Uniqueness of Tangent Cones to Positive-(p, p) Integral Cycles

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Abstract: *We prove that every positive-(p, p) integral cycle in an arbitrary almost complex manifold possesses at every point a unique tangent cone. The argument relies on an algebraic blow up perturbed in order to face the analysis issues of this problem in the almost complex setting.*
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1 Introduction

The notion of calibration appeared in the foundational paper [13] in 1982, after that key features of calibrations had been observed in some particular cases in the previous decades (see [19] for a historical overview).

An immediate impact of calibrated currents was in connection with Plateu's problem, since these objects are mass-minimizers in their homology class and thus provide plenty of interesting and explicit examples of volume-minimizers. In the last fifteen years, however, calibrations have appeared surprisingly in many other geometric or physical problems, for example (see [9], [26], [27], [28], [29]) theory of invariants, Yang-Mills fields, String theory, etc. Typically an essential issue in these studies is to understand regularity properties of calibrated currents.

Already raised in [13], one of the long-standing regularity questions is whether calibrated integral currents admit unique tangent cones. The issue is still open, except for currents of dimension 2. Let us recall a few notions and the state of the art, before passing to the results of the present work.

Given a m-form ϕ on a Riemannian manifold (M, g), the comass of ϕ is

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defined to be

$$||\phi||^* := \sup\{\langle \phi_x, \xi_x \rangle : x \in M, \xi_x \text{ is a unit simple } m\text{-vector at } x\}.$$

A form ϕ of comass one is called a *calibration* if it is closed ($d\phi = 0$). We will be dealing also with non-closed forms of unit comass, which will be referred to as *semi-calibrations*, following the terminology in [20].

Let ϕ be a (semi-)calibration; among the oriented m-dimensional planes of the Grassmannians $G(m, T_x M)$, we pick those on which ϕ agrees with the m-dimensional volume form. Representing oriented m-dimensional planes as unit simple m-vectors, we are thus selecting the subfamily of the so-called *m-planes calibrated by ϕ*:

$$\mathcal{G}(\phi) := \cup_{x \in M} \{\xi_x \in G(m, T_x M) : \langle \phi_x, \xi_x \rangle = 1\}.$$

An integral current C of dimension m is said to be ϕ-(semi)calibrated if, $\mathcal{H}^m$-almost everywhere, its (oriented) approximate tangent planes belong to $\mathcal{G}$. Equivalently this means that the m-volume agrees with ϕ on C, i.e. we recover the mass of C by testing the current on the (semi)calibration, $C(\phi) = M(C)$. A simple argument (see [13]) then shows that calibrated currents are homologically mass-minimizing, while semi-calibrated ones are almost minimizers (or λ-minimizers, using the terminology of [10]).

The interest in allowing for semi-calibrations rather than only calibrations is for instance related, as described in section 6 of [28], to the possibility of using non-closed forms to define anti self-dual instantons and compactify the corresponding moduli spaces to define new geometric invariants. Moreover, a regularity theory for semi-calibrated integral cycles can be expected to be considerably nicer than that for general almost-minimizers (see e.g. the case of Special Legendrians in [2], [3]).

Examples of well-known calibrations are the symplectic form ω in an almost Kähler manifold, its normalized powers $\frac{1}{p!}\omega^p$, the Special Lagrangian calibration in Calabi-Yau m-folds, the Associative calibration, and many others.

If we drop the closedness assumption on ω in the definition of almost Kähler manifold, we get what is called an almost Hermitian manifold, and ω and $\frac{1}{p!}\omega^p$ are then semi-calibrations. We will refer to these as almost-complex semi-calibrations.

When dealing with a boundaryless integral current C, also called integral cycle, or simply when we localize the current to an open set in which the boundary is zero, it turns out that calibrated currents satisfy an important *monotonicity* formula for the mass ratio: for any x_0, the quantity $\frac{M(C \llcorner B_r(x_0))}{r^m}$ is a weakly increasing function of r. This is a classical result for mass-minimizers (see [11], [23] or [19]), proved for constant calibrations in [13].

When we turn our attention to almost-minimizers, what we get is an *almost-monotonicity* formula, see [10] and [23]. Almost-monotonicity was proved for C^1 semi-calibrations in [20]: it states that the mass ratio at scale r, i.e. the quantity $\frac{M(C \llcorner B_r(x_0))}{r^m}$, is given by a weakly increasing function of r plus a perturbation term, that is infinitesimal of r. The perturbation term is bounded in modulus independently of x_0.

Immediate consequences of (almost) monotonicity are:

(i) the *density* of the current is well-defined for every point x_0 as the limit

$$\nu(x_0) := \lim_{r \to 0} \frac{M(C \llcorner B_r(x_0))}{\alpha_m r^m},$$

where α_m is the m-dimensional volume[1] of the unit ball B^m.

(ii) the density is an *upper semi-continuous* function.

(iii) the density of a semi-calibrated integral cycle is, everywhere on the support[2], bounded by 1 from below [3].

Monotonicity further yields the existence of *tangent cones*: this is a first step in the study of regularity of calibrated currents. The notion of tangent cone to a current C at a point x_0 is defined by the following procedure, called the *blow up limit*, whose idea goes back to De Giorgi [8]. Dilate C around x_0 by a factor r; in normal coordinates around x_0 this amounts to pushing forward C via the map $\frac{x - x_0}{r}$:

$$(C_{x_0,r} \llcorner B_1)(\psi) := \left[\left(\frac{x - x_0}{r} \right)_* C \right] (\chi_{B_1} \psi) = C \left(\chi_{B_r(x_0)} \left(\frac{x - x_0}{r} \right)^* \psi \right). \tag{1}$$

The fact that $\frac{M(C \llcorner B_r(x_0))}{r^m}$ is monotonically almost-decreasing as $r \downarrow 0$ gives that, for $r \leq r_0$ (for a small enough r_0), we are dealing with a family of

[1] Recall that an arbitrary integral current C is defined by assigning on an oriented m-rectifiable set $\mathcal{C}$ an integer-valued *multiplicity function* $\theta \in L^1(\mathcal{C}, \mathbb{N})$. For an arbitrary integral current, the density ν is well-defined $\mathcal{H}^m$-a.e. and agrees $\mathcal{H}^m$-a.e. with θ. What we get for semi-calibrated cycles is that the density ν is well-defined everywhere, and we can take ν as the "precise representative" for the multiplicity θ.

[2] The support of a current C is the complement of the largest open set in which the action of the current is zero.

[3] This fails for arbitrary integral currents, as the example of the current of integration on a cone (counted with multiplicity 1) shows: the density at the vertex, although well-defined, depends on the opening angle of the cone (the narrower the cone is, the lower the density is). If we take, instead of a cone, a surface with a cusp point, the density there is 0.

cycles $\{C_{x_0,r} \llcorner B_1\}$ in B_1 that are equibounded in mass. Therefore Federer-Fleming's compactness theorem (see e.g. [12] page 141) gives that there exist weak limits of $C_{x_0,r}$ as $r \to 0$. Every such limit C_∞ is an integer multiplicity rectifiable boundaryless current which turns out to be a cone[4] calibrated by ω_{x_0} and is called a tangent cone to C at x_0. The density of each tangent cone at the vertex is the same as the density of C at x_0 (see [13]).

The natural first question, raised already in [13], is whether Federer-Fleming's compactness theorem can yield different sequences of radii with different cones as limits, i.e. whether the tangent cone at an arbitrary point is *unique* or not. The answer is positive for semi-calibrated integral cycles of dimension 2, as proved in [20]. The uniqueness is also known for mass-minimizing integral currents of dimension 2, thanks to [30]. In some other cases, which also follow from either of the aforementioned [30] or [20], the proof has been achieved using techniques of positive intersection, namely for integral pseudo-holomorphic cycles in dimension 4 ([27], [21]) and for integral Special Legendrian cycles in dimension 5 ([2], [3]). In [22] the uniqueness for pseudo holomorphic integral 2-dimensional cycles is achieved in arbitrary codimension. In [24] it is proved that if a tangent cone to a minimal integral current has multiplicity one and has an isolated singularity, then it is unique.

In dimensions higher than two the uniqueness of tangent cones is an open question. In this work we give a positive answer in the case of pseudo-holomorphic integral currents (i.e. semi-calibrated by the almost complex semi-calibration $\frac{1}{p!}\omega^p$) of arbitrary dimension and codimension.

The uniqueness cannot be obtained merely as a consequence of the monotonicity of the mass ratio. Indeed, the notion of being calibrated by ω can be extended from integral currents to normal ones (then it is usually called *ω-positiveness*, as in [13]). Normal ω-positive cycles still fulfil the same monotonicity formula, but it was proved in [17] that they might have non-unique tangent cones.

The approach presented in this work relies on an algebraic blow up technique, adapted to the almost complex setting, that shows how, for almost-complex semi-calibration, uniqueness of tangent cones can indeed be obtained for <u>integral</u> semi-calibrated cycles just as a consequence of almost monotonicity and of the fact that the density is bounded by 1 from below.

An interesting aspect of this proof is that it does not require the study of the rate of convergence of $\frac{M(C \llcorner B_r(x_0))}{\alpha_m r^m}$ to the density $\nu(x_0)$. The understanding of this "rate of decay" was instead essential in [20], [24], [30].

The technique of a *"pseudo holomorphic blow up"* that we present is used in [4] for pseudo-holomorphic non-rectifiable currents of dimension 2. Here we extend that technique to higher dimensional pseudo holomorphic currents.

[4]A current is said to be a cone with vertex p if it is invariant under homotheties centered at p.

These are also referred to as *positive-(p,p) currents*, as we will see in the next section, where we describe more closely the setting and the result.

2 The main result

Let $(\mathcal{M}, J)$ be an almost complex manifold of dimension $2n+2$, where J is an almost complex structure. We will be interested in integral cycles of even dimension (say $2p$) with the property that almost all approximate tangents are positively oriented J-invariant $2p$-planes. Recall that the orientation on $\mathcal{M}$ is induced by J (see e.g. [18]). Such currents are called positive-(p,p) integral cycles. The cycle condition (absence of boundary) for a current C means that it holds, for any compactly supported $(2p-1)$-form α, $(\partial C)(\alpha) := C(d\alpha) = 0$.

In the case of an integrable J a complete picture of the positive-(p,p) integral cycles is known, [16]. In fact, positivity can be dropped, [15], [1]. In these works it is shown that such cycles are a sum of holomorphic subvarieties, each one counted with an integer multiplicity (a positive integer multiplicity if we keep the positivity assumption).

Although such a complete picture might be unattainable for a non-integrable J, the goal is to first formulate (not always easy) and the prove to the extent possible results which generalize the integrable case. The uniqueness of tangent cones is a first and difficult step (where the formulation is easy). Remark that, at the moment, even in the integrable case there is no direct proof of the uniqueness of tangent cones: such a result can only be deduced starting from the more difficult characterization obtained in [16], [15], [1]. The proof given in the present work can be considerably shortened and simplified to provide a direct and new proof of the uniqueness of tangent cones in the integrable case.

There are geometric motivations for the extension of the aforementioned regularity results from the complex to the almost complex setting: an example, as explained in [21], is the understanding of the regularity properties of pseudo-holomorphic maps between almost complex manifolds. As described in [22], the conjectured bound on the size of the singular set of such maps would lead to the characterization of stable-bundle, almost complex structures over almost Kähler manifolds.

Positive-(p,p) integral cycles also appear as blow-up sets for some sequences of stationary harmonic maps or Yang-Mills fields (see examples in [28], [29]) and their regularity is essential for the compactification of the corresponding moduli spaces.

A further and considerably more difficult problem related to these geometric issues would be understanding the structure of the set of points of strictly positive density for a positive-(p,p) *normal* cycle (see examples in [29], [4], [5]). It is conjectured that a result analogous to the one obtained for the integrable case in [25] should hold in the non-integrable case. Concentrating on integral cycles seems to be a reasonable step before addressing the generic case of normal cycles.

The generalizations from the integrable to the non-integrable case turn out to extremely difficult and require to a large extent new ideas and techniques (as it is the case in [27], [21], [22] or in the present work); indeed, in the integrable case complex analysis can be directly brought to bear, but such tools are no longer avaliable in the non-integrable case.

Let us now analyze how the notion of being positive-(p,p) is intimately related to semi-calibrations. Given an almost complex manifold $(\mathcal{M}, J)$, it is locally always possible to find a non-degenerate differential form ω of degree 2 compatible with J. The compatibility relies in the fact that $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on $\mathcal{M}$. The tensor $h = g - i\omega$ is called a Hermitian metric on $(\mathcal{M}, J)$. If $d\omega = 0$ then we have a symplectic form, but in general closedness cannot be expected in dimension higher than 4: an example was exhibited on S^6 in [7]. The triple $(\mathcal{M}, J, g)$ is an almost Hermitian manifold; when the associated form ω is closed, we get an almost Kähler manifold. The word "almost" refers to the fact that J can be non-integrable.

The form ω on $(\mathcal{M}, J)$ has pointwise unit comass for the associated metric $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$. The same holds for the differential form $\Omega := \frac{1}{p!}\omega^p$, where p is any fixed integer $p \in \{1, 2, ...n\}$. This is nothing else but Wirtinger's inequality, [31]. We have therefore that Ω is a semi-calibration on $(\mathcal{M}, g)$. If ω is closed, then so is Ω and we get a calibration. Recall that the family $\mathcal{G}(\Omega)$ of *$2p$-planes calibrated by Ω* is

$$\mathcal{G}(\Omega) := \cup_{x\in\mathcal{M}}\, \mathcal{G}_x := \cup_{x\in\mathcal{M}}\{\xi_x \in G_{2p}(x, T_x\,\mathcal{M}) : \langle \Omega_x, \xi_x\rangle = 1\}.$$

By Wirtinger's theorem [31] $\mathcal{G}_x$ is made exactly of the $2p$-dimensional J_x-complex subspaces of $T_x\,\mathcal{M}$. For this reason the $2p$-planes in $\mathcal{G}(\Omega)$ are exactly the positive-(p,p) vectors and thus a **positive-(p,p)** integral cycle is **semi-calibrated** by Ω.

Remark 2.1. It is worthwile stressing the fact that the property of being a complex subspace of $(T_x\,\mathcal{M}, J_x)$ is not affected by choosing different couples (g_x, ω_x) and (g'_x, ω'_x) compatible with J_x in $T_x\,\mathcal{M}$. Therefore a positive-(p,p) integral cycle is semi-calibrated by $\Omega := \frac{1}{p!}\omega^p$ in $(\mathcal{M}, g)$ for any choice of hermitian metric $h = g - i\omega$ compatible with J. This flexibility in the choice of (ω, g) on $\mathcal{M}$ will be of key importance for our proof.

The issues we will be dealing with, namely tangent cones, are local: we will be only interested in the asymptotic behaviour of currents around a point, so we can assume to work in a chart rather than on a manifold. Ω-positive normal cycles in $\mathbb{R}^{2n+2}$ satisfy the following important *almost monotonicity property* for the mass-ratio at any point x_0.

Proposition 1 (**Almost-monotonicity of the mass ratio, e.g. [20]**). *Let $\mathbb{R}^{2n+2}$ be endowed with a Riemannian metric g and a non-degenerate two-form ω of unit comass. Denote by Ω the semi-calibration $\frac{1}{p!}\omega^p$. Let the $2p$-dimensional normal cycle T be Ω-positive and let x_0 be an arbitrary point. Denote by $B_r(x_0)$ the geodesic ball around x_0 of radius r.*

For an arbitrarily chosen point x_0, the mass ratio $\frac{M(T \llcorner B_r(x_0))}{r^{2p}}$ is an almost-increasing function in r, i.e. $\dfrac{M(T \llcorner B_r(x_0))}{r^{2p}} = R(r) + O(r)$ for a function R which is monotonically non-increasing as $r \downarrow 0$ and a function $O(r)$ which is infinitesimal.

This is proved in [20], Proposition 1. It is important to notice that the same proof works if we assume the semi-calibration to be just Lipschitz continuous rather than C^1, see the appendix of [4]. The perturbation term $O(r)$ is bounded, <u>independently</u> of x_0, by $C \cdot L \cdot r$, where C is a dimensional constant and L is the Lipschitz constant of the semi-calibration. As mentioned in the introduction, such an almost-monotonicity formula guarantees the existence of tangent cones, but not the uniqueness.

In this work we prove:

Theorem 2.1. *Suppose that T is a positive-(p,p) integral cycle in an almost complex manifold. Then for any point x_0 the tangent cone to T at x_0 is unique.*

Recall once again Remark 2.1: there is freedom on the choice of hermitian metric compatible with J. From the discussion in this section the theorem can be equivalently formulated in the following way:

Theorem 2.2. *Let $(\mathcal{M}, g, J)$ be a $(2n+2)$-dimensional almost Hermitian manifold and let ω be the associated semi-calibration, i.e. $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. Denote by Ω the semi-calibration $\Omega := \frac{1}{p!}\omega^p$, for a fixed $p \in \{1, 2, ..., n\}$. Let T be a an integral $2p$-cycle semi-calibrated by Ω.*

Then for any x_0 the tangent cone to T at x_0 is unique.

At first sight the statements just given might seem very special, in that they are conditioned to the fact that we are in an almost complex manifold. This turns out however to be a wrong impression, as we are about to describe. In [6] we show that, roughly speaking, given any semi-calibrated 2-current in a Riemannian manifold, we can locally cook up an almost complex structure that makes the current pseudo holomorphic. This allows, among other applications, to give a way more general setting in which the argument for the uniqueness of tangent cones that we give in the present paper can be carried out. For the sake of clarity we now proceed, borrowing from [6], to explain the geometric ideas behind this generalization: this will lead us to Theorem 2.3 below.

Let ω be a two-form on a $(2n+2)$-dimensional Riemannian manifold $(\mathcal{M}, g)$, where g denotes the metric, and assume that ω has unit comass and that it is non-degenerate (i.e. $\omega^{n+1} \neq 0$ everywhere).

A standard construction (see e.g. [18]) provides the existence of an almost complex structure J compatible with ω and consequently of a Riemannian metric g_J that is uniquely defined by ω and by the almost complex structure. The construction is done pointwise on $\mathcal{M}$ and we briefly recall it here. We will be working in the tangent space to $\mathcal{M}$ at an arbitrary point x. Define an endomorphism $A : T_x \mathcal{M} \to T_x \mathcal{M}$ by setting

$$\omega(v, w) = g(Av, w) \text{ for any } v, w \in T_x \mathcal{M} .$$

Then A is skew-adjoint with respect to g, i.e. $g(Av, w) = -g(v, Aw)$. The g-positive definite endomorphism $-A^2 = A^*A = P$ is diagonalizable and there exists a square root, i.e. a positive definite $Q : T_x \mathcal{M} \to T_x \mathcal{M}$ such that $Q^2 = P$. It follows that $(Q^{-1}A)^2 = -Id$. Doing this for any point x we thus obtain that $J := Q^{-1}A$ is an almost complex structure on $\mathcal{M}$ compatible with ω in the sense that we can define the Riemannian metric $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ on $\mathcal{M}$. Remark that ω is a semi-calibration also with respect to the metric g_J, since we are now in the situation of an almost Hermitian manifold.

We have not yet used the fact that ω has unit comass with respect to the metric g. By exploiting this we will draw, in the next proposition, a very important piece of information on the relation between the two metrics g and g_J.

Proposition 2. *(i) Let $v, w \in T_x \mathcal{M}$ be such that the 2-plane $v \wedge w$ is calibrated by $\omega(x)$ in $(T_x \mathcal{M}, g)$. Then g agrees with the metric g_J constructed above when they are restricted to the 2-plane $v \wedge w$. In particular $v \wedge w$ belongs to the family of $\omega(x)$-calibrated planes also in $(T_x \mathcal{M}, g_J)$.*

(ii) $g_J \leq g$, i.e. the bilinear form $g - g_J$ is positive semi-definite.

proof of Proposition 2. (i) The first important observation is the following. Let $t \in T_x \mathcal{M}$ be g-orthogonal to the 2-plane $v \wedge w$; then $\omega(v, t) = \omega(w, t) = 0$. This can be seen by noticing that, as t varies among all possible vectors orthogonal to $v \wedge w$, the 2-planes of the form $v \wedge t$ and $w \wedge t$ span the tangent space to the Grassmannian $G(2, T_x \mathcal{M})$ at the point $v \wedge w$. From the fact that ω restricted to $G(2, T_x \mathcal{M})$ realizes its maximum at $v \wedge w$ we have that $\omega(v, t) = \omega(w, t) = 0$, as desired. This fact is known as the "first cousin principle", see e.g. [14].

With this in mind it follows, by the definition of A, that $g(Av, t) = g(Aw, t) = 0$ for any $t \in T_x \mathcal{M}$ that is g-orthogonal to $v \wedge w$, therefore A restricts to an endomorphism of the 2-plane $v \wedge w$. Now from the fact that $v \wedge Av$ is calibrated we will infer that g and g_J agree on this 2-plane. Indeed, the fact that $v \wedge Av$ is calibrated by ω in $(T_x\mathcal{M}, g)$ can be expressed by the equality

$$g(v,v)g(Av,Av) - g(v,Av)^2 = \omega(v,Av)^2$$

and by construction $g(v, Av) = \omega(v, v) = 0$. Recalling that (see above for the construction of g_J) the g-adjoint of A is $-A$ we rewrite the calibrating condition as

$$g(v,v) = g(-A^2v, v).$$

It follows that the only possible eigenvalue for the endomorphism $-A^2$ restricted to the 2-plane $v \wedge w$ is 1. Therefore $A^2 = -Id$ (the Q constructed above is the identity on the 2-plane $v \wedge w$) and $J = A$ on $v \wedge w$. In particular $g = g_J$ on this 2-plane.

(ii) The fact that the comass of ω with respect to g is 1 yields that for any vector v

$$g(v,v)g(Av,Av) - g(v,Av)^2 \geq \omega(v,Av)^2,$$

so arguing as in part *(i)* we obtain that any eigenvalue of Q^2 (and thus of Q) must belong to the interval $]0,1]$. Remark that if v is an eigenvector of Q^2 then Av is an eigenvector for the same eigenvalue. Since $g(v, Av) = 0$ we can choose a g-orthogonal eigenbasis of Q^2 (and thus of Q) of the form $\{v_1, Av_1, ..., v_{n+1}, Av_{n+1}\}$. For any egenvector v of Q it holds

$$g_J(Av,Av) = \omega(Av, Q^{-1}A^2v) = \omega(Av, -Qv) =$$
$$= \omega(Qv, Av) \leq \omega(v, Av) = g(Av, Av).$$

By letting v vary in the set of vectors forming the chosen eigenbasis we obtain the result. □

In view of this proposition, given a non-degenerate semi-calibration ω on a $(2n+2)$-dimensional Riemannian manifold $(\mathcal{M}, g)$, we can change the metric in a coherent way, in the sense that any cycle T that is semi-calibrated by ω in $(\mathcal{M}, g)$ will be also semi-calibrated by ω in $(\mathcal{M}, g_J)$. The advantage of the new metric relies in the fact that there exists an almost complex structure J that satisfies the compatibility conditions with ω and g_J. The classical Wirtinger's inequality tells us that $\Omega := \frac{1}{p!}\omega^p$ (for a fixed $p \in \{1, 2, ..., n\}$) is a semi-calibration in $(\mathcal{M}, g_J)$. More precisely, using our Proposition 2 together with Theorem 6.11 of [13] we can see that, given a non-degenerate semi-calibration ω on a $(2n+2)$-dimensional Riemannian manifold $(\mathcal{M}, g)$, the $2p$-form $\Omega := \frac{1}{p!}\omega^p$ is a semi-calibration both in $(\mathcal{M}, g)$ and in $(\mathcal{M}, g_J)$ and the set of calibrated $2p$-planes in $(\mathcal{M}, g)$ is contained in the set of calibrated $2p$-planes in $(\mathcal{M}, g_J)$. We thus get, from Theorem 2.2, the following

Theorem 2.3. *Let $(\mathcal{M}, g)$ be a $(2n+2)$-dimensional Riemannian manifold, endowed with a non-degenerate two-form ω of comass 1. Denote by Ω the semi-calibration $\Omega := \frac{1}{p!}\omega^p$, for a fixed $p \in \{1, 2, ..., n\}$. Let T be a an integral $2p$-cycle semi-calibrated by Ω.*

Then for any x_0 the tangent cone to T at x_0 is unique.

When[5] $p = 1$ then Theorem 2.3 is the special case of [20] in the case when the semi-calibration is non-degenerate. Always for $p = 1$, if we moreover assume $d\omega = 0$, then the result follows from [30].

In the rest of this work we will prove the uniqueness of tangent cones in the situation described in Theorem 2.2, where we have the three structures ω, J and g. With a suitable choice of coordinates we can identify the tangent space $T_{x_0}\mathcal{M}$, endowed with the complex structure J_{x_0}, with $\mathbb{C}^{n+1}$: then every tangent cone T_∞ to T at x_0 is a positive-(p,p) cone in $\mathbb{C}^{n+1}$: such a cone is uniquely defined by a holomorphic $(p-1, p-1)$ integral cycle L_∞ in $\mathbb{CP}^n$.

Using the regularity theory for holomorphic integral cycles ([16], [15], [1]) we can deduce that L_∞ is in fact the sum of a finite number of holomorphic algebraic varieties[6], each one taken with a constant integer multiplicity, but we will not need this result.

We will prove first the following

Lemma 2.1. *Let $(\mathcal{M}, g, J)$ be a $(2n+2)$-dimensional almost Hermitian manifold and let ω be the associated semi-calibration, i.e. $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. Denote by Ω the semi-calibration $\Omega := \frac{1}{p!}\omega^p$, for a fixed $p \in \{1, 2, ..., n\}$. Let T be a Ω-semi-calibrated integral cycle and x_0 an arbitrary point. Then all tangent cones to T at x_0 have a uniquely determined support.*

Once this lemma is achieved, the uniqueness of tangent cones (i.e. Theorem 2.2) follows with a few extra considerations (without making use of the results in [16], [15], [1]) developed in Section 4, namely: **(i)** the space of tangent cones to T at x_0 is closed and connected in the space of $2p$-integral cycles, **(ii)** the density is continuous under convergence of calibrated integral cycles sharing the same support.

As for Lemma 2.1, the key idea for its proof is the analysis implementation, in the almost complex setting in which we are working, of the classical algebraic blow up. This was already used in [4] for positive-$(1,1)$ normal cycles and is here generalized to higher dimensional pseudo-holomorphic currents. The technique clearly shows that the uniqueness in Theorem 2.2 holds

[5] Remark that the generalization obtained in [6] is even wider in applicability, in that we can actually drop the assumptions of even dimensionality of the ambient manifold and of non-degeneracy of the semi-calibration. We stick however here to Theorem 2.3 so that we can keep the digression short, but still effective in order to show the flexibility of Theorems 2.1 and 2.2.

[6] We are slighlty abusing language here: these are algebraic varieties that are holomorphic away from their possible singular set.

just for "density reasons" (recall that the uniqueness can fail when we look at non-rectifiable currents, where the density is allowed to take any values ≥ 0, see [17], [4] and [5]).

3 Strategy and tools for the proof of the theorem

The first important remarks are contained in the following

Lemma 3.1. *Let T be as in Theorem 2.2 and be $\mathcal{T}$ the support of T. Assume that there exists a sequence of points $x_m \in \mathcal{T}$ with $x_m \to x_0$ and $x_m \neq x_0$ such that $\frac{x_m - x_0}{|x_m - x_0|} \to y \in S^{2n+1}$. Then there exists a tangent cone to T at x_0, say T_∞, such that the point y belongs to the support of T_∞.*

On the other hand, if $y \in S^{2n+1}$ belongs to the support of a tangent cone T_∞ to T at x_0, then there exists a sequence $x_m \to x_0$ (with $x_m \neq x_0$) of points x_m in the support of T such that $\frac{x_m - x_0}{|x_m - x_0|} \to y$.

Remark 3.1. Let $C_k \rightharpoonup C_\infty$ be a sequence of ϕ_k-semi-calibrated integral cycles ($k \in \mathbb{N} \cup \{\infty\}$), where ϕ_k are semi-calibrations with respect to the metrics g_k, and assume that the ϕ_k converge uniformly to ϕ_∞, g_k converge uniformly to g_∞ and the C_k's have equibounded masses. Then $M(C_k \llcorner B) \to M(C_\infty \llcorner B)$ for any open set B. This follows since computing the mass for a semicalibrated current amounts to testing the current on the semi-calibration, so the convergence of the masses follows from the definition of weak*-convergence of currents.

Remark 3.2. Recall that, as a consequence of monotonicity, a point belongs to the support of a semi-calibrated integral cycle if and only if its density is ≥ 1.

proof of Lemma 3.1. The first statement follows by choosing the sequence of radii $r_m := |x_m - x_0|$ and by looking at the sequence T_{x_0, r_m}. Up to a subsequence we may assume that $T_{x_0, r_m} \rightharpoonup T_\infty$. Each x_m is of density ≥ 1 for T by assumption and, for any m, the point $\frac{x_m - x_0}{|x_m - x_0|}$ is of density ≥ 1 for T_{x_0, r_m}. Since $\frac{x_m - x_0}{|x_m - x_0|} \to y$, analogously to Remark 3.1 we can get

$$M\left(T_{x_0, r_m} \llcorner B_R\left(\frac{x_m - x_0}{|x_m - x_0|}\right)\right) \to M(T_\infty \llcorner B_R(y))$$

for any $R > 0$. By the almost monotonicity formula $M(T_\infty \llcorner B_R(y)) \geq \alpha_{2p} R^{2p}$ and so y is a point of density ≥ 1 for T_∞.

Let now $y \in S^{2n+1}$. If there exists no sequence $x_m \neq x_0$ such that $x_m \in \mathcal{T}$, $x_m \to x_0$ and $\frac{x_m - x_0}{|x_m - x_0|} \to y$, then we can assume to have a ball $B_a^{2n+1}(y) \subset S^{2n+1}$ such that the cone $0 \sharp B_a^{2n+1}(y)$ is disjoint from $\mathcal{T} \cap B_R^{2n+2}(0)$, for some small $R > 0$. But then, for any dilation $T_{x_0, r}$ with $r < R$ we have $M(T_{x_0, r} \llcorner B_a^{2n+2}(y)) = 0$. Since the mass passes to the limit for convergence of semi-calibrated cycles (Remark 3.1), we deduce that y is a point of density 0 for any limit of the family $T_{x_0,r}$, therefore it cannot appear as a point in the support of any tangent cone. $\square$

In order to achieve Lemma 2.1, it suffices, thanks to Lemma 3.1, to analyze limits of $\frac{x_m - x_0}{|x_m - x_0|} \to y$ for $x_m \in \mathcal{T}$, $x_m \to x_0$. More precisely, recalling that each tangent cone is a holomorphic (p,p)-cone, if $y \in S^{2n+1}$ belongs to the support of a tangent cone T_∞, then every point in the Hopf fiber $\{e^{i\theta} y\}_{\theta \in [0,2\pi)}$ is also a point whose density for T_∞ equals that of y. In other words, if y is in the support of T_∞, so is the whole fiber $\{e^{i\theta} y\}_{\theta \in [0,2\pi)}$. Denote by $H : S^{2n+1} \to \mathbb{CP}^n$ the standard Hopf projection. Then, in order to prove Lemma 2.1, we actually need to show the following

Proposition 3. *Let T be a positive-(p,p) integral cycle. Let $\{x_m\}$ be a sequence of points such that $x_m \in \mathcal{T}$ with $x_m \to x_0$, $x_m \neq x_0$ and $H\left(\frac{x_m - x_0}{|x_m - x_0|}\right) \to y \in \mathbb{CP}^n$. Then the support of any tangent cone to T at x_0 must contain the Hopf circle $H^{-1}(y)$.*

This proposition will be proved by employing a "*pseudo holomorphic blow up*" of the semi-calibrated current T at x_0, i.e. a procedure inspired by the classical algebraic blow up and adapted to the almost complex setting. We now shortly recall the notations and the construction, which is developed in more detail in [4].

Since tangent cones to T at a point x_0 are a local issue, we can assume straight from the beginning to work in the unit geodesic ball, in normal coordinates centered at x_0; for this purpose it is enough to start with the current T already dilated enough around x_0. Always up to a dilation, without loss of generality we can actually start with the following situation.

T is a Ω-positive normal cycle in the ball $B_2^{2n+2}(0)$, the coordinates are normal with respect to the origin, J is the standard complex structure J_0 at the origin, ω is the standard symplectic form ω_0 at the origin, $\|\omega - \omega_0\|_{C^{2,\nu}(B_2^{2n+2})}$, $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})}$ and $\|\Omega - \Omega_0\|_{C^{2,\nu}(B_2^{2n+2})}$ are small enough, where $\Omega = \frac{1}{p!}\omega^p$ and $\Omega_0 = \frac{1}{p!}\omega_0^p$.

How to blow up the origin. We shall be using standard coordinates $(z_0, z_1, ..., z_n)$ in $B_2^{2n+2}(0) \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ and the following notations as in [4]:

$$\mathcal{S} := \{(z_0, z_1, ... z_n) \in B_1^{2n+2} \subset \mathbb{C}^{n+1} : |(z_1, ..., z_n)| < |z_0|\},$$

$$\mathcal{V} \subset \mathbb{CP}^n, \ \mathcal{V} := \{[z_0, z_1, ..., z_n] : |(z_1, ..., z_n)| < |z_0|\}.$$

Using homogeneous coordinates on $\mathbb{CP}^n$, for $X = [Z_1, ..., Z_{n+1}] \in \mathcal{V}$

D^X is the "straight" 2-plane made of all points$\{\zeta(Z_1, ...Z_{n+1}) : \zeta \in \mathbb{C}\}$.

As shown in Section 3 of [4], by constructing (via a fixed point theorem) a *pseudo-holomorphic polar foliation* we can produce an appropriate $C^{2,\nu}$-diffeomorphism

$$\Psi : \mathcal{S} \to \Psi(\mathcal{S}) \approx \mathcal{S}, \tag{2}$$

which is close to the identity on $\mathcal{S}$, and which (by pulling-back[7] the problem via Ψ) allows us to make an extra assumption on the almost complex structure J: namely the "straight 2-planes" D^X are J-pseudo holomorphic for all $X \in \mathcal{V}$. Figure 1 in [4] visually explains the behaviour of Ψ.

With this extra assumption on J, we can proceed to blow up the origin of $\mathbb{C}^{n+1}$ as follows.

Reminder: *algebraic blow up* (from *symplectic geometry* and *algebraic geometry*, see e.g. [18]). Define $\widetilde{\mathbb{C}}^{n+1}$ to be the submanifold of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ made of the pairs $(\ell, (z_0, ...z_n))$ such that $(z_0, ...z_n) \in \ell$.

Denote by I_0 the complex structure that $\widetilde{\mathbb{C}}^{n+1}$ inherits from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let $\Phi : \widetilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ be the projection map $(\ell, (z_0, ...z_n)) \to (z_0, ...z_n)$. This map is holomorphic for the standard complex structures J_0 on $\mathbb{C}^{n+1}$ and I_0 on $\widetilde{\mathbb{C}}^{n+1}$ and is a diffeomorphism between $\widetilde{\mathbb{C}}^{n+1} \setminus (\mathbb{CP}^n \times \{0\})$ and $\mathbb{C}^{n+1} \setminus \{0\}$. Moreover the inverse image of $\{0\}$ is $\mathbb{CP}^n \times \{0\}$.

We will endow $\widetilde{\mathbb{C}}^{n+1}$ with other almost complex structures, different from I_0, so $\widetilde{\mathbb{C}}^{n+1}$ should be thought of just as an oriented manifold and the structure on it will be specified in every instance. The transformation Φ^{-1} (called *proper transform*) sends the point $0 \neq (z_0, ...z_n) \in \mathbb{C}^{n+1}$ to the point $([z_0, ...z_n], (z_0, ...z_n)) \in \widetilde{\mathbb{C}}^{n+1} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$.

We will keep using the same letters Φ and Φ^{-1} to denote the same maps restricted to the sets

$$\mathcal{S} \subset B^{2n+2} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \text{ and } \mathcal{A} := \Phi^{-1}(\mathcal{S}) \subset \widetilde{\mathbb{C}}^{n+1},$$

(i.e. $\mathcal{A}$ is the inverse image of $\mathcal{S}$ via the projection Φ) also when we look at these spaces just as oriented manifolds (not complex ones). We will make use of the notation

$$\mathcal{S}^\rho := \mathcal{S} \cap B^{2n+2}_\rho \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \text{ and } \mathcal{A}^\rho := \Phi^{-1}(\mathcal{S}^\rho) \subset \widetilde{\mathbb{C}}^{n+1},$$

[7] The idea is to consider the pull-backs via the map Ψ of the current $T \llcorner \Psi(\mathcal{S})$, of the almost complex structure J and of the metric g and then work on those new objects in $\mathcal{S}$ as if they were the ones we started from. The fact that Ψ is by construction pseudo holomorphic w.r.t. the structures $\Psi^* J$ and J yields that the pull-back current in $\mathcal{S}$ is also positive-(p,p) w.r.t. $\Psi^* J$. Moreover the "straight 2-planes" D^X in $\mathcal{S}$ are pseudo holomorphic w.r.t. $\Psi^* J$. In [4] it is shown that everything is well-defined.

i.e. $\mathcal{A}^\rho$ is the inverse image of $\mathcal{S}^\rho$ via the projection Φ.

Let g_0 denote the standard metric[8] on $\mathcal{A}$ as a subset of $\widetilde{\mathbb{C}}^{n+1} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ and ϑ_0 be the standard symplectic form on $\mathcal{A}$, uniquely defined by $\vartheta_0(\cdot,\cdot) := g_0(\cdot, -I_0\cdot)$.

Define on $\mathcal{A} \setminus (\mathbb{CP}^n \times \{0\})$:

- the almost complex structure $I := \Phi^* J$, i.e. $I(\cdot) := \left(\Phi^{-1}\right)_* J\Phi_*(\cdot)$,
- the metric $\mathbf{g}(\cdot,\cdot) := \frac{1}{2}\left(g_0(\cdot,\cdot) + g_0(I\cdot, I\cdot)\right)$,
- the non-degenerate two-form $\vartheta(\cdot,\cdot) := \mathbf{g}(\cdot, -I\cdot)$.

The triple $(I, \mathbf{g}, \vartheta)$ makes $\mathcal{A} \setminus (\mathbb{CP}^n \times \{0\})$ an almost Hermitian manifold and from [4] we have

Lemma 3.2. *The triple $(I, \mathbf{g}, \vartheta)$, extended to $\mathcal{A}$ by setting it to be equal to (I_0, g_0, ϑ_0) on $\mathcal{V} \times \{0\}$, is Lipschitz continuous on $\mathcal{A}$ and fulfils*

$$|I - I_0|(\cdot) \leq c\,dist_{g_0}(\cdot, \mathbb{CP}^n \times \{0\}),$$

$$|\mathbf{g} - g_0|(\cdot) \leq c\,dist_{g_0}(\cdot, \mathbb{CP}^n \times \{0\}),$$

$$|\vartheta - \vartheta_0|(\cdot) \leq c\,dist_{g_0}(\cdot, \mathbb{CP}^n \times \{0\}),$$

for some constant $c > 0$, which is $o(1)$ of $|J - J_0|$.

Remark 3.3. The choice of the J-pseudo holomorphic polar foliation as building block needed to implement the "pseudo holomorphic blow up" aimed exactly to obtain the Lipschitz extension across $\mathbb{CP}^n \times \{0\}$, which could fail on the vertical vectors (see the proof in [4]) if we were working with an arbitrary polar foliation (for example if we used the one made with flat J_0-holomorphic 2-planes).

Set $\Theta := \frac{1}{p!}\vartheta^p$ on $\mathcal{A}$. The aim is now to translate our original problem to the new space $(\mathcal{A}, I, \mathbf{g}, \vartheta)$. For any $\rho > 0$ we can take the proper transform of $T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)$, since $\left(\Phi^{-1}\right)$ is a diffeomorphism away from the origin:

$$P_\rho := \left(\Phi^{-1}\right)_* \left(T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)\right).$$

The current P_ρ is clearly positive-(p,p) in $(\mathcal{A}, I, \mathbf{g}, \vartheta)$. What happens when $\rho \to 0$? Here is the answer.

[8] The standard metric on $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ is the product of the Fubini-Study metric on $\mathbb{CP}^n$ and the flat metric on $\mathbb{C}^{n+1}$.

Lemma 3.3. *(i) The current* $P := \lim_{\rho \to 0} P_\rho = \lim_{\rho\to 0} \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$ *is well-defined as the limit of currents of equibounded mass. The mass of* P *(both with respect to* $\mathbf{g}$ *and to* g_0*) is bounded by a dimensional constant* C *times the mass of* T*.*

(ii) *P it is an integral cycle in $\mathcal{A}$ and it is semi-calibrated by Θ.*

The proof is analog to the one in [4], but we need to take care of the fact that the dimension of the current is higher. With the same notations as in [4], for any ρ consider the dilation $\lambda_\rho(\cdot) := \frac{\cdot}{\rho}$, sending B_ρ to B_1, and the map

$$\Lambda_\rho : \mathcal{A}^\rho \to \mathcal{A}, \;\; \Lambda_\rho := \Phi^{-1} \circ \lambda_\rho \circ \Phi, \tag{3}$$

which in the coordinates of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (the ambient space in which $\mathcal{A}$ is embedded) reads $\Lambda_\rho(\ell, z) = \left(\ell, \frac{z}{\rho}\right)$.

proof of Lemma 3.3 (i). *1st step: it is enough to uniformly bound the masses of* P_ρ. Each $P_\rho = \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$ is Θ-positive by construction, so $M(P_\rho) = P_\rho(\Theta)$, where the mass is computed here with respect to $\mathbf{g}$. The currents P_ρ and $P_{\rho'}$ for $\rho > \rho'$ coincide on $\mathcal{A} \setminus \overline{\mathcal{A}^\rho}$. Therefore, in order to study the limit as $\rho \to 0$, it is enough to have a sequence $\rho_k \to 0$ and a current P such that $P_{\rho_k} \rightharpoonup P$: this guarantees that actually $P_\rho \rightharpoonup P$ as $\rho \to 0$. Therefore we only need to prove a uniform bound (independent of ρ) for the masses $M(P_\rho)$: then the compactness theorem and the remark just made will guarantee the existence of a unique limit for P_ρ as $\rho \to 0$.

2nd step: uniform bound on the masses. We use in $\mathcal{A}$ standard coordinates inherited from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, i.e. we have $2n$ horizontal variables (from $\mathbb{CP}^n$) and $2n+2$ vertical variables. We want to estimate $M(P_\rho) = P_\rho(\Theta) = P_\rho(\Theta_0) + P_\rho(\Theta - \Theta_0)$, where $\Theta_0 := \frac{1}{p!}\vartheta_0^p$ on $\mathcal{A}$. From Lemma 3.2 we get that $|\Theta - \Theta_0|(p) < c \text{ dist}_{g_0}(p, \mathbb{CP}^n \times \{0\})$ (we keep denoting the constant by c, although it is generally different that the one in Lemma 3.2; what is important is that it is still controlled by $|J - J_0|$).

Recall that $\Theta_0 := \frac{1}{p!}\vartheta_0^p$. The domain $\mathcal{S}$ is a submanifold of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Therefore, using coordinates from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, the standard form ϑ_0 is $\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}}$: here $\vartheta_{\mathbb{CP}^n}$ is the standard symplectic form on $\mathbb{CP}^n$ extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (so independent of the $2n+2$ "vertical variables") and $\vartheta_{\mathbb{C}^{n+1}}$ is the symplectic two-form on $\mathbb{C}^{n+1}$, extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (so independent of the $2n$ "horizontal variables").

Taking the p-th wedge power we get

$$\Theta_0 := \frac{1}{p!}\vartheta_0^p = \frac{1}{p!}(\vartheta_{\mathbb{CP}^n})^p + \sum_{m=1}^{p} \frac{1}{(p-m)!m!}(\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m. \tag{4}$$

Let us begin with the first term on the right-hand side of (4). It is proved in [4] (proof of Lemma 4.2) that the two-form $\left(\Phi^{-1}\right)^* \theta_{\mathbb{CP}^n}$ on $\mathcal{S}$ coincides with the two-form $\frac{(\omega_0)_t}{|z|^2}$, where ω_0 is the standard Kähler form of $\mathbb{C}^{n+1}$ restricted to $\mathcal{S}$ and the tangential part of ω_0 is by definition $(\omega_0)_t := \frac{\partial}{\partial r} \lrcorner\, (dr \wedge \omega_0)$. As a consequence we have the equality

$$\frac{1}{p!}\left(\left(\Phi^{-1}\right)^* \theta_{\mathbb{CP}^n}\right)^p = \frac{1}{p!}\frac{((\omega_0)_t)^p}{|z|^{2p}}. \tag{5}$$

Using the decomposition $\omega_0 = (\omega_0)_t + dr \wedge (\omega_0)_n$ into its tangential and normal parts (see [12]) we can see that

$$dr \wedge \Omega_0 = dr \wedge ((\omega_0)_t + dr \wedge (\omega_0)_n)^p = dr \wedge ((\omega_0)_t)^p\,,$$

which readily gives

$$\frac{(\Omega_0)_t}{|z|^{2p}} = \frac{1}{p!}\frac{((\omega_0)_t)^p}{|z|^{2p}}. \tag{6}$$

In[9] the same way $\omega_t := \frac{\partial}{\partial r} \lrcorner\, (dr \wedge \omega)$ so similarly as above the decomposition into tangential and normal parts $\omega = \omega_t + dr \wedge \omega_n$ yields

$$dr \wedge \Omega = dr \wedge (\omega_t + dr \wedge \omega_n)^p = dr \wedge (\omega_t)^p\,,$$

so that

$$|\Omega_t - (\Omega_0)_t|(z) \leq C|z|, \;\; \text{where } C = C(p, n, \|\nabla\omega\|_\infty). \tag{7}$$

Recall now the almost monotonicity formula (see [13], [20]) for the Ω-semicalibrated cycle T:

$$(T \llcorner B_1)(\Omega) - \nu(0) = \int_{B_1} \left\langle \vec{T}, \frac{\Omega_t}{|z|^{2p}} \right\rangle d\|T\|. \tag{8}$$

In view of the fact that the integrand on the right hand side is everywhere non-negative, we have the finiteness of the integrand on the right hand side also when we restrict to a sector, i.e.

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[9]Thanks to equalities (5) and (6) we get that the two-form $\frac{1}{p!}\left(\left(\Phi^{-1}\right)^* \theta_{\mathbb{CP}^n}\right)^p$ is exactly the one appearing on the right hand side of the monotonicity formula in the case that T is a Ω_0-calibrated cycle ([13]):

$$(T \llcorner B_1)(\Omega_0) - \nu(0) = \int_{B_1} \left\langle \vec{T}, \frac{(\Omega_0)_t}{|z|^{2p}} \right\rangle d\|T\|.$$

The finiteness of the (improper) integral on the right-hand side readily gives that the action of P_ρ on $\frac{1}{p!}(\theta_{\mathbb{CP}^n})^p$ is finite independently of ρ. In order to face the case under study, where T is semi-calibrated by Ω, we need a perturbation of this argument.

$$\int_{\mathcal{S}\setminus B_\rho(0)} \left\langle \vec{T}, \frac{\Omega_t}{|z|^{2p}} \right\rangle d\|T\| \tag{9}$$

is the integration of a non-negative quantity and the integral is bounded from above independently of ρ. Split this last integral as

$$\int_{\mathcal{S}\setminus B_\rho(0)} \left\langle \vec{T}, \frac{(\Omega_0)_t}{|z|^{2p}} \right\rangle d\|T\| + \int_{\mathcal{S}\setminus B_\rho(0)} \left\langle \vec{T}, \frac{\Omega_t - (\Omega_0)_t}{|z|^{2p}} \right\rangle d\|T\|. \tag{10}$$

We will now show that

$$\int_{\mathcal{S}\setminus B_\rho(0)} \frac{\left|\left\langle \vec{T}, \Omega_t - (\Omega_0)_t \right\rangle\right|}{|z|^{2p}} d\|T\|$$

is bounded from above by a constant independent of ρ. For this purpose use a dyadic decomposition $\mathcal{S} = \cup_{j=0}^{\infty} A_j$, where $A_j = \mathcal{S} \cap \left(B_{\frac{1}{2^j}} \setminus B_{\frac{1}{2^{j+1}}}\right)$. It holds (by the almost monotonicity formula for T) $M(T \llcorner A_j) \leq K \frac{1}{2^{2pj}}$. On the other hand, in the same annulus A_j, the integrand $\frac{\left|\left\langle \vec{T}, \Omega_t - (\Omega_0)_t \right\rangle\right|}{|z|^{2p}}$ is bounded, thanks to (7), by $C 2^{(2p-1)(j+1)}$ and therefore

$$\int_{\mathcal{S}\setminus B_\rho(0)} \frac{\left|\left\langle \vec{T}, \Omega_t - (\Omega_0)_t \right\rangle\right|}{|z|^{2p}} d\|T\| \leq C\, K 2^{2p-1} \sum_{j=0}^{\infty} \frac{1}{2^j} = 2^{2p} C\, K.$$

This last bound, recalling that (9) stays finite as $\rho \to 0$ proves that the first term

$$\int_{\mathcal{S}\setminus B_\rho(0)} \left\langle \vec{T}, \frac{(\Omega_0)_t}{|z|^{2p}} \right\rangle d\|T\|$$

in (10) is also bounded in modulus from above independently of ρ. Therefore we get that

$$\left| P_\rho \left(\frac{1}{p!} (\vartheta_{\mathbb{CP}^n})^p \right) \right| = \left| (T \llcorner (\mathcal{S} \setminus B_\rho)) \left(\frac{1}{p!} \left(\left(\Phi^{-1}\right)^* \theta_{\mathbb{CP}^n} \right)^p \right) \right|$$

is bounded independently of ρ, as desired.

We pass now to estimating the action of P_ρ on the other wedge products

$$\sum_{m=1}^{p} \frac{1}{m!(p-m)!} (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^{n+1}})^m$$

left from (4); the key observation is that $\left(\Phi^{-1}\right)^*(\vartheta_{\mathbb{C}^{n+1}})$ has unit comass, and therefore the forms $\left(\Phi^{-1}\right)^*\left((\vartheta_{\mathbb{CP}^{n+1}})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m\right)$ for $m \in \{1,...,p\}$ all have comasses bounded by $\frac{K}{|z|^{2p-2}}$, where $|z|$ is the distance from the origin and K is a universal constant. We then argue again using a dyadic decomposition for the estimate on $|P_\rho(\Theta - \Theta_0)|$, as follows.

Break up $\mathcal{S} = \cup_{j=0}^{\infty} A_j$, where $A_j = \mathcal{S} \cap \left(B_{\frac{1}{2^j}} \setminus B_{\frac{1}{2^{j+1}}}\right)$. It holds by almost monotonicity that $M(T \llcorner A_j) \leq K \frac{1}{2^{2pj}}$. On the other hand, in the same annulus A_j, the comass of the form $\left(\Phi^{-1}\right)^*\left(\sum_{m=1}^{p}(\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m\right)$ is $\leq K(p,n)\, 2^{2(p-1)(j+1)}$, for a constant $K(p,n)$ which only depends on the dimensions involved.

Therefore summing on all j's we can bound

$$\left|P_\rho\left(\sum_{m=1}^{p}(\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m\right)\right| =$$

$$= \left|(T \llcorner \mathcal{S})\left(\left(\Phi^{-1}\right)^*\left(\sum_{m=1}^{p}(\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m\right)\right)\right| \leq$$

$$\leq K(p,n)\sum_{j=0}^{\infty} 2^{2(p-1)(j+1)}\frac{1}{2^{2pj}} = K(p,n)\sum_{j=0}^{\infty} 2^{2p-2-2j} < \infty,$$

therefore $\left|P_\rho\left(\sum_{m=1}^{p}\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m\right)\right|$ is also equibounded independently of ρ.

So far we have thus shown that $|P_\rho(\Theta_0)|$ is bounded independently ρ. To conclude the proof of part (i) of Lemma 3.3 (see the beginning of step 2), we must still prove that $|P_\rho(\Theta - \Theta_0)|$ stays finite as $\rho \to 0$. Thanks to the Lipschitz control on $\vartheta - \vartheta_0$, which also yields $|\Theta - \Theta_0|(\cdot) \leq c\mathrm{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\})$, the form $\left(\Phi^{-1}\right)^*(\Theta - \Theta_0)$ in $\mathcal{S}$ has comass $\leq \frac{K}{|z|^{2p-1}}$, where $|z|$ is the distance from the origin. Arguing with a dyadic decomposition as done above, we find that also $|P_\rho(\Theta - \Theta_0)|$ is bounded independently of ρ.

We have thus obtained that $M(P_\rho)$ are uniformly bounded as $\rho \to 0$ and therefore there exists a current P in $\mathcal{A}$ such that $P_\rho \rightharpoonup P$.

$\square$

proof of Lemma 3.3 (ii). *1st step: choice of the sequence.* Since $P_{\rho_k} \rightharpoonup P$ for a well-defined current P, independently of the sequence $\rho_k \to 0$, we will now choose a particular $\{\rho_k\}$ to prove that $\partial P_{\rho_k} \rightharpoonup 0$. This will thus show that P has zero boundary.

Denote by $\langle T, |z| = r\rangle$ the slice of T with the sphere ∂B_r. Choose ρ_k so to ensure

- (i) $T_{\rho_k} \rightharpoonup T_\infty$ in $\mathcal{S}$ for a certain cone T_∞,
- (ii) $M(\langle T_{\rho_k}, |z| = 1\rangle)$ are equibounded by $4K$,
  or, equivalently, $M\left(\langle T, |z| = \rho_k\rangle\right) \leq 4K\rho_k^{2p-1}$.

This is just like step 1 of Lemma 4.3 in [4].

2nd step. Let us think of the currents P and $P_\rho := \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$ as currents in the open set $\mathcal{A}$ in the manifold $\widetilde{\mathbb{C}}^{n+1}$. Given the sequence $\rho_k \to 0$, we want to observe the boundaries ∂P_{ρ_k}. By step 1 we assume that $T_{0,\rho_k} \rightharpoonup T_\infty$ for a certain cone. Then the boundaries ∂P_{ρ_k} satisfy, as $k \to \infty$, by the definition (3) of Λ_{ρ_k}:

$$(\Lambda_{\rho_k})_*(\partial P_{\rho_k}) = -\left(\Phi^{-1}\right)_* \langle T_{0,\rho_k}, |z| = 1\rangle \rightharpoonup -\left(\Phi^{-1}\right)_* \langle T_\infty, |z| = 1\rangle. \tag{11}$$

Recall that we are viewing P_{ρ_k} as currents in the open set $\mathcal{A}$, so also $T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)$ should be thought of as a current in the open set $\mathcal{S}$: this is why the only boundary comes from the slice of T with $|z| = \rho_k$.

Moreover, always by the choice of the sequence ρ_k made in the 1st step, we have that $(\Lambda_{\rho_k})_*(\partial P_{\rho_k})$ have equibounded masses, since so do the $\partial(T_{0,\rho_k})$'s and Φ^{-1} is a diffeomorphism on ∂B_1.

The current T_∞ has a special form: it is a positive-(p,p)-cone, so the $(2p-1)$-current $\langle T_\infty, |z| = 1\rangle$ has an associated $(2p-1)$-vector field that always contains the direction tangent to the Hopf fibers[10] of S^{2n+1}.

3rd step. We want to show that P is a cycle in $\mathcal{A}$, i.e. that $\partial P_{\rho_k} \rightharpoonup 0$ as $k \to \infty$. The boundary in the limit could possibly appear on $\mathbb{CP}^n \times \{0\}$ and we can exclude that as follows.

Let α be a $(2p-1)$-form of comass one with compact support in $\mathcal{A}$ and let us prove that $\partial P_{\rho_k}(\alpha) \to 0$. Since $\mathcal{A}$ is a submanifold in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, we can extend α to be a form in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let us write, using horizontal coordinates $\{t_j\}_{j=1}^{2n}$ on $\mathbb{CP}^n$ and vertical ones $\{s_j\}_{j=1}^{2n+2}$ for $\mathbb{C}^{n+1}$,

$$\alpha = \alpha_h + \alpha_{v1} + \alpha_{v2} + \ldots \alpha_{v(2p-1)},$$

where α_h is a form only in the dt_j's, and each $\alpha_{v\ell}$ (for $\ell = 1, 2, \ldots, (2p-1)$) contains wedge products of $(2p-1-\ell)$ of the dt_j's and ℓ of the ds_j's. Rewrite, viewing P_{ρ_k} as currents in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$,

[10] Recall that the Hopf fibration is defined by the projection $H : S^{2n+1} \subset \mathbb{C}^{n+1} \to \mathbb{CP}^n$, $H(z_0, \ldots, z_n) = [z_0, \ldots, z_n]$. The Hopf fibers $H^{-1}(p)$ for $p \in \mathbb{CP}^n$ are maximal circles in S^{2n+1}, namely the links of complex lines of $\mathbb{C}^{n+1}$ with the sphere.

$$\partial P_{\rho_k}(\alpha) = [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] \left((\Lambda^{-1}_{\rho_k})^* \alpha \right).$$

The map $\Lambda^{-1}_{\rho_k}$ is expressed in our coordinates by $(t_1, ..., t_{2n}, s_1, ...s_{2n+2}) \to (t_1, ..., t_{2n}, \rho_k s_1, ...\rho_k s_{2n+2})$, therefore

$$(\Lambda^{-1}_{\rho_k})^* \alpha = \alpha^k_h + \alpha^k_{v1} + \alpha^k_{v2} + ...\alpha^k_{v(2p-1)},$$

where the decomposition is as above and with $\|\alpha^k_h\|^* \approx \|\alpha_h\|^*$ and $\|\alpha^k_{v\ell}\|^* \lesssim (\rho_k)^\ell \|\alpha_v\|^*$. The signs $\approx$ and $\lesssim$ mean respectively equality and inequality of the comasses up to a dimensional constant, so independently of the index k of the sequence.

As $k \to \infty$ it holds $\alpha^k_h \to \alpha^\infty_h$ in some C^ℓ-norm, where $\|\alpha^\infty_h\|^* \lesssim 1$ and α^∞_h is a form in the dt_j's [11]. We can write

$$\left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] \, (\alpha^k_h) \right| \leq \left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] \, (\alpha^k_h - \alpha^\infty_h) \right| + \left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] \, (\alpha^\infty_h) \right|$$

and both terms on the r.h.s. go to 0. The first, since $M((\Lambda_{\rho_k})_*(\partial P_{\rho_k}))$ are equibounded and $\|\alpha^k_h - \alpha^\infty_h\|_\infty \to 0$; the second because we can use (11) and $\left(\Phi^{-1}\right)_* \partial(T_\infty)$ has zero action on a form that only has the dt_j's components, as remarked in step 1.

Moreover

$$\left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] \, (\alpha^k_{v\ell}) \right| \to 0$$

for any $\ell \in \{1, 2, ...(2p-1)\}$, because the currents $(\Lambda_{\rho_k})_*(\partial P_{\rho_k}) = -\left(\Phi^{-1}\right)_* \langle T_{0,\rho_k}, |z| = 1 \rangle$ have equibounded masses by the choice of ρ_k, while the comasses $\|\alpha^k_{v\ell}\|^* \lesssim (\rho_k)^\ell \|\alpha_v\|^*$ go to 0.

Therefore no boundary appears in the limit and P is an integral cycle in $\mathcal{A}$. The fact that it is semi-calibrated by Θ follows easily by the fact that so are the currents P_ρ, as remarked just before Lemma 3.3.

$\square$

Summarizing, Lemmas 3.2 and 3.3 provide the analytic tools for the implementation of the *pseudo holomorphic blow up*. Thus we are now able to take the proper transform of an integral cycle T semi-calibrated by Ω in $\mathcal{S} \subset B^{2n+2}_1$ and get an integral cycle P in $\mathcal{A}$ that is semi-calibrated by Θ, where the semicalibration Θ is Lipschitz (and actually smooth away from $\mathbb{CP}^n \times \{0\}$). Therefore the almost monotonicity formula holds true for P in $\mathcal{A}$.

[11] More precisely α^∞_h coincides with the restriction of α_h to $\mathbb{CP}^n \times \{0\}$, extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ independently of the s_j-variables.

4 Proof of the result

With the assumptions in Proposition 3 we have to observe a sequence $(\lambda_{r_k})_* T$ as $r_k \to 0$. Recall that we have assumed, see (2), that the "straight" 2-planes D^X are pseudo-holomorphic for J.

Take any converging sequence $T_{0,r_k} := (\lambda_{r_k})_* T \to T_\infty$ for $r_k \to 0$. Take the proper transform of each T_{0,r_k} and denote it by P_k. Remark that P_k is a Θ_k-semi-calibrated cycle, for a semicalibration Θ_k that is smooth away from $\mathbb{CP}^n \times \{0\}$ and Lipschitz-continuous, with $|\Theta_k - \Theta_0| < c_k \mathrm{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\})$ and the constants c_k go to 0 as $k \to \infty$ (lemma 3.2).

From Lemma 3.3, the masses of P_k are uniformly bounded in k, since so are the masses of T_{0,r_k} (by almost monotonicity).

So by compactness, up to a subsequence that we do not relabel, we can assume $P_k \rightharpoonup P_\infty$ for a normal cycle P_∞.

Lemma 4.1. *P_∞ is a Θ_0-semi-calibrated cycle; more precisely it is the proper transform of T_∞.*

proof of Lemma 4.1. Θ_0-positiveness follows straight from the Θ_k-positiveness of P_k and $|\Theta_k - \Theta_0| < c_k \mathrm{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\})$, $c_k \to 0$.

The proper transform is a diffeomorphism away from the origin, thus

$$P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}^\rho) = \lim_k \left(\Phi^{-1}\right)_* T_{0,r_k} \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho) = \left(\Phi^{-1}\right)_* T_\infty \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho),$$

so in order to conclude that P_∞ is the proper transform of $\left(\Phi^{-1}\right)_* T_\infty$ we only need to show $P_\infty = \lim_{\rho \to 0} P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}^\rho)$, i.e. that $M(P_\infty \llcorner \mathcal{A}^\rho) \to 0$ as $\rho \to 0$.

Recall that $\vartheta_0 = \vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}}$; we want to estimate $M(P_\infty \llcorner \mathcal{A}^\rho) = (P_\infty \llcorner \mathcal{A}^\rho)(\Theta_0) = \lim_{k \to \infty} (P_k \llcorner \mathcal{A}^\rho)(\Theta_0)$. Write

$$\begin{aligned} (P_k \llcorner \mathcal{A}^\rho)(\Theta_0) = \frac{1}{p!} (P_k \llcorner \mathcal{A}^\rho)((\vartheta_{\mathbb{CP}^n})^p) + \\ + (P_k \llcorner \mathcal{A}^\rho) \left(\sum_{m=1}^{p} \frac{1}{m!(p-m)!} (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^{n+1}})^m \right). \end{aligned} \tag{12}$$

The second term on the r.h.s. is bounded as follows:

$$\begin{aligned} (P_k \llcorner \mathcal{A}^\rho) \left(\sum_{m=1}^{p} (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^{n+1}})^m \right) = \\ = (\Lambda_\rho)_* (P_k \llcorner \mathcal{A}^\rho) \left((\Lambda_r^{-1})^* \left(\sum_{m=1}^{p} (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^{n+1}})^m \right) \right). \end{aligned}$$

The current $(\Lambda_\rho)_*(P_k \llcorner \mathcal{A}^\rho)$ is the proper transform of $T_{0,\rho r_k}$, therefore $M\left((\Lambda_\rho)_*(P_k \llcorner \mathcal{A}^\rho)\right) \leq K$ independently of k; the form in brackets has comass bounded by ρ^2. Altogether

$$(P_k \llcorner \mathcal{A}^\rho)\left(\sum_{m=1}^{p}(\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}^{n+1}})^m\right) \leq C(p,n)\rho^2.$$

To bound the first term on the r.h.s. of (12), let P be the proper transform of T; using $(\Lambda_r)^*(\vartheta_{\mathbb{CP}^n})^p = (\vartheta_{\mathbb{CP}^n})^p$ we can write

$$(P_k \llcorner \mathcal{A}^\rho)((\vartheta_{\mathbb{CP}^n})^p)) = (P \llcorner \mathcal{A}^{r_k\rho})(\vartheta_{\mathbb{CP}^n})^p \leq M\left(P \llcorner \mathcal{A}^{r_k\rho}\right) \leq M\left(P \llcorner \mathcal{A}^\rho\right)$$

so it goes to 0 as $\rho \to 0$ (uniformly in k). Summarizing we get that $(P_k \llcorner \mathcal{A}^\rho)(\Theta_0)$ is $o(1)$ of $\rho \to 0$ uniformly in k; therefore so is $M(P_\infty \llcorner \mathcal{A}^\rho) = \lim_{k\to\infty}(P_k \llcorner \mathcal{A}^\rho)(\Theta_0)$ and we can write $P_\infty = \lim_{\rho\to 0} P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}^\rho)$.

□

For a positive-(p,p) integral cycle in $\mathbb{R}^{2n+2}$, as we have already mentioned, each tangent cone is determined by a $(p-1,p-1)$ integral cycle L_∞ in $\mathbb{CP}^n$ that is calibrated by $\frac{(\theta_{\mathbb{CP}^n})^{p-1}}{(p-1)!}$, the normalized power of the Kähler form.

The previous lemma tells us that, for a sequence $r_k \to 0$ such that $T_{0,r_k} \rightharpoonup T_\infty$, the proper transforms $P_k := \left(\Phi^{-1}\right)_* T_{0,r_k}$ converge to $\left(\Phi^{-1}\right)_* T_\infty$, i.e. to the current $L_\infty \times [\![D^2]\!]$. Indeed, the fact that a cone T_∞ is radially invariant translates into the fact that its proper transform is invariant under the action of Λ_ρ for any $\rho > 0$.

proof of Proposition 3. Let $T_{0,r_k} \rightharpoonup T_\infty$, a possible tangent cone. Let L_∞ be the holomorphic $(p-1,p-1)$-integral cycle in $\mathbb{CP}^n$ that identifies T_∞.

If y_0 is a point in the support of L_∞, then there exists a sequence of points $0 \neq y_j \to 0$ such that $H\left(\frac{y_j}{|y_j|}\right) \to y_0$ (where H is the Hopf projection) and radii δ_j such that each ball $B_{\delta_j}(y_j)$ contains a set $\mathcal{C}_j$ of strictly positive $\mathcal{H}^{2p}$-measure, $\mathcal{C}_j$ is contained in the support of T_{0,r_k} and the balls $B_{\delta_j}(y_j)$ are disjoint.

If we take a different sequence $T_{0,R_k} \rightharpoonup \tilde{T}_\infty$, we still find a sequence of points as before, since $\frac{y_j}{|y_j|}$ is not changed under radial dilations. Take the proper transforms $\tilde{P}_k$ of T_{0,R_k}. The density is preserved almost everywhere under the push-forward via a diffeomorphism. For each $\tilde{P}_k$ we find that, by upper semi-continuity of the density, y_0 is a point of density ≥ 1 for $\tilde{P}_k$ (for all k). Therefore y_0 is of density ≥ 1 for the limit of the currents $\tilde{P}_k$, i.e. for $\tilde{P}_\infty = \left(\Phi^{-1}\right)_* \tilde{T}_\infty$: this follows from the monotonicity formula, with an argument as in Remark 3.1.

This proves that any point in the support of L_∞ is also in the support of $\tilde{L}_\infty$, the holomorphic $(p-1,p-1)$-integral cycle in $\mathbb{CP}^n$ that identifies $\tilde{T}_\infty$. Since $\tilde{T}_\infty$ is an arbitrary tangent cone this concludes the proposition and therefore Lemma 2.1 is proved, i.e. all tangent cones must have the same support. $\square$

Now that the **support** of any tangent cone is **uniquely determined**, we have to make sure that any two links L_∞ and $\tilde{L}_\infty$ (obtained by a blow up with different sequences r_k and R_k) have **multiplicities** that **agree a.e.**

The following lemma should be known, but we recall it for sake of completeness.

Lemma 4.2. *Let C be a semicalibrated cycle of dimension m in $\mathbb{R}^n$ (or in an arbitrary Riemannian manifold). For any x_0 the set of tangent cones to C at x_0 is a closed and connected subset (for the flat distance, which metrizes the weak*-topology on currents of equibounded mass and boundary mass, see [12]).*

proof of Lemma 4.2. Let Υ be the set of all possible tangent cones at x_0. Given a sequence $\{T_k\}_{k=1}^{\infty}$ in Υ assume that $T_k \rightharpoonup T$. We want to show that $T \in \Upsilon$. The assumption $T_k \in \Upsilon$ means that there exists a sequence $r_j^k \to 0$ such that as $j \to \infty$ we have $C_{x_0, r_j^k} \rightharpoonup T_k$. With a diagonal argument we get $T \in \Upsilon$.

Now, to prove connectedness, assume by contradiction that $\Upsilon = \Upsilon_1 \cup \Upsilon_2$, where Υ_1 and Υ_2 are closed and disjoint. Then there exist (in the space of currents) A_1 and A_2 open disjoint neighbourhoods respectively of Υ_1 and Υ_2. The family of currents $C_{x_o,r}$ $(r \geq 0)$ is continuous and should therefore accumulate (as $r \to 0$) also somewhere outside A_1 and A_2, contradiction. $\square$

The following lemma will be applied to a sequence of possible tangent cones at a chosen point.

Lemma 4.3. *Let C_n and C be integral cycles of dimension k calibrated by a k-form ω (in an arbitrary Riemannin manifold). Assume that $C_n \rightharpoonup C$ and that the support $\mathcal{C}$ is the same for all C_n and C and it is compact. Let $\nu_n(x)$ denote the density at x for C_n and $\nu(x)$ analogously the density at x for C (dealing with calibrated cycles, each ν_n or ν is well-defined everywhere).*

Then for every $x \in \mathcal{C}$ it holds $\nu(x) = \lim_{n\to\infty} \nu_n(x)$.

proof of Lemma 4.3. We will achieve the proof in three steps.

Claim (i) for every $x \in \mathcal{C}$ it holds $\nu(x) \geq \limsup_{n\to\infty} \nu_n(x)$.

This follows from the monotonicity formula. Indeed, let $B_r(x)$ be the ball around x with radius r. By Remark 3.1, the weak convergence $C_n \rightharpoonup C$ yields $M(C_n \llcorner B_r(x)) \to M(C \llcorner B_r(x))$. By monotonicity we have $M(C_n \llcorner B_r(x)) \geq \alpha_k \nu_n(x) r^k$, thus it must hold, for all $r > 0$,

$$M(C \llcorner B_r(x)) \geq \alpha_k (\limsup_{n\to\infty} \nu_n(x)) r^k.$$

Since $\nu(x) = \lim_{r\to 0} \frac{M(C \llcorner B_r(x))}{\alpha_k r^k}$ we can conclude **claim (i)**.

Claim (ii) There exists $L > 0$ such that $\nu_n, \nu \leq L$ everywhere on $\mathcal{C}$.

For each fixed C_n (resp. C), the density ν_n (resp. ν) is a bounded function: this follows from the facts that the mass is locally finite, the monotonicity formula holds and the density is upper semi-continuous. So, in order to prove claim (ii), assume by contradiction that here exist points $p_n \in \mathcal{C}$ such that $\nu_n(p_n) \uparrow +\infty$ as $n \to \infty$. Up to a subsequence that we do not relabel we can assume $p_n \to p$ for a point p in $\mathcal{C}$. Choose a ball $B_R(p)$ and let $m > 0$ be chosen so that $M(C \llcorner B_R(p)) = \alpha_k \cdot m \cdot R^2$. Choose n_0 large enough so that for all $n \geq n_0$ it holds (i) $\theta_n(p_n) \geq 3m$ and (ii) $|p_n - p| < \frac{R}{10}$. Then consider the balls $B_{\frac{9R}{10}}(p_n)$: they are contained in $B_R(p)$.

By the monotonicity formula applied at p_n for C_n, we get

$$M(C_n \llcorner B_{\frac{9R}{10}}(p_n)) \geq \alpha_k (3m) \frac{9^2 R^2}{10^2}$$

and therefore

$$M(C_n \llcorner B_R(p)) \geq \alpha_k (3m) \frac{9^2 R^2}{10^2}.$$

By Remark 3.1 we must have $M(C_n \llcorner B_R(p)) \to M(C \llcorner B_R(p))$, so we can write

$$M(C \llcorner B_R(p)) \geq \alpha_k (3m) \frac{9^2 R^2}{10^2}.$$

Since $\frac{3 \cdot 9^2}{10^2} > 1$ we contradicted the assumption that $M(C \llcorner B_R(p)) = \alpha_k m R^2$.

Claim (iii) for every $x \in \mathcal{C}$ it holds $\nu(x) = \lim_{n\to\infty} \nu_n(x)$.

It suffices to show, for an arbitrary x, that

$$\nu(x) = \limsup_{n\to\infty} \nu_n(x). \tag{13}$$

Once this is achieved, choose a subsequence n_k such that $\liminf_{n\to\infty} \nu_n(x) = \lim_{n_k\to\infty} \nu_{n_k}(x)$ and apply (13) to the sequence of currents C_{n_k} to show that $\nu(x) = \lim_{n_k\to\infty} \nu_{n_k}(x) = \liminf_{n\to\infty} \nu_n(x)$.

Again the main ingredient for (13) is the monotonicity formula, which for an arbitrary $x \in \mathcal{C}$ states

$$
\begin{aligned}
\frac{1}{R^k}M(C_n \llcorner B_R(x)) &= \nu_n(x) + \int_{B_R(x)} \frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k}\nu_n(y)d\mathcal{H}^k(y)\llcorner\mathcal{C} \\
\frac{1}{R^k}M(C \llcorner B_R(x)) &= \nu(x) + \int_{B_R(x)} \frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k}\nu(y)d\mathcal{H}^k(y)\llcorner\mathcal{C},
\end{aligned} \tag{14}
$$

where the unit simple k-vector $\vec{C}_y$ represents the approximate tangent to $\mathcal{C}$ at y with the orientation given on C_n and ∂_r is the radial unit vector (with respect to the point x). Therefore the function $\frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k}$ is independent of n (since the underlying $\mathcal{C}$ is always the same and $\pm\vec{C}_y$ both yield the same value for $|\vec{C}_y \wedge \partial_r|$).

Let μ be the finite measure $\dfrac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k} \cdot \mathcal{H}^k(y)\llcorner(\mathcal{C} \cap B_R(x))$. By **claim (ii)** we can apply Fatou's lemma to $L - \nu_n$ and $L - \nu$ to get

$$\int \limsup_n \nu_n(y)d\mu(y) \geq \limsup_n \int \nu_n(y)d\mu(y),$$

which together with **claim (i)** yields

$$\int \nu(y)d\mu(y) \geq \limsup_n \int \nu_n(y)d\mu(y).$$

We can now use this last inequality together with claim (i) and the fact that $M(C_n \llcorner B_R(x)) \to M(C \llcorner B_R(x))$ to pass to the limit in (14) as $n \to \infty$: we get that necessarily we have the equality $\nu(x) = \limsup_{n\to\infty} \nu_n(x)$.

$\square$

proof of Theorem 2.2. Let Υ be the family of possible tangent cones to T at x_0. The elements of Υ are integral $(p-1,p-1)$-cycles (in $\mathbb{CP}^n$, the projective space of $\mathbb{C}^{n+1} \equiv T_x\mathcal{M}$) calibrated by $\frac{(\theta_{\mathbb{CP}^n})^{p-1}}{(p-1)!}$ and by Lemma 2.1 they all have the same support.

First we are going to prove that there exists a subset $\Upsilon_d \subset \Upsilon$ that is countable and dense in Υ, i.e. Υ is separable. This is achieved as follows.

All currents in Υ are supported on the same rectifiable set $\mathcal{C}$ and they can only differ by the choice of the density function. We can represent $\mathcal{C} \setminus \tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ is a $\mathcal{H}^{2p}$-null set, as the image of a Borel subset K of $\mathbb{R}^{2p}$ via a Lipschitz map taking values in $\mathbb{CP}^n$ and with Lipschitz constant $\frac{1}{2} \leq L \leq 2$. To obtain this representation, recall (see [19]) that $\mathcal{C} \setminus \tilde{\mathcal{C}}$ is a countable union of disjoint pieces, each piece being the image, via a Lipschitz map with constant close to 1, of a compact subset of $\mathbb{R}^{2p}$. We can freely change by translation the position of these countably many compact subset of $\mathbb{R}^{2p}$ and make them disjoint, so by denoting their union with K we get the desired representation for $\mathcal{C} \setminus \tilde{\mathcal{C}}$.

For each current in Υ the density on the rectifiable set $\mathcal{C}$ is an L^1 function on $(\mathcal{C}, \mathcal{H}^{2p})$. Using the previous representation of $\mathcal{C}$, we can record these densities as L^1 functions on $\mathbb{R}^{2p}$ that are zero outside of K. The family Υ therefore yields a family $\{g_a\}_{a\in\Upsilon}$ of such $L^1(\mathbb{R}^{2p})$ functions. Every such L^1 function on $\mathbb{R}^{2p}$ is associated to a current supported on $\mathcal{C}$.

The family $\{g_a\}$ is compact in L^1: indeed the L^1-convergence for a sequence in $\{g_a\}$ yields the (weak*) convergence for the corresponding currents. So $\{g_a\}$ is closed with respect to the L^1-norm, because Υ is closed with respect to the weak*-topology. Moreover $\{g_a\}$ is bounded in L^1 because $\int_{\mathbb{R}^{2p}} g_a d\mathcal{L}^{2p}$ is comparable (up to a factor 2, recall the condition on L) to the mass of the corresponding current which is fixed for all elements of Υ.

We conclude that, as a compact subspace of the separable normed space $L^1(\mathbb{R}^{2p})$, $\{g_a\}$ is also separable. The corresponding countable set of currents is the desired Υ_d.

Except on a $\mathcal{H}^{2p}$-null set $\tilde{\mathcal{C}}'$, all points of $\mathcal{C}\setminus\tilde{\mathcal{C}}'$ have integer densities for all currents in Υ_d.

Let now $x \in \mathcal{C}\setminus\tilde{\mathcal{C}}'$ and observe the function F from Υ_d to $\mathbb{R}$ assigning to every current $P \in \Upsilon_d$ the value $F(P) := \nu_P(x)$, where ν_P is the density of P. By Lemma 4.3 the function F is continuous on the metric space Υ_d, but since it is also integer-valued it must be locally constant on Υ_d.

Υ_d is dense in Υ, so for every current $P' \in \Upsilon$ the value $\nu_{P'}(x)$ is also locally constant by Lemma 4.3. Since Υ is connected, $\nu_{P'}(x)$ must then be globally constant for $P' \in \Upsilon$. The point $x \in \mathcal{C}\setminus\tilde{\mathcal{C}}'$ was arbitrary, therefore all currents in Υ have a fixed density at all points except on the null set $\tilde{\mathcal{C}}'$ and this makes them equal as currents. A posteriori also the density on $\tilde{\mathcal{C}}'$ is fixed.

We have therefore obtained that Υ is made of one single element and we can conlcude the uniqueness Theorem 2.2 for tangent cones. $\square$

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