\hat{f}\hat{g} \in I(V)$. The polynomials $\hat{f}$ and $\hat{g}$ might not be homogeneous, but we can split each one as a sum of homogeneous polynomials:

$$\hat{f} = \hat{f}_d + ... + \hat{f}_0 \qquad \text{and} \qquad \hat{g} = \hat{g}_e + ... + \hat{g}_0$$

At least one of the $\hat{f}_i$ is not in $I(V)$, or $\hat{f}$ would be in $I(V)$. Let s be the smallest number such that $\hat{f}_s \notin I(V)$. Similarly let t be the smallest number such that $\hat{g}_t \notin I(V)$. Since $\hat{f}\hat{g} \in I(V)$ every homogeneous summand of $\hat{f}\hat{g}$ also lies in $I(V)$; we saw this in the proof of Lemma 15.13. The summand of degree $s + t$ is:

$$(\hat{f}\hat{g})_{s+t} = \hat{f}_0\hat{g}_{s+t} + \hat{f}_1\hat{g}_{s+t-1} + ... + \hat{f}_s\hat{g}_t + ... + \hat{f}_{s+t}\hat{g}_0$$

In every term except $\hat{f}_s\hat{g}_t$ at least one factor lies in $I(V)$ by assumption, hence $\hat{f}_s\hat{g}_t$ must also lie in $I(V)$. □

Proposition 16.5. *A projective variety $\mathbb{V} \subset \mathbb{P}^n$ is irreducible if and only if the corresponding affine cone $V \subset \mathbb{A}^{n+1}$ is irreducible.*

Proof. If $\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2$ then it follows immediately that the cone decomposes as the union $V = V_1 \cup V_2$ of the cones for $\mathbb{V}_1$ and $\mathbb{V}_2$. Conversely, suppose that V is reducible. By the previous lemma we can find homogeneous polynomials $f, g \notin I(V)$ such that $fg \in I(V)$, then setting $\mathbb{V}_1 = \mathbb{V} \cap \mathbb{V}(f)$ and $\mathbb{V}_2 = \mathbb{V} \cap \mathbb{V}(g)$ gives a non-trivial decomposition of $\mathbb{V}$. □

Corollary 16.6. *Any projective variety decomposes into a finite number of irreducible components $\mathbb{V} = \mathbb{V}_1 \cup ... \cup \mathbb{V}_k$, and the decomposition is unique up to ordering.*

Proof. Pass to the affine cone V and apply the result for affine varieties (Proposition 5.6). □

Definition 16.7. Let $\mathbb{V} \subset \mathbb{P}^n$ be a projective variety. A **Zariski-closed** subset of $\mathbb{V}$ is a projective subvariety $\mathbb{W} \subset \mathbb{V}$, and a **Zariski open** subset of $\mathbb{V}$ is the complement $\mathbb{U} = \mathbb{V} \setminus \mathbb{W}$ of a Zariski-closed subset.

As for affine varieties this defines a topology on $\mathbb{V}$ (exercise) called the *Zariski topology.*

Lemma 16.8. *Let $\mathbb{V} \subset \mathbb{P}^n$ be a projective variety and let $V_i \subset \mathbb{A}^n$ be the intersection of $\mathbb{V}$ with one of the standard charts. Then a subset $U \subset V_i$ is Zariski open if and only if $U = V_i \cap \mathbb{U}$ for some Zariski open subset $\mathbb{U} \subset \mathbb{V}$.*

Proof. Exercise. □

This says that V_i is a *subspace* of $\mathbb{V}$ in the Zariski topology.

Definition 16.9. A **quasi-projective variety** is a Zariski open subset of a projective variety.

Note that $\mathbb{A}^n$ is a quasi-projective variety, because $\mathbb{V}(x_n) \subset \mathbb{P}^n$ is Zariski-closed and its complement is the first standard chart:

$$\mathbb{A}^n = \{x_n \neq 0\} \subset \mathbb{P}^n$$

Moreover if $V \subset \mathbb{A}^n$ is any affine variety then it has a projective completion $\mathbb{V} \subset \mathbb{P}^n$ such that $V = \mathbb{V} \setminus \mathbb{V}(x_n)$, so V is a quasi-projective variety. It follows that any quasi-affine variety is also quasi-projective.

We've now met four kinds of varieties:

$$\{\text{affine}\} \subset \{\text{quasi-affine}\} \subset \{\text{quasi-projective}\} \supset \{\text{projective}\}$$

You many be wondering if there's a general definition of "variety" of which these are all special cases. There is indeed such a definition, but it's quite hard and not that helpful. In practice most varieties we care about are quasi-projective so that's general enough.

Note that most quasi-projective varieties are neither quasi-affine nor projective.

Example 16.10. Let $\mathbb{U} = \mathbb{P}^2 \setminus 0\!:\!0\!:\!1$, this is a quasi-projective variety. It is not contained in any of the standard charts (or non-standard charts), so it doesn't look like it's quasi-affine. It's possible to prove that it's not isomorphic to any quasi-affine variety, or any projective variety. △

Now let's do tangent spaces, singularities, and dimension for projective varieties. The simplest approach is to look at your projective variety in charts: given a point $p \in \mathbb{V}$, choose a standard chart $\mathbb{A}^n \subset \mathbb{P}^n$ containing p, and look at the affine variety $V = \mathbb{V} \cap \mathbb{A}^n$. Then we already know how to define the tangent space $T_p V$, so we can use this to define singular points of $\mathbb{V}$, and the dimension of $\mathbb{V}$. However, before we can do this we must check what happens if we change charts.

Let $\mathbb{V} \subset \mathbb{P}^2$ be a projective variety (we're working in $\mathbb{P}^2$ for simplicity but the same argument will work for $\mathbb{P}^n$). Let's write V_1, V_2, V_3 for the three affine varieties we get by intersecting $\mathbb{V}$ with the three standard charts, *i.e.*

$$V_1 = \mathbb{V} \cap \{z \neq 0\} \subset \mathbb{A}^2_{x,y}$$
$$V_2 = \mathbb{V} \cap \{y \neq 0\} \subset \mathbb{A}^2_{x,z}$$
$$V_3 = \mathbb{V} \cap \{x \neq 0\} \subset \mathbb{A}^2_{y,z}$$

Changing charts from the first to the second chart gives the function

$$\Psi_{12} : \mathbb{A}^2_{x,y} \setminus V(y) \xrightarrow{\sim} \mathbb{A}^2_{x,z} \setminus V(z)$$
$$(x, y) \mapsto \left(\frac{x}{y}, \frac{1}{y}\right)$$

(see exercises). This is an isomorphism of quasi-affine varieties, or a birational equivalence from $\mathbb{A}^2$ to $\mathbb{A}^2$. Restricting to $\mathbb{V}$ gives an isomorphism

$$\Psi_{12} : V_1 \setminus V(y) \xrightarrow{\sim} V_2 \setminus V(z)$$

between a Zariski open subset of V_1 and a Zariski open subset of V_2.

Now pick a point $p \in \mathbb{V}$ lying in $\{z \neq 0\} \cap \{y \neq 0\}$, so it defines points $p_1 \in V_1$ and $p_2 \in V_2$. Then $\Psi_{12}(p_1) = p_2$ so we have a linear isomorphism

$$(D\Psi_{12})_{p_1} : T_{p_1} V_1 \xrightarrow{\sim} T_{p_2} V_2$$

(because Corollary 12.17 also holds for regular maps between quasi-affine varieties; see the exercises). Hence $\dim T_{p_1} V_1 = \dim T_{p_2} V_2$, and p_1 is a singular point of V_1 iff p_2 is a singular point of V_2. Therefore the following definition makes sense:

Definition 16.11. Let $p \in \mathbb{V}$ be a point of a projective variety, such that p lies in the ith standard chart $\mathbb{A}^n \subset \mathbb{P}^n$. We say p is a **singular point** of $\mathbb{V}$ if it gives a singular point of the affine variety $V_i = \mathbb{V} \cap \mathbb{A}^n$. Otherwise we say p is a **non-singular** (or **regular**) point.

We can also define an integer

$$d_p = \dim T_p V_i$$

for the point $p \in \mathbb{V}$; this is well-defined because it's independent of which chart you choose. But defining an actual vector space $T_p\mathbb{V}$ is a bit more tricky and we won't do it.

We can use the function d to give a slightly different characterization of singular points, in the style of Definition 12.10.

Lemma 16.12. *A point $p \in \mathbb{V}$ is non-singular if and only if there exists a Zariski open neighbourhood $p \in \mathbb{U} \subset \mathbb{V}$ such that $d_q = d_p$ for all $q \in \mathbb{U}$.*

Proof. Pick a standard chart $p \in \mathbb{A}^n \subset \mathbb{P}^n$ and let $V = \mathbb{V} \cap \mathbb{A}^n$ be the corresponding affine variety. If such a neighbourhood $\mathbb{U}$ exists then $\mathbb{U} \cap V$ is a Zariski open subset of V and it follows that p is a non-singular point of V. Conversely if p is a non-singular point of V then there is a Zariski open neighbourhood $p \in U \subset V$ on which the function d is constant, and U is also a Zariski open subset of $\mathbb{V}$ by Lemma 16.8. □

Recall that for affine varieties the dimension only really makes sense if the variety is irreducible. If $\mathbb{V}$ is an irreducible projective variety then the intersection $V_i = \mathbb{V} \cap \mathbb{A}^n$ of $\mathbb{V}$ with a standard chart is also irreducible (Proposition 16.2) so we can define

$$\dim \mathbb{V} = \dim V_i$$

and it doesn't matter which chart we choose. Equivalently we can define $\dim \mathbb{V}$ to be the minimum value of the function d.

Instead of intersecting $\mathbb{V}$ with a chart we could look at the affine cone $V \subset \mathbb{A}^{n+1}$. Recall that V is irreducible iff $\mathbb{V}$ is irreducible (Proposition 16.5). Their dimensions are related as follows:

Proposition 16.13. *Let $\mathbb{V} \subset \mathbb{P}^n$ be an irreducible projective variety and let $V \subset \mathbb{A}^{n+1}$ be the corresponding affine cone. Then:*

$$\dim \mathbb{V} = \dim V - 1$$

As we said in the previous section hopefully this is intuitively clear. But the proof takes a little ingenuity.

Proof. To simplify notation let's set $n = 2$ and use co-ordinates x, y, z. The same proof will work for a general n with co-ordinates $x_1, ..., x_n$.

Let $V_1 \subset \mathbb{A}^2$ be the intersection of $\mathbb{V}$ with the first standard chart $\{z \neq 0\}$. Points in V_1 correspond to points of V lying in the Zariski open subset $U = V \setminus V(z)$, *i.e.* the regular map

$$\begin{aligned} \widehat{F} : \mathbb{A}^3 \setminus V(z) &\to \mathbb{A}^2 \\ (x, y, z) &\mapsto (x/z, y/z) \end{aligned}$$

induces a regular map $F : U \to V_1$. Now fix a point $p \in U$ and let $q = F(p) \in V_1$. Differentiating gives a map:

$$DF_p : T_pV \to T_qV_1$$

We claim that this is a surjection with 1-dimensional kernel. This implies that

$$\dim T_qV_1 = \dim T_pV - 1$$

and since this hold for all points $p \in U$ the proposition follows immediately.

First we prove that the kernel of DF_p is 1-dimensional. Let (α, β, γ) be the co-ordinates of our point p. It's an easy computation that $\operatorname{Ker} D\widehat{F}_p$ is 1-dimensional and spanned by the vector $(\alpha, \beta, \gamma) \in \mathbb{C}^3$; this proves that $\operatorname{Ker} DF_p$

is at most 1-dimensional, but to know that it is actually 1-dimensional we need to show that (α, β, γ) lies in T_pV.

If $f \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree k then

$$x\partial_x f + y\partial_y f + z\partial_z f = kf$$

(this is sometimes called *Euler's homogeneous function theorem*). In particular if f vanishes at our point p then the vector (α, β, γ) lies in the kernel of the derivative Df_p. Since $I(V)$ can be generated by homogeneous polynomials this implies that the vector $(\alpha, \beta, \gamma) \in T_pV$, as required.

Finally we prove that DF_p is surjective. Consider the regular map:

$$\begin{aligned} \widehat{G} : \mathbb{A}^2 &\to \mathbb{A}^3 \setminus V(z) \\ (x, y) &\mapsto (\gamma x, \gamma y, \gamma) \end{aligned}$$

It induces a regular map $G : V_1 \to U$ such that $G(q) = p$ and $F \circ G = 1_{V_1}$. Then the chain rule (Lemma 12.16(ii)) gives that $DF_p \circ DG_q$ is the identity on T_qV_1 which means that DF_p must be a surjection. □

17 Regular and rational functions

Our next task is to define regular functions between projective varieties. As we shall see they are closely related to *rational* functions on affine varieties; this is one of the reasons for developing the theory of rational functions.

Let's start by thinking about functions from $\mathbb{A}^n$ to $\mathbb{P}^1$. Such a function is something of the form

$$\mathbf{x} \mapsto f(\mathbf{x}):g(\mathbf{x})$$

for scalar functions f and g, and if we want the function to be 'algebraic' we should probably insist that f and g are polynomials. So it looks like we can get a function $\mathbb{A}^n \to \mathbb{P}^1$ from a pair of polynomials, and we could write this function as $f:g$. But rescaling both sides does nothing, so for any polynomial h

$$\mathbf{x} \mapsto h(\mathbf{x})f(\mathbf{x}) : h(\mathbf{x})g(\mathbf{x})$$

defines the same function as $f : g$. So we are really talking about equivalence classes (f, g) of polynomials, for the equivalence relation generated by

$$(f, g) \sim (hf, hg)$$

for any polynomial h. This is almost exactly the same thing a rational function f/g on $\mathbb{A}^n$.

To see what the difference is, recall that for a rational function $\psi = f/g$ on $\mathbb{A}^n$:

(i) A point $\mathbf{x} \in \mathbb{A}^n$ is a regular point if $g(\mathbf{x}) \neq 0$ (after cancelling common factors), and ψ defines a regular function $\mathbb{A}^n \setminus V(g) \to \mathbb{C}$.

(ii) We do not allow $g = 0$.

Now compare this to our supposed 'function' $f:g$ from $\mathbb{A}^n$ to $\mathbb{P}^1$.

(i) If $g(\mathbf{x}) = 0$, but $f(\mathbf{x}) \neq 0$, then

$$f(\mathbf{x}){:}g(\mathbf{x}) \; = \; 1{:}0 \; \in \mathbb{P}^1$$

is a perfectly good point in $\mathbb{P}^1$. So these points $\mathbf{x}$ are OK. Similarly points $\mathbf{x}$ where $f(\mathbf{x}) = 0$ but $g(\mathbf{x}) \neq 0$ are OK. But if $f(\mathbf{x}) = 0$ and $g(\mathbf{x}) = 0$ then our 'function' doesn't make sense.

(ii) If $g = 0$ is the zero polynomial but $f \neq 0$, then $f : g$ is equivalent to $1{:}0$, where 1 and 0 here mean constant polynomials. This is fine, it is a constant function mapping all of $\mathbb{A}^n$ to the point $1{:}0 \in \mathbb{P}^1$.

So our expression $f{:}g$ does define a function to $\mathbb{P}^1$, but only on the open set $\mathbb{A}^n \setminus V(f, g)$.

Definition 17.1. A **rational function** $\Psi : \mathbb{A}^n \dashrightarrow \mathbb{P}^1$ is a pair of polynomials (f, g), with at least one of f, g non-zero, up to the equivalence relation generated by

$$(f, g) \sim (hf, hg)$$

for all polynomials h. We write $\Psi = f{:}g$ for the equivalence class.

A point $\mathbf{x} \in \mathbb{A}^n$ is called a **base-point** of Ψ if (after cancelling common factors in (f, g)) we have $f(\mathbf{x}) = g(\mathbf{x}) = 0$. If not $\mathbf{x}$ is called a **regular point**.

Ψ defines an actual function $\Psi : \mathbb{A}^n \setminus \{\text{base-points}\} \longrightarrow \mathbb{P}^1$, and the domain here is a Zariski open set. Note:

- Say $\Psi = f{:}g$. Assuming $g \neq 0$, we have a rational function:

  $$\psi = f/g \; \in \mathbb{C}(x_1, ..., x_n)$$

  If $\mathbf{x}$ is a regular point for ψ then it is a regular point for Ψ. But the converse is not true, because we could have $g(\mathbf{x}) = 0$ but $f(\mathbf{x}) \neq 0$. In this setting we are allowed points where "$\psi(\mathbf{x}) = \infty$".

- If $f \neq 0$ then we have a second rational function $\varphi = g/f \in \mathbb{C}(x_1, ..., x_n)$, and if neither f nor g are zero then obviously $\varphi = 1/\psi$. Clearly a point $\mathbf{x}$ is a regular point for Ψ iff it is a regular point for at least one of ψ and φ.

  What we're doing here is choosing either the first or second standard chart on the target $\mathbb{P}^1$, and then trying to write down Ψ as a function from $\mathbb{A}^n$ to $\mathbb{A}^1$. But this function is not defined at base-points of Ψ, and it is also not defined at points whose image lies outside our chosen chart.

- We could try and generalize this definition by allowing the components of Ψ to be rational functions instead of polynomials, *i.e.* considering expressions of the form:

  $$\Psi = \frac{f_1}{f_2} : \frac{g_1}{g_2}$$

  But since we can rescale we can just clear the denominators and get

  $$\Psi = f_1 g_2 : g_1 f_2$$

  so we don't gain anything.

Example 17.2. The first standard chart is a rational function

$$\begin{aligned} \mathbb{A}^1 &\dashrightarrow \mathbb{P}^1 \\ x &\mapsto x : 1 \end{aligned}$$

from $\mathbb{A}^1$ to $\mathbb{P}^1$. It has no base-points. △

Example 17.3. The map 'quotient by rescaling'

$$\begin{aligned} \mathbb{A}^2 &\dashrightarrow \mathbb{P}^1 \\ (x, y) &\mapsto x : y \end{aligned}$$

is a rational function, with a single base-point at $(0, 0)$. △

Example 17.4. Consider the rational function:

$$\begin{aligned} \Psi : \mathbb{A}^3 &\dashrightarrow \mathbb{P}^1 \\ (x, y, z) &\mapsto xy : xz \end{aligned}$$

At first sight this seems to have base-points on the plane $\{x = 0\}$ and also on the line $\{y = z = 0\}$. But Ψ is equivalent to $y : z$, so only the points on the line are genuinely base-points.

If we write Ψ using the standard charts on $\mathbb{P}^1$ we get the two rational functions:

$$\begin{aligned} \psi &= xy/xz & \varphi &= xz/xy \\ &\sim y/z & &\sim z/y \quad \in \mathbb{C}(x, y, z) \end{aligned}$$

We can see ψ is regular outside $\{z = 0\}$ and φ is regular outside $\{y = 0\}$. △

It's easy to generalize this definition to allow the target to be $\mathbb{P}^k$.

Definition 17.5. A **rational map** $\Psi : \mathbb{A}^n \dashrightarrow \mathbb{P}^k$ is a $(k + 1)$-tuple of polynomials $(f_0, ..., f_k)$, with at least one non-zero, up to the equivalence relation generated by

$$(f_0, ..., f_k) \sim (hf_0, ..., hf_k)$$

for all polynomials h. We write $\Psi = f_0 : ... : f_k$ for the equivalence class.

A point $\mathbf{x} \in \mathbb{A}^n$ is called a **base-point** of Ψ if (after cancelling common factors) we have $f_0(\mathbf{x}) = ... = f_k(\mathbf{x}) = 0$, and a **regular point** otherwise.

Here are two observations, which we leave as exercises:

- From Ψ we can construct $k + 1$ rational maps

  $$\Phi_i : \mathbb{A}^n \dashrightarrow \mathbb{A}^k$$

  and $\mathbf{x}$ is a regular point of Ψ iff it is a regular point for at least one of the Φ_i.

- $(f_0, ..., f_k)$ and $(g_0, ..., g_k)$ define the same rational map Ψ iff we have

  $$f_i g_j = f_j g_i$$

  for each $i, j \in [0, k]$.

Now let's think about maps from $\mathbb{P}^n$ to $\mathbb{P}^k$. Such a map is exactly the same thing as a map $\Psi : \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^k$ which is invariant under rescaling, *i.e.* such that $\Psi(\lambda\mathbf{x}) = \Psi(\mathbf{x})$ for all $\mathbf{x}$ and λ. To get an (algebraic) map with this property we need that the components of Ψ are all homogeneous polynomials of the same degree d. Because then:

$$\begin{aligned}\Psi(\lambda\mathbf{x}) = & \ f_0(\lambda\mathbf{x}) : ... : f_k(\lambda\mathbf{x}) \\ = & \ \lambda^d f_0(\mathbf{x}) : ... : \lambda^d f_k(\mathbf{x}) \\ = & \ f_0(\mathbf{x}) : ... : f_k(\mathbf{x}) \quad = \Psi(\mathbf{x}) \quad \in \mathbb{P}^k\end{aligned}$$

You could think of $(f_0, ..., f_k)$ as defining a regular map $F : \mathbb{A}^{n+1} \to \mathbb{A}^{k+1}$, and then F has the property that the whole line through a point $\mathbf{x}$ gets mapped to the line through the point $F(\mathbf{x})$. So we nearly get a map on the sets of lines, from $\mathbb{P}^n$ to $\mathbb{P}^k$. But this isn't quite true because there could be base-points: if $F(\mathbf{x}) = 0$ for some $\mathbf{x} \neq 0$ then the whole line through $\mathbf{x}$ gets mapped to the origin in $\mathbb{A}^{k+1}$, and we don't get a point in $\mathbb{P}^k$.

Definition 17.6. A **rational map** $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ is a $(k+1)$-tuple $(f_0, ..., f_k)$ of homogeneous polynomials of the same degree, with at least one f_i non-zero, up to the equivalence relation generated by rescaling by any homogeneous polynomial h.

We can cancel all common factors to get a 'minimal' representative (unique up to multiplication by a scalar), and we can think of this as a regular map $F : \mathbb{A}^{n+1} \to \mathbb{A}^{k+1}$. A **base-point** of Ψ is a line which gets mapped to the origin by F, the other points are **regular points**. If you fail to cancel common factors then you'll miss some regular points. A rational map with no base-points is called a **regular map**.

The **degree** of Ψ is the degree if its component polynomials.

Example 17.7. Consider the rational map of degree one:

$$\begin{aligned}\Psi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2 \\ x\!:\!y \mapsto x\!:\!y\!:\!0\end{aligned}$$

There is a corresponding regular map $F : \mathbb{A}^2 \to \mathbb{A}^3$ sending (x, y) to $(x, y, 0)$, and the only point that maps to $(0, 0, 0)$ under F is the origin. Hence Ψ has no base-points, it's a regular map. Its image is the complement of the third standard chart. △

Any rational map $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ also defines a rational map $\widehat{\Psi} : \mathbb{A}^{n+1} \dashrightarrow \mathbb{P}^k$. The map $\widehat{\Psi}$ will always have a base-point at the origin (unless it has degree zero), the other base-points come in lines and correspond to the base-points of Ψ.

Example 17.8. Consider the rational map

$$\begin{aligned}\widehat{\Psi} : \mathbb{A}^2 \dashrightarrow \mathbb{P}^2 \\ (x, y) \mapsto x\!:\!y\!:\!0\end{aligned}$$

associated to the Ψ from Example 17.7. This $\widehat{\Psi}$ has a unique base-point at the origin. △

Let's look at some other examples of rational maps between projective spaces.

Example 17.9. The map

$$\Psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$
$$x:y:z \mapsto x:y$$

has a single base-point at $0:0:1$. △

Example 17.10. The map

$$\Psi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$
$$s:t \mapsto s^2:t^2:st$$

has no base-points, it's a regular map. The image of this map lies inside the projective hypersurface $\mathbb{V} = \mathbb{V}(xy - z^2) \subset \mathbb{P}^2$. We claimed in Example 15.6 that this $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$, and we'll prove soon that this map Ψ is indeed an isomorphism. △

Example 17.11. The map

$$\Psi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$$
$$s:t \mapsto s^3:s^2t:st^2:t^3$$

is regular, and its image lies in the (projective) twisted cubic:

$$\mathbb{V} = \mathbb{V}(xz - y^2,\ yw - z^2,\ xw - yz) \ \subset \mathbb{P}^3$$

In fact Ψ is an isomorphism from $\mathbb{P}^1$ to $\mathbb{V}$, as we shall see shortly. In Example 12.14 we studied the affine cone on $\mathbb{V}$, we saw that has dimension 2 but cannot be defined by two polynomials. Similarly $\mathbb{V}$ itself has dimension 1, but cannot be cut out of $\mathbb{P}^3$ by less than three polynomials. △

Example 17.12. Generalizing the previous two examples, we have the *Veronese embedding*

$$\Psi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$$
$$s:t \mapsto \ s^n : s^{n-1}t : \ldots : t^n$$

which is also regular map. △

If we have a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ and we pick a chart $\{x_i \neq 0\} \subset \mathbb{P}^n$ then we get a rational map $\Phi' : \mathbb{A}^n \dashrightarrow \mathbb{P}^k$ by setting the variable x_i to 1. At points where Φ is regular we are simply restricting the function to the chart. Note that the base-points of Φ' are exactly the base-points of Φ that lie in our chosen chart.

Example 17.13. If we write the Veronese embedding in the first chart on $\mathbb{P}^1$ we get

$$\Psi' : \mathbb{A}^1 \dashrightarrow \mathbb{P}^n$$
$$s \mapsto \ s^n : s^{n-1} : \ldots : 1$$

which has no base-points. △

Example 17.14. Let

$$\begin{aligned} \Psi : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ x{:}y{:}z &\mapsto x{:}y \end{aligned}$$

as in Example 17.9. If we restrict to the chart $\{y \neq 0\}$ we get

$$\begin{aligned} \Psi' : \mathbb{A}^2 &\dashrightarrow \mathbb{P}^1 \\ (x, z) &\mapsto x{:}1 \end{aligned}$$

which has no base-points. But in the chart $\{z \neq 0\}$ we get

$$\begin{aligned} \Psi'' : \mathbb{A}^2 &\dashrightarrow \mathbb{P}^1 \\ (x, y) &\mapsto x{:}y \end{aligned}$$

which has a base-point at the origin. △

If we fix a projective variety $\mathbb{W} \subset \mathbb{P}^k$ then it's easy to define a regular map from $\mathbb{P}^n$ to $\mathbb{W}$; it's simply a regular map to $\mathbb{P}^k$ whose image lies in $\mathbb{W}$. But we should also be able to pick a projective variety $\mathbb{V} \subset \mathbb{P}^n$ and talk about regular maps from $\mathbb{V}$ to $\mathbb{W}$. This is more complicated.

Example 17.15. Let

$$\mathbb{V} = \mathbb{V}(xy - z^2) \subset \mathbb{P}^2$$

as in Example 15.6. In Example 17.10 we wrote down a regular map $\Psi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $s{:}t \mapsto s^2{:}t^2{:}st$, and we claimed that Ψ was an isomorphism. Let's try and construct the inverse function.

Suppose $x:y:z \in \mathbb{V}$ and that $x \neq 0$. Then this point is the image of the point $x{:}z \in \mathbb{P}^1$, because:

$$x^2{:}z^2{:}xz \;=\; x^2{:}xy{:}xz \;=\; x{:}y{:}z$$

This suggests that we consider the rational map:

$$\begin{aligned} \Phi_1 : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ x{:}y{:}z &\mapsto x{:}z \end{aligned}$$

This map is only regular away from the point $0{:}1{:}0$. Unfortunately this point lies in $\mathbb{V}$, so we only get a function on the Zariski open subset:

$$\mathbb{U}_1 = \mathbb{V} \setminus 0{:}1{:}0 = \mathbb{V} \setminus \mathbb{V}(x, z) = \mathbb{V} \setminus \mathbb{V}(x)$$

Alternatively suppose $x{:}y{:}z \in \mathbb{V}$ is a point where $y \neq 0$. Then this point is the image of $z{:}y \in \mathbb{P}^1$, since:

$$z^2{:}y^2{:}yz \;=\; xy{:}y^2{:}yz \;=\; x{:}y{:}z$$

So maybe we should consider the rational map

$$\begin{aligned} \Phi_2 : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ x{:}y{:}z &\mapsto z{:}y \end{aligned}$$

which is regular at all points of $\mathbb{V}$ except for $1:0:0$. This Φ_2 defines a function on $\mathbb{U}_2 = \mathbb{V} \setminus 1:0:0$.

These two open sets between them cover the whole of $\mathbb{V}$. What's more, the functions Φ_1 and Φ_2 agree on the overlap $\mathbb{U}_1 \cap \mathbb{U}_2$. If $x:y:z \in \mathbb{V}$ and neither x nor y are zero then:

$$x:z \ = \ xy:yz \ = \ z^2:yz \ = z:y$$

So we can define a function $\Phi : \mathbb{V} \to \mathbb{P}^1$ by

$$\Phi : \ x:y:z \mapsto \begin{cases} \Phi_1(x:y:z) & \text{if } \ x:y:z \in \mathbb{U}_1 \\ \Phi_2(x:y:z) & \text{if } \ x:y:z \in \mathbb{U}_2 \end{cases}$$

and this function is the inverse to Ψ. In each open subset the function Φ is the restriction of a rational function on $\mathbb{P}^2$, but there is no single rational function on $\mathbb{P}^2$ that works everywhere. △

This should remind you of what happened when we studied regular functions on quasi-affine varieties, like in Example 8.12. There is a good reason for this: a function from $\mathbb{V}$ to $\mathbb{P}^1$ is exactly the same thing as a function on the quasi-affine variety

$$U = V(xy - z^2) \setminus (0,0,0) \longrightarrow \mathbb{P}^1$$

which is invariant under rescaling. Our Zariski open subsets correspond to a cover $U = U_1 \cup U_2$, where we cut out $V(x)$ or $V(y)$. However, note that we have *not* defined a function from U to $\mathbb{A}^2$. If you think of Φ_1 and Φ_2 as functions to $\mathbb{A}^2$ then they do not agree on the overlap $U_1 \cap U_2$; they only agree once we quotient by rescaling in the target.

Now we can give the definition of a regular map between two projective varieties, just by copying Definition 8.13.

Definition 17.16. Let $\mathbb{V} \subset \mathbb{P}^n$ and $\mathbb{W} \subset \mathbb{P}^k$ be projective varieties. A function $\Psi : \mathbb{V} \to \mathbb{W}$ is **regular** if there exists a finite cover

$$\mathbb{V} = \mathbb{U}_1 \cup ... \cup \mathbb{U}_t$$

of $\mathbb{V}$ by Zariski open subsets, and rational functions $\Psi_i : \mathbb{P}^n \dashrightarrow \mathbb{W}$, such that for every i the rational function Ψ_i is regular at all points of $\mathbb{U}_i$ and $\Psi_i|_{\mathbb{U}_i} \equiv \Psi|_{\mathbb{U}_i}$.

Now we know what it means for two projective varieties to be **isomorphic**.

Example 17.17. Let $\mathbb{V} = \mathbb{V}(z) \subset \mathbb{P}^2$ and let:

$$\Psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$
$$x:y:z \mapsto x:y$$

Ψ only has a base-point at $0:0:1$ so it is regular at every point of $\mathbb{V}$. Hence its restriction to $\mathbb{V}$ defines a regular map $\Psi : \mathbb{V} \to \mathbb{P}^1$. In this example we do not need to split $\mathbb{V}$ up into open subsets.

Obviously Ψ is the inverse map to the inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ from Example 17.7, this proves that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$. △

Example 17.18. Let $\mathbb{V} = \mathbb{V}(xy - z^2) \subset \mathbb{P}^2$, and let $\Phi : \mathbb{V} \to \mathbb{P}^1$ be the function we constructed in Example 17.15. Then Φ is a regular map. It is the inverse to the regular map $\Psi : \mathbb{P}^1 \to \mathbb{V}$ from Example 17.10, which shows that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$. As promised! △

Example 17.19. Let $\mathbb{V}$ be the twisted cubic

$$\mathbb{V} = \mathbb{V}(xz - y^2,\, yw - z^2,\, xw - yz)\ \subset \mathbb{P}^3$$

and let Ψ be the regular map

$$\begin{aligned}\Psi : \mathbb{P}^1 &\dashrightarrow \mathbb{V}\\ s:t &\mapsto s^3:s^2t:st^2:t^3\end{aligned}$$

from Example 17.11. Now consider the following two rational functions:

$$\begin{aligned}\Phi_1 : \mathbb{P}^3 &\dashrightarrow \mathbb{P}^1 \\ x:y:z:w &\mapsto x:y\end{aligned} \qquad\qquad \begin{aligned}\Phi_2 : \mathbb{P}^3 &\dashrightarrow \mathbb{P}^1 \\ x:y:z:w &\mapsto z:w\end{aligned}$$

Φ_1 defines a function on the open set $\mathbb{U}_1 = \mathbb{V} \setminus \mathbb{V}(x, y)$ and Φ_2 defines a function on the open set $\mathbb{U}_2 = \mathbb{V} \setminus \mathbb{V}(z, w)$. Clearly $\mathbb{U}_1$ and $\mathbb{U}_2$ cover $\mathbb{V}$.

We leave it as an exercise to check that Φ_1 and Φ_2 agree on $\mathbb{U}_1 \cap \mathbb{U}_2$, and that the resulting regular function is the inverse to Ψ. This shows that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$. △

The previous example generalizes to the Veronese embedding of any degree (Example 17.12).

We've now defined regular maps between two projective varieties $\mathbb{V}$ and $\mathbb{W}$ but we haven't defined rational maps in this generality. We're not going to do it, but hopefully it's clear that we could define a rational map to be a 'partially defined regular map' in the style of Definition 11.2. Of course we must assume that $\mathbb{V}$ is irreducible.

If $\mathbb{V}$ is irreducible then the affine cone $V \subset \mathbb{A}^{n+1}$ is also irreducible (Proposition 16.5) so there should be a second approach to rational maps based on the function field $\mathbb{C}(V)$, in the style of Definition 11.1. We're not going to do this either, but let's look at an example to get an indication of how this approach works.

Recall that a rational map $f : g$ from $\mathbb{A}^3$ to $\mathbb{P}^1$ defines two rational functions f/g and g/f in $\mathbb{C}(x, y, z)$, and a point is a base-point for $f : g$ when it is not regular for either of these rational functions.

Example 17.20. The affine cone associated to the projective hypersurface $\mathbb{V} = \mathbb{V}(xy - z^2) \subset \mathbb{P}^2$ is the ODP singularity:

$$V = V(xy - z^2) \subset \mathbb{A}^3$$

In the field $\mathbb{C}(V)$ we have a rational function:

$$\phi = \frac{x}{z} = \frac{z}{y}$$

This is obviously regular on the set $V \setminus V(y)$, and in fact this is exactly the set of regular points (see exercises). The reciprocal of ϕ is

$$\frac{1}{\phi} = \frac{z}{x} = \frac{y}{z}$$

which is regular on $V \setminus V(x)$. The only point where neither are regular is the origin. So it looks like we have a rational map $\widehat{\Phi} : V \dashrightarrow \mathbb{P}^1$ which we can express either as

$$\widehat{\Phi} : \ x{:}y{:}z \ \mapsto \ x{:}z \qquad \text{or} \qquad \widehat{\Phi} : \ x{:}y{:}z \ \mapsto \ z{:}y$$

and that $\widehat{\Phi}$ only has a base-point at the origin.

The components of $\widehat{\Phi}$ are homogeneous and all of the same degree, so there should be an induced rational map

$$\Phi : \mathbb{V} \dashrightarrow \mathbb{P}^1$$

on the projective variety. And since $\widehat{\Phi}$ was regular away from the origin, Φ should be a regular map. △

If you define everything properly then the argument in the above example is valid, and produces the regular map Φ from Example 17.15.

18 Plane curves

A hypersurface $\mathbb{V} = \mathbb{V}(f)$ in $\mathbb{P}^2$ is called a (projective) *plane curve*. The *degree* of $\mathbb{V}$ means the degree of the defining polynomial f.

Plane curves of degree 1 are very easy. A polynomial f of degree 1 is exactly a linear map $f : \mathbb{C}^3 \to \mathbb{C}$, and after changing basis you can assume that $f = z$. So $\mathbb{V}(f) \cong \mathbb{V}(z)$, and this is isomorphic to $\mathbb{P}^1$ as we verified in Example 17.17.

What about curves of degree 2? One example is:

$$\mathbb{V} = \mathbb{V}(xy - z^2) \subset \mathbb{P}^2$$

We proved in Example 17.18 that this $\mathbb{V}$ is also isomorphic to $\mathbb{P}^1$. But this is essentially the only example of a degree 2 curve! Because...

A brief digression on quadratic forms

A homogeneous polynomial of degree 2 is called a *quadratic form*. Quadratic forms in n variables are the same thing as symmetric $n \times n$ matrices, for example in 2 variables a general quadratic form can be written as

$$f(x,y) = ax^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the quadratic form $xy - z^2$ can be written as:

$$xy - z^2 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In general a quadratic form can be written as $f(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$, where $\mathbf{x}$ here denotes a column vector, for a unique symmetric matrix Q.

Now suppose we perform a linear change-of-co-ordinates (a change of basis):

$$\mathbf{x}' = A\mathbf{x}$$

In the new co-ordinates our quadratic form becomes:

$$g(\mathbf{x}') = f(A^{-1}\mathbf{x}') = \mathbf{x}'^T A^{-T} Q A^{-1}\mathbf{x}'$$

Example 18.1. Let $f(x, y, z) = x^2 + y^2 + z^2$, this is the quadratic form associated to the 3×3 identity matrix I. Change co-ordinates to:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + iy \\ x - iy \\ iz \end{pmatrix} = \begin{pmatrix} 1 & i & 0 \\ 1 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The quadratric form becomes

$$g(x', y', z') = \begin{pmatrix} x' & y' & z' \end{pmatrix} A^{-T} A^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = x'y' - (z')^2$$

(the inverse transformation is easier to verify). △

So quadratic forms up to changes-of-basis are the same thing as symmetric matrices up to *congruence*:

$$Q \mapsto A^T Q A$$

Proposition 18.2. *Any symmetric complex matrix Q is congruent to a diagonal matrix*

$$Q' = \operatorname{diag}(1, 1, ..., 1, 0, ..., 0)$$

where the number of 1's is the rank of Q.

Proof. Gram-Schmidt algorithm. □

You might be more familiar with the real version of this result, Sylvester's Law of Inertia. It states that symmetric real matrix is congruent to a diagonal matrix $\operatorname{diag}(1, ..., 1, -1, ..., -1, 0, ..., 0)$, so real quadratric forms have two invariants, the rank and the *index*. Over the complex numbers we can multiply by i so we can change the -1's to 1's.

Now we can apply this result to understand plane curves of degree 2.

Corollary 18.3. *Let $\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^2$ be a plane curve of degree 2. Then up to a linear co-ordinate change, either:*

(i) $f(x, y, z) = xy - z^2$

(ii) $f(x, y, z) = xy$

(iii) $f(x, y, z) = x^2$

The three cases correspond to the rank of f being 3, 2 or 1. In case (i) we've shown $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$. This case is typical, because 'most' 3×3 matrices have rank 3. In algebraic geometry we say case (i) is *generic*.

We looked at case (ii) in Example 16.3, here $\mathbb{V}$ is the union of two copies of $\mathbb{P}^1$, meeting at a single point.

Case (iii) is really a curve of degree 1, because $\mathbb{V}(x^2) = \mathbb{V}(x)$, so $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$.

We've seen that all plane curves of degree 1, and all (generic) plane curves of degree 2, are isomorphic to $\mathbb{P}^1$. Topologically, this means they look like the 2-sphere. What can we say about plane curves of higher degree?

Fact: If f is generic (*i.e.* almost always) the plane curve $\mathbb{V}(f)$ has no singular points. It is a real 2-dimensional oriented manifold.

Topologically, $\mathbb{V}(f)$ is a closed oriented surface. There are not very many of these, one of the big theorems of 19th century geometry is that they are classified by their *genus*: the 2-sphere has genus 0, the torus has genus 1, a 'torus with two holes' has genus 2, etc. Here is a very nice theorem, which unfortunately we won't prove:

Theorem 18.4 (Degree-genus formula)**.** *If $\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^2$ is a generic plane curve of degree d then $\mathbb{V}$ is a surface of genus:*

$$g = \tfrac{1}{2}(d-1)(d-2)$$

This theorem shows a beautiful interplay between algebra and topology. It's the baby case of a deeper theorem called the Riemann-Roch theorem, which is arguably the single most important result in algebraic geometry.

If you plug in $d = 1$ or $d = 2$ you get that $\mathbb{V} \cong S^2$, as we've already seen. The case $d = 3$ is about elliptic curves, it says that they have genus 1, so topologically an elliptic curve is a torus. If we set $d = 4$ then we get a surface of genus 3.

Not every value of g can occur, we don't get $g = 2$ for example. But this doesn't mean that these surfaces don't show up as projective varieties; it just means you need to look inside $\mathbb{P}^n$ for $n > 2$, so they're no longer hypersurfaces. If we do this then it's possible to find surfaces of any genus.

19 Multi-projective space

Some nice geometry arises when we think about products of projective spaces, for example:

$$\begin{aligned}\mathbb{P}^1 \times \mathbb{P}^1 &= \big\{(p,q),\ p,q \in \mathbb{P}^1\big\} \\ &= \frac{\big\{(x,y,s,t),\ (x,y) \neq (0,0),\ (s,t) \neq (0,0)\big\}}{(x,y,s,t) \sim (\lambda x, \lambda y, \mu s, \mu t), \quad \forall \lambda, \mu \in \mathbb{C}^*}\end{aligned}$$

We can cover $\mathbb{P}^1 \times \mathbb{P}^1$ with four 'standard charts', *e.g.* the subset

$$\{y \neq 0,\ t \neq 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

bijects with $\mathbb{A}^2$, using the map:

$$(x\!:\!y, s\!:\!t) \mapsto (x/y, s/t)$$

This chart misses the set

$$1\!:\!0 \times \mathbb{P}^1 \ \cup\ \mathbb{P}^1 \times 1\!:\!0$$

which is a union of two $\mathbb{P}^1$'s meeting at the single point $(1\!:\!0, 1\!:\!0)$.

We can get subvarieties in $\mathbb{P}^1 \times \mathbb{P}^1$ from affine varieties in $\mathbb{A}^4$ which are invariant under *both* rescaling operations, the λ one and the μ one. This means that the defining polynomials must be homogeneous in the (x, y) variables (with some degree d_1) and also homogeneous in the (s, t) variables (with some degree d_2). We say such a polynomial has *bidegree* (d_1, d_2).

Example 19.1. Let $f(x, y, s, t) = x^2s^3 + 2xyst^2 - y^2t^3$. Every term in f has degree 2 in (x, y) and degree 3 in (s, t), so f has bidegree $(2, 3)$. It follows that:

$$f(\lambda x, \lambda y, \mu s, \mu t) = \lambda^2\mu^3 f(x, y, s, t)$$

So the set $V(f) \subset \mathbb{A}^4$ is invariant under both rescaling operations, and gives us a well-defined subset:

$$\mathbb{V}(f) \subset \mathbb{P}^1 \times \mathbb{P}^1$$

Note that this is *not* the product of two projective varieties $\mathbb{V}_1, \mathbb{V}_2 \subset \mathbb{P}^1$, because that would have to be a finite set. It's not the product of any two subsets of $\mathbb{P}^1$.

Fact: if f is a generic polynomial of bidegree $(2, 3)$ then $\mathbb{V}(f)$ is topologically a surface of genus 2.

△

Clearly we could generalize the whole theory of projective varieties to "multi-projective varieties". But it turns out that you don't get anything new. Here's why.

Example 19.2. The following map is called the *Segre embedding*:

$$F : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$(x\!:\!y, s\!:\!t) \mapsto xs\!:\!xt\!:\!ys\!:\!yt$$

This is a regular map, because if we look at the associated rational map

$$\widehat{F} : \mathbb{A}^4 \dashrightarrow \mathbb{P}^3$$

it only has base-points where $(x, y) = (0, 0)$ or $(s, t) = (0, 0)$, and these are not points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let's write $a\!:\!b\!:\!c\!:\!d$ for the co-ordinates on the target $\mathbb{P}^3$. The image of F is obviously contained in the hypersurface:

$$\mathbb{V} = \mathbb{V}(ad - bc) \subset \mathbb{P}^3$$

We claim that F is actually an isomorphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{V}$.

To see this, consider the rational maps:

$$\Phi_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \qquad\qquad \Phi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$$
$$a\!:\!b\!:\!c\!:\!d \mapsto a\!:\!c \qquad\qquad a\!:\!b\!:\!c\!:\!d \mapsto b\!:\!d$$

The regular points of Φ_1 and Φ_2 together cover the whole of $\mathbb{V}$ (in fact the whole of $\mathbb{P}^3$), and at any point in $\mathbb{V}$ where both maps are regular they give the same answer. So together they define a regular map:

$$\Phi : \mathbb{V} \longrightarrow \mathbb{P}^1$$

This should look familiar, it's just another point-of-view on Examples 8.12 and 9.2.

In a similar way we can define a regular map $\Psi : \mathbb{V} \longrightarrow \mathbb{P}^1$ which is given by:

$$\Psi : \; a{:}b{:}c{:}d \;\mapsto \begin{cases} a{:}b & \text{if } (a,b) \neq (0,0) \\ c{:}d & \text{if } (c,d) \neq (0,0) \end{cases}$$

The product of Φ and Ψ defines a regular map from $\mathbb{V}$ to $\mathbb{P}^1 \times \mathbb{P}^1$, and it's easy to verify that this map is the inverse to F. Hence $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{V}$. △

It follows that a "multi-projective variety" in $\mathbb{P}^1 \times \mathbb{P}^1$ is always going to be isomorphic to some projective variety in $\mathbb{P}^3$. The Segre embedding can be generalized to the map

$$\mathbb{P}^n \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{(n+1)(k+1)-1}$$
$$(x_0{:}\ldots{:}x_n,\; s_0{:}\ldots{:}s_k) \;\mapsto\; x_0 s_0{:}x_0 s_1{:}\ldots{:}x_n s_k$$

which an isomorphism onto its image. So any "multi-projective variety" can be written as a projective variety.

A by-product of the previous example is that we now understand all degree two hypersurfaces in $\mathbb{P}^3$, *i.e.* all projective varieties of the form

$$\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^3$$

where f is a quadratic form in x, y, z, w. This kind of $\mathbb{V}$ is called a *quadric surface*.

Assuming f has rank 4 (which is the generic case) Proposition 18.2 tells us that we can change basis to make $f = xy - zw$, and then Example 19.2 shows that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

There are three more 'increasingly degenerate' cases, where f has rank 3, 2 or 1:

(3) Here we can change basis to make $f(x, y, z, w) = xy - z^2$. This surface has a singular point at $0{:}0{:}0{:}1$ which looks like the ODP singularity, and contains a copy of $\mathbb{P}^1$ at $w = 0$ (see exercise sheets).

(2) Here we can assume $f = xy$. Then $\mathbb{V} = \mathbb{V}(x) \cup \mathbb{V}(y)$ which is two copies of $\mathbb{P}^2$, meeting along a $\mathbb{P}^1 = \mathbb{V}(x, y)$.

(1) Here $f = x^2$, so $\mathbb{V} = \mathbb{V}(x) \cong \mathbb{P}^2$.

20 Blow-ups

Consider the space

$$\mathbb{A}^2 \times \mathbb{P}^1 = \frac{\big\{(x,y,s,t),\ (s,t) \neq (0,0)\big\}}{(x,y,s,t) \sim (x,y,\lambda s,\lambda t),\ \forall \lambda \in \mathbb{C}^*}$$

This is a Zariski open set in $\mathbb{P}^2 \times \mathbb{P}^1$, so using the Segre embedding it's a quasi-projective variety in $\mathbb{P}^5$.

Subvarieties in this space are cut out by polynomials in $\mathbb{C}[x,y,s,t]$ which are homogeneous in (s,t). An important example is:

$$\mathbb{B} = \mathbb{V}(xt - ys) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

There is a regular map:

$$\pi : \mathbb{B} \longrightarrow \mathbb{A}^2$$
$$(x,y,s\!:\!t) \mapsto (x,y)$$

What are the fibres (level sets) of π? Fix a point $(x,y) \in \mathbb{A}^2$, then $\pi^{-1}(x,y)$ is the set of points $s\!:\!t \in \mathbb{P}^1$ satisfying:

$$xt - ys = \begin{pmatrix} -y & x \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = 0$$

If $(x,y) \neq (0,0)$ this linear map has a 1-dimensional kernel so then there is a unique line of solutions, namely $s\!:\!t = x\!:\!y$, the line through (x,y). So $\pi^{-1}(x,y)$ is the single point $(x,y,x\!:\!y)$.

But if $(x,y) = (0,0)$ then $s:t$ can be anything, so $\pi^{-1}(0,0) = (0,0) \times \mathbb{P}^1$. Over $(0,0)$ the fibre of π jumps from being a single point to the whole of $\mathbb{P}^1$. From this, we see another way to describe $\mathbb{B}$ is:

$$\mathbb{B} = \big\{(x,y,s\!:\!t) \in \mathbb{A}^2 \times \mathbb{P}^1,\ (x,y) \in s\!:\!t\big\}$$

A point in $\mathbb{B}$ consists of a line in $\mathbb{C}^2$, plus a point lying on that line. The map π just forgets the line $s\!:\!t$ and remembers the point (x,y). If $(x,y) \neq (0,0)$ then there's a unique line through (x,y), but if $(x,y) = (0,0)$ then it lies on all lines.

The space $\mathbb{B}$ looks like $\mathbb{A}^2$, except that the origin has been replaced with a copy of $\mathbb{P}^1$. We call $\mathbb{B}$ the **blow-up** of $\mathbb{A}^2$ at $(0,0)$.

We can understand some more about $\mathbb{B}$ by looking in charts. There are two standard charts on $\mathbb{A}^2 \times \mathbb{P}^2$, namely $\{s \neq 0\}$ and $\{t \neq 0\}$. In the first chart $\mathbb{B}$ becomes:

$$V(xt - y) \subset \mathbb{A}^3_{x,y,t}$$

This is the graph of a function, so it's isomorphic to $\mathbb{A}^2$. Similarly in the second chart $\mathbb{B}$ becomes

$$V(x - ys) \subset \mathbb{A}^3_{x,y,s}$$

which is also isomorphic to $\mathbb{A}^2$. So $\mathbb{B}$ has no singular points, and it is two copies of $\mathbb{A}^2$ glued together somehow.

Using these charts it's easy to see that the map π is an isomorphism:

$$\pi : \mathbb{B} \setminus V(x,y) \xrightarrow{\sim} \mathbb{A}^2 \setminus (0,0)$$

In fact π is a birational equivalence between $\mathbb{B}$ and $\mathbb{A}^2$, but we leave it as an exercise to write down the rational inverse.

If you construct $\mathbb{B}$ over the real numbers instead then the space you get is an infinite Möbius strip. See if you can visualise this.

Now let's see what effect replacing $\mathbb{A}^2$ with $\mathbb{B}$ has on subvarieties $V \subset \mathbb{A}^2$.

Example 20.1. Let $V = V(xy) \subset \mathbb{A}^2$, the node. Now let $\mathbb{W} = \pi^{-1}(V) \subset \mathbb{B}$. What is $\mathbb{W}$?

Points in V are either

- $(x, 0)$ with $x \neq 0$. Then $\pi^{-1}(x, 0) = (x, 0, 1\!:\!0) \in \mathbb{B}$, a single point.
- $(0, y)$ with $y \neq 0$. Then $\pi^{-1}(0, y) = (0, y, 0\!:\!1) \in \mathbb{B}$, a single point.
- $(0, 0)$. Then $\pi^{-1}(0, 0) = (0, 0) \times \mathbb{P}^1 \subset \mathbb{B}$.

Indeed:

$$\begin{aligned}
\mathbb{W} &= \mathbb{V}(xt - ys,\, xy) \ \subset \mathbb{A}^2 \times \mathbb{P}^1 \\
&= \mathbb{V}(x, ys) \cup \mathbb{V}(y, xt) \\
&= \mathbb{V}(x, s) \cup \mathbb{V}(x, y) \cup \mathbb{V}(y, t) \\
&= \{(0, y, 0\!:\!1)\} \cup \{(0, 0, s\!:\!t)\} \cup \{(x, 0, 1\!:\!0)\}
\end{aligned}$$

The first and third irreducible components are copies of $\mathbb{A}^1$, and the second is a copy of $\mathbb{P}^1$. The first two meet at the point $(0, 0, 0\!:\!1)$ and the second two meet at $(0, 0, 1\!:\!0)$. The map π just collapses the $\mathbb{P}^1$ component down to $(0, 0)$. △

Now take any affine variety $V \subset \mathbb{A}^2$ with $(0, 0) \in V$. Then $\mathbb{W} = \pi^{-1}(V) \subset \mathbb{B}$ contains $\mathbb{V}(x, y) = \mathbb{P}^1$. Usually $\mathbb{W}$ will be reducible, and this $\mathbb{P}^1$ is one of the irreducible components (as in the example above). We define the **proper transform** of V to be the union of the remaining irreducible components (to be more precise, it's the Zariski-closure of $\pi^{-1}(V) \setminus \mathbb{V}(x, y)$).

The larger variety $\mathbb{W}$ is sometimes called the **total transform** of V.

Example 20.2. The proper transform of $V = V(xy) \subset \mathbb{A}^2$ is:

$$\mathbb{W}' = \mathbb{V}(x, s) \cup \mathbb{V}(y, t) \subset \mathbb{B}$$

This is a disjoint union of two copies of $\mathbb{A}^2$. Note that $\mathbb{W}' \cap \mathbb{V}(x, y)$ is the two points $0\!:\!1, 1\!:\!0 \in \mathbb{P}^1$. These are the two lines that lie in V. When we form the proper transform of V we pull the two lines apart, and we get a variety which has no singular points. △

What happens if V is not a cone?

Example 20.3. Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$, the cusp singularity. The total transform of V is:

$$\mathbb{W} = \pi^{-1}(V) = \mathbb{V}(xt - ys,\, y^2 - x^3) \ \subset \mathbb{A}^2 \times \mathbb{P}^1$$

We claim that $\mathbb{W}$ splits as two irreducible components:

$$\mathbb{W} = \mathbb{V}(x, y) \cup \mathbb{V}(xt - ys,\, t^2 - s^2 x)$$

We can check this claim by looking in charts.

- In $\{s \neq 0\}$ we have

$$\begin{aligned} V(xt - y,\, y^2 - x^3) &= V(xt - y,\, x^2t^2 - x^3) \\ &= V(x, y) \cup V(xt - y,\, t^2 - x) \;\subset \mathbb{A}^3_{x,y,t} \end{aligned}$$

  which agrees with our claim (to see the final equality note that if $x = 0$ then $y = 0$ and if $x \neq 0$ then $t^2 = x$). Inside this chart we can see two non-singular irreducible components meeting at the single point $(0, 0, 0)$.

- In $\{t \neq 0\}$ we have:

$$\begin{aligned} V(x - ys,\, y^2 - x^3) &= V(x - ys,\, y^2 - y^3s^3) \\ &= V(x, y) \cup V(x - ys,\, 1 - ys^3) \;\subset \mathbb{A}^3_{x,y,s} \end{aligned}$$

  In this chart we see two non-singular irreducible components, and they're disjoint.

So the proper transform of V is:

$$\mathbb{W}' = \mathbb{V}(xt - ys,\, t^2 - s^2x) = \mathbb{B} \cap \mathbb{V}(t^2 - s^2x) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

From the charts we see that $\mathbb{W}'$ has no singular points. Also note that:

(i) The intersection $\mathbb{W}' \cap \mathbb{V}(x, y)$ is the single point $1 : 0 \in \mathbb{P}^1$. This is the unique line in $\mathbb{A}^2$ which is *tangent* to V at the point $(0, 0)$.

   This observation generalizes to any $V \subset \mathbb{A}^2$. The intersection of the proper transform of V with the $\mathbb{P}^1$ lying over $(0, 0)$ is exactly the set of lines that are tangent to V at the origin. So to get the proper transform we delete the point $(0, 0)$ from V, and replace it with the set of tangent lines. You should compare this to the process of projective completion, where we added in the set of lines that were tangent to V 'at infinity'.

(ii) In fact $\mathbb{W}'$ is contained inside the chart $\{s \neq 0\}$ (because setting $s = 0$ in the equations for $\mathbb{W}'$ forces $t = 0$, and $s = t = 0$ is impossible). So $\mathbb{W}'$ is the affine variety in $\mathbb{A}^2_{x,y,t}$ cut out by:

$$\mathbb{W}' = V(xt - y,\, t^2 - x) = V(y - t^3,\, x - t^2)$$

   Evidently $\mathbb{W}'$ is isomorphic to $\mathbb{A}^1$, using the map $t \mapsto (t^2, t^3, t)$. The map π is actually the map

$$\begin{aligned} \mathbb{A}^1 &\longrightarrow V \\ t &\mapsto (t^2, t^3) \end{aligned}$$

   which we've seen before (Example 6.12).

$\triangle$

In both the previous examples (the cusp and the node) we found that the proper transform $\mathbb{W}'$ of V was non-singular and the map

$$\pi : \mathbb{W}' \to V$$

is an isomorphism away from the singular points of V. A map like this is called a **resolution of singularities**, and they're very useful.

Example 20.4. For a final example, let's use blow-ups to construct a resolution-of-singularities for the ODP singularity $V = V(xy - z^2) \subset \mathbb{A}^3$.

First we must define the blow up of $\mathbb{A}^3$ at the origin. Extrapolating from the $\mathbb{A}^2$ case, the space we're looking for is:

$$\mathbb{B} = \left\{\text{a line } s\!:\!t\!:\!u \in \mathbb{P}^2 \text{ and a point } (x,y,z) \in \mathbb{A}^3 \text{ lying on } s\!:\!t\!:\!u\right\}$$

The equations defining $\mathbb{B}$ are:

$$\mathbb{B} = \mathbb{V}(xt - ys,\, xu - zs,\, yu - zt) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{P}^2_{s:t:u}$$

These are the determinants of the three maximal minors of $\left(\begin{smallmatrix} x & y & z \\ s & t & u \end{smallmatrix}\right)$, and they vanish exactly when the rows are collinear. If we examine $\mathbb{B}$ in the chart $\{s \neq 0\}$ we get

$$V(xt - y,\, xu - z,\, yu - zt) \subset \mathbb{A}^5_{x,y,z,t,u}$$

which is isomorphic to $\mathbb{A}^3$ (the third equation is redundant). The other two charts look similar, so $\mathbb{B}$ has no singular points, and consists of three copies of $\mathbb{A}^3$ glued together.

We have a regular map

$$\begin{aligned} \pi : \mathbb{B} &\longrightarrow \mathbb{A}^3 \\ (x,y,z,s\!:\!t\!:\!u) &\mapsto (x,y,z) \end{aligned}$$

which just forgets the line. If $(x,y,z) \neq (0,0,0)$ there is a unique line through this point, so $\pi^{-1}(x,y,z)$ is the single point $(x,y,z,x\!:\!y\!:\!z)$. But $\pi^{-1}(0,0,0)$ is the whole of $\mathbb{P}^2$.

Now let $V = V(xy - z^2) \subset \mathbb{A}^3$. The total transform of V is

$$\pi^{-1}(V) = \mathbb{B} \cap \mathbb{V}(xy - z^2) \subset \mathbb{A}^3 \times \mathbb{P}^2$$

and we claim this the union of two irreducible components:

$$\pi^{-1}(V) = \mathbb{V}(x,y,z) \cup \big(\mathbb{B} \cap \mathbb{V}(st - u^2)\big)$$

This can be verified in charts (exercise). The first component is the $\mathbb{P}^2$ lying over the origin in $\mathbb{A}^3$, the second component is (by definition) the proper transform $\mathbb{W}'$ of V. We leave it as another exercise to check that $\mathbb{W}'$ has no singular points.

We know that the map $\pi : \mathbb{W}' \longrightarrow V$ is a bijection away from the origin in V. But at the origin we have

$$\pi^{-1}(0,0,0) = \mathbb{V}(x,y,z) \cap \mathbb{W}' = \mathbb{V}(st - u^2) \ \subset \mathbb{P}^2$$

which is a degree 2 plane curve, and know it is isomorphic to $\mathbb{P}^1$.

So in this construction we replace the singular point in V with a copy of $\mathbb{P}^1$, and we get the non-singular quasi-projective variety $\mathbb{W}'$. $\triangle$

A Technical results on regular functions

Let $U \subset \mathbb{A}^n$ be a quasi-affine variety. In Definition 8.13 we said that a function $F : U \to \mathbb{C}$ was regular if we can cover U by Zariski open subsets $U_1, ..., U_k$ and in each one find a rational function f_i/g_i which agrees with F.

But now suppose that U is a Zariski open subset of $\mathbb{A}^n$. In this case we have a simpler definition of a 'regular function' on U (Definition 8.7), we said it was just a rational function f/g such that g doesn't vanish anywhere in U. Since we have two competing definitions here, we must prove that they're equivalent.

Lemma A.1. *Let $U \subset \mathbb{A}^n$ be a Zariski open subset. Suppose $F : U \to \mathbb{C}$ is a regular function in the sense of Definition 8.13. Then there is a single rational function $f/g \in \mathbb{C}(x_1, ..., x_n)$ such that g doesn't vanish on U and $F = (f/g)|_U$.*

Proof. By assumption we have a Zariski open cover $U = U_1 \cup ... \cup U_k$ and rational functions $f_i/g_i \in \mathbb{C}(x_1, ..., x_n)$ such that $F|_{U_i} = (f_i/g_i)|_{U_i}$. Pick any two i and j. Then $U_i \cap U_j$ is a non-empty Zariski open subset of $\mathbb{A}^n$ by Lemma 7.6. The polynomial

$$f_i g_j - f_j g_i$$

vanishes on $U_i \cap U_j$, so it vanishes on the Zariski-closure of $U_i \cap U_j$, and by Lemma 7.11 this is the whole of $\mathbb{A}^n$. So $f_i g_j - f_j g_i$ is actually the zero polynomial, hence f_i/g_i and f_j/g_j are equivalent rational functions in $\mathbb{C}(x_1, ..., x_n)$, and if we write them both in lowest terms we get the same expression f/g. Moreover g cannot vanish at any point in $U_i \cup U_j$. Since this holds for any i, j we see that g cannot vanish in U, and F is the restriction of f/g. $\square$

Now suppose instead that $V \subset \mathbb{A}^n$ is an affine variety. In this case we again have two competing definitions of a 'regular function; on V: the complicated one in Definition 8.13, and the simpler one where a regular function is the just the restriction of a polynomial (Definition 4.1). So we must prove that these two definitions are equivalent.

Lemma A.2. *Let $V \subset \mathbb{A}^n$ be an affine variety, and let $F : V \to \mathbb{C}$ be a regular function in the sense of Definition 8.13. Then there are polynomials $f_1, ..., f_k$ and $g_1, ..., g_k$ such that*

$$U_1 = V \setminus V(g_1), \quad ..., \quad U_k = V \setminus V(g_k)$$

form a Zariski open cover of V, and such that

$$F|_{U_i} \equiv (f_i/g_i)|_{U_i}$$

for each i.

This looks just like Definition 8.13 except that each U_i is not an arbitrary open set, it's the complement of $V(g_i)$.

Proof. By assumption we can find a Zariski open cover $U_1, ..., U_k$ of V such that on each piece of the cover F agrees with some rational function $\hat{f}_i/\hat{g}_i \in \mathbb{C}(x_1, ..., x_n)$. We can assume that each U_i is of the form

$$U_i = V \setminus V(h_i)$$

for a polynomial h_i; this is because every Zariski open subset is a finite union of subsets of this form, so we can make it true by refining our open cover.

Consider the polynomial $\hat{g}_1$. By assumption it does not vanish at any point in $U_1 = V \setminus V(h_1))$. So if $\hat{g}_1(x) = 0$, and $x \in V$, then $h_1(x) = 0$. This says that h_1 vanishes on the affine variety $V \cap V(\hat{g}_1)$. Nullstellensatz (Theorem 3.8) tells us that we can write

$$h_1^k = p + q\hat{g}_1$$

for some $p \in I(V)$, some $k \in \mathbb{N}$, and some polynomial q. So inside the set U_1 we have:

$$F|_{U_1} \equiv \left.\frac{\hat{f}_1}{\hat{g}_1}\right|_{U_1} \equiv \left.\frac{q\hat{f}_1}{h_1^k}\right|_{U_1}$$

Since $U_1 = V \setminus V(h_1) = V \setminus V(h_1^k)$, if we set $f_1 = q\hat{f}_1$ and $g_1 = h_1^k$ then we have expressed F in the required form within U_1. We can do the same in each U_i. □

Proposition A.3. *Let $V \subset \mathbb{A}^n$ be an affine variety, and let $F : V \to \mathbb{C}$ be a regular function in the sense of Definition 8.13. Then there is a single polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ such that $F = f|_V$.*

So Definitions 8.13 and 4.1 agree for affine varieties.

Proof. By Lemma A.2 we can cover V by open sets $U_i = V \setminus V(g_i)$ and find polynomials f_i such that F is given by f_i/g_i inside U_i. Pick any two i and j and consider the intersection:

$$U_i \cap U_j = V \setminus V(g_i g_j)$$

The expressions f_i/g_i and f_j/g_j define the same function in this subset, so the polynomial $f_i g_j - f_j g_j$ vanishes in this subset, which implies that

$$(f_i g_j - f_j g_i)g_i g_j = f_i g_i g_j^2 - f_j g_j g_i^2$$

vanishes at all points of V.

Now the intersection $V \cap V(g_1^2, ..., g_k^2)$ is empty because the U_i cover V. By Weak Nullstellensatz (Corollary 3.13) we can find polynomials $h_0, h_1, ..., h_k$ with $h_0 \in I(V)$ such that:

$$h_0 + h_1 g_1^2 + ...h_k g_k^2 = 1 \tag{A.4}$$

We set f to be the polynomial

$$f = f_1 g_1 h_1 + ... + f_k g_k h_k$$

and we claim that F is the restriction of f.

To see the claim take a point $x \in U_1$. Here $F(x) = f_1(x)/g_1(x)$, so multiplying by (A.4) we get:

$$F|_x = \left.\left(f_1 h_1 g_1 + \frac{f_1 h_2 g_2^2}{g_1} + ... + \frac{f_1 h_k g_k^2}{g_1} \right)\right|_x = f|_x$$

If we take x in any other U_i we get the same result. □