Tangent cones to positive-$(1,1)$ De Rham currents Les cônes tangents des courants positifs $(1,1)$ de De Rham

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Abstract: *We show a uniqueness result for tangent cones to positive-*$(1,1)$ *De Rham currents at non-isolated points of positive density in an arbitrary almost complex manifold.*

Résumé: *Nous démontrons un resultat d'unicité du cône tangent à un courant positif* $(1,1)$ *de De Rham aux points de densité strictement positive non isolés dans une varieté presque complexe quelconque.*

Field: Geometry / Analysis of PDEs.

Version française abrégée

Dans une variété complexe quelconque l'ensemble des points à nombre de Lelong strictement positif d'un courant arbitraire positif $(1,1)$ (où $(1,1)$ est la bidimension), de masse finie et sans bord, a une structure très particulière: cet ensemble est une union de variétés algébriques, comme l'a démontré Y.-T. Siu dans [19] (ces variétés algébriques sont des sous-variétés holomorphes en dehors éventuellement de points singuliers isolés).

Il est important d'insister sur le fait qu'un tel "théorème de structure" n'est valable que pour l'ensemble des points à nombre de Lelong strictement positif: une représentation analogue pour le courant complet n'est pas vraie. En effet il existe des paires de courants positifs $(1,1)$ distincts deux à deux (sans bord, de masse finie) qui coïncident sur un ouvert de la variété complexe ambiante (et même, qui sont tous deux nuls sur un ouvert dense)!

Cette absence de la *propriété de continuation unique* a permis à C. O. Kiselman de construire dans [11] des contre-exemples à la propriète d'*unicité du cône tangent* en un point pour un courant $(1,1)$ positif.

Le résultat de Siu implique que l'absence d'unicité du cône tangent ne peut se produire qu'en un point x_0 pour lequel il existe $\delta > 0$ tel que le nombre de Lelong ν satisfait $\nu(x_0) \geq \nu(x) + \delta$ pour tout point x dans un voisinage de x_0.

Dans le présent travail nous étudions dans quelle mesure les résultats précédents s'étendent au cadre des variétés presque complexe où la structure n'est plus nécessairement intégrable. Dans ce cadre bien plus général la régularité de l'ensemble des points à nombre de Lelong strictement positif semble ne pas avoir été jusqu'à présent analysée de façon systèmatique. Les études locales de la régularité des courants positifs $(1,1)$ das le cas intégrable, à l'instar du travail de Siu, repose fondamentalement sur la caractérisation des ces courants au moyen des fonctions plurisousharmoniques. Une telle caractérisation n'existe plus dans le cadre général des structures presque complexe et on perd là un outil très puissant. Il est donc nécessaire d'adopter une stratégie complètement différente afin d'aborder dans le cadre presque complexe les questions résolues dans [19] pour les structures intégrables.

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Nous insistons sur le fait que la problématique d'étendre ces résultats du cadre intégrable au cadre non intégrable est fortement motivée par des applications géométriques diverses, où la structure complexe doit être perturbée en structure presque complexe afin de satisfaire des conditions génériques de transversalité (voire [6], [17], [20], [21]).

Le résultat principal de ce travail est le suivant:

Théorème 1. *Soit T un cycle $(1,1)$ normal positif dans une variété presque complexe $(\mathcal{M}, J)$. Soit x_0 un point de densité (nombre de Lelong) $\nu(x_0)$ strictement positif tel qu'il existe une suite $x_m \neq x_0$ convergeant vers x_0 avec $\nu(x_m) > 0$ et $\nu(x_m)$ convergeant vers $\nu(x_0)$.*

Alors le cône tangent à x_0 est unique et est donné par $\nu(x_0)$ fois le courant d'intégration sur un disque D plat et J_{x_0}-holomorphe, c'est à dire $\nu(x_0)[\![D]\!]$.

L'idée principale de la démonstration du théorème 1, donnée dans [1], repose sur une implémentation analytique de la procédure classique de *blow up* en géométrie algébrique et symplectique. En fait nos arguments permettent de démontrer le résultat plus fort suivant, qui est une sorte de propriété d'unicité partielle du cône tangent à un cycle $(1,1)$ positif.

Théorème 2. *Soit T un cycle $(1,1)$ positif normal dans une variété presque complexe $(\mathcal{M}, J)$. Soit x_0 un point de densité (nombre de Lelong) $\nu(x_0)$ strictement positif tel qu'il existe une suite $x_m \neq x_0$ convergeant vers x_0 et satisfaisant $\kappa = \underline{\lim}\, \nu(x_m) > 0$.*

Soit $H : S^{2n-1} \to \mathbb{CP}^{n-1}$ la projection d'Hopf, où S^{2n-1} est la sphère unitaire dans l'espace $T_{x_0}\mathcal{M}$ munie des cordonneés pour les quelles la structure complexe J_{x_0} est standard. Soit $\{y_\alpha\}_{\alpha \in A}$ l'ensemble d'accumulation dans $\mathbb{CP}^{n-1}$ de la suite $H\left(\frac{x_m - x_0}{|x_m - x_0|}\right)$. Soit D_α le disque plat J_{x_0}-holomorphe dans $T_{x_0}\mathcal{M}$, centré en 0 et contenant le cercle $H^{-1}(y_\alpha)$.

Alors l'ensemble A est fini et si T_∞ est un cône tangent en x_0 il doit "contenir" la somme des courants d'intègration sur les D_α multiplié par κ: en d'autres termes $T_\infty - \oplus_{\alpha \in A} \kappa [\![D_\alpha]\!]$ est $(1,1)$ positif.

English version

Positive currents in complex manifolds have been studied quite extensively, since the work of Lelong [13]: they appeared at first in relation with the study of plurisubharmonic functions and with integration on analytic varieties (see the surveys [10] and [15]). Striking results, some of which we will now recall, have been obtained since then. We will concentrate on positive-$(1,1)$ currents.

For any positive-$(1,1)$ current of finite mass and without boundary (also called positive-$(1,1)$ normal cycle) in a complex manifold, the set of points with stricly positive Lelong number has a very regular structure: this set is a union of algebraic varieties, as shown by Y.-T. Siu in [19]: these varieties are holomorphic outside of possible singular points.

Such a "structure theorem" only holds, however, for the set of points with strictly positive Lelong number: there is no analogous representation for the global current. Indeed, two different positive-$(1,1)$ cycles of finite mass can coincide on an open set but not globally. This lack uf unique continuation allowed C. O. Kiselman in [11] to produce couterexamples to the *uniqueness of tangent cones* at an arbitrary point of a positive-$(1,1)$ normal cycle.

Siu's result implies that this failure can only happen at a point x_0 where there exists $\delta > 0$ so that the Lelong number ν fulfils $\nu(x_0) \geq \nu(x) + \delta$ for all points x in a neghbourhood of x_0.

If we turn to an almost complex manifold and consider positive-$(1,1)$ currents with respect to a possibly non-integrable almost complex structure J, the structure of the set of points with positive Lelong number has not been investigated much. The approach in the case of complex manifolds (see [19], [4], [12], [14]) relies a lot on a connection with plurisubharmonic functions, not avaliable in an almost complex manifold; due to the loss of such a tool, the strategy for a structure theorem (analogous to [19]) in the almost complex setting would require the use of totally different approaches and techniques.

With that aim in mind, in this work we concentrate on the issue of tangent cones, providing a result that allows their understading in terms of the Lelong number. Let us first recall the setting and basic notions.

Let $(\mathcal{M}, J)$ be a smooth almost complex manifold of dimension $2n$, endowed with a non-degenerate 2-form ω compatible with J. If $d\omega = 0$ then we have a symplectic form, but we will not need to assume closedness. Let g be the associated Riemannian metric, $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$.

The form ω is a semi-calibration (following the terminology of [16]) on $\mathcal{M}$ for the metric g, i.e. the comass $\|\omega\|^*$ is 1; recall that the comass of ω is defined to be

$$||\omega||^* := \sup\{\langle \omega_x, \xi_x\rangle : x \in \mathcal{M},\, \xi_x \text{ is a unit simple 2-vector at } x\}.$$

If ω is closed, then we have a calibration, as in the landmark paper [9].

The set $\mathcal{G}_x$ of 2-*planes calibrated by* ω_x is defined to be the subfamily of the Grassmannian $G(2, T_x\mathcal{M})$ made of the oriented 2-planes that, represented as unit simple 2-vectors ξ_x, realize $\langle \omega_x, \xi_x\rangle = 1$. Set

$$\mathcal{G}(\omega) := \cup_{x\in\mathcal{M}}\, \mathcal{G}_x := \cup_{x\in\mathcal{M}}\{\xi_x \in G(x, T_x\mathcal{M}) : \langle \omega_x, \xi_x\rangle = 1\}.$$

We will consider a ω-**positive** normal 2-cycle T in $\mathcal{M}$. The definition, as in [9], is as follows.

T is a 2-dimensional De Rham current T in $\mathcal{M}$ with finite mass and zero boundary (we refer to [7] of [8] for notions from Geometric Measure Theory). The finiteness of the mass allows a representation of T by integration, i.e. there exist

(i) a positive Radon measure $\|T\|$,

(ii) a generalized tangent space $\vec{T_x} \in \Lambda_2(T_x\mathcal{M})$, that is defined $\|T\|$-a.e., is $\|T\|$-measurable and has mass-norm 1 (the latter means that it belongs to the convex envelope of unit simple 2-vectors),

such that the action of T on any 2-form β with compact support is expressed as follows

$$T(\beta) = \int_{\mathcal{M}} \langle \beta, \vec{T}\rangle d\|T\|.$$

The cycle condition is: $(\partial T)(\alpha) := T(d\alpha) = 0$, for any compactly supported one-form α.

The positiveness of T with respect to ω is defined by the following equivalent conditions:

- $\vec{T} \in$ convex hull of $\mathcal{G}(\omega)$ $\quad\|T\|$-a.e.
- $\langle \omega, \vec{T}\rangle = 1 \quad \|T\|$-a.e.
- $T(\omega) := \int_{\mathcal{M}} \langle \omega, \vec{T}\rangle d\|T\| = M(T)$.

In the case when ω is closed, a ω-positive T is (locally) homologically mass-minimizing (see [9]). In the case of a non-closed ω, the same argument shows that a ω-positive cycle T is locally an almost-minimizer of the mass (also called λ-minimizer).

A useful equivalent characterization for the fact that a unit simple 2-vector at x is in $\mathcal{G}_x$, i.e. it is ω_x-calibrated is as follows: a 2-plane is in $\mathcal{G}_x$ if an only if it is J_x-invariant.

Therefore an equivalent way to express ω-positiveness is that $\|T\|$-a.e. $\vec{T}$ belongs to the convex hull of J-holomorphic simple unit 2-vectors, in particular $\vec{T}$ itself is J-invariant. For this reason ω-positive normal cycles are also called positive-$(1,1)$ normal cycles. Remarkably the $(1,1)$-condition only depends on J, so a positive-$(1,1)$ cycle is ω-positive for any J-compatible couple (ω, g).

Positive cycles satisfy an important *almost monotonicity property*: at any point x_0, denoting by $B_r(x_0)$ the geodesic ball around x_0 of radius r, the mass ratio $\frac{M(T \llcorner B_r(x_0))}{\pi r^2}$ is an almost-increasing function of r, i.e. it is expressable as a weakly increasing function of r plus an infinitesimal of r.

Monotonicity yields at any point x_0 a well-defined limit

$$\nu(x_0) := \lim_{r\to 0} \frac{M(T \llcorner B_r(x_0))}{\pi r^2}.$$

This is called the (two-dimensional) **density** of the current T at the point x_0 (*Lelong number* in the classical literature, see [13]).

The following procedure is called the *blow up limit* and the idea goes back to De Giorgi [5]. Consider a dilation of T around x_0 of factor r which, in normal coordinates around x_0, is expressed by the push-forward of T under the action of the map $\frac{x-x_0}{r}$:

$$(T_{x_0,r} \llcorner B_1)(\psi) := \left[\left(\frac{x-x_0}{r}\right)_* T\right](\chi_{B_1}\psi) = T\left(\chi_{B_r(x_0)}\left(\frac{x-x_0}{r}\right)^* \psi\right). \tag{1}$$

The fact that $\frac{M(T \llcorner B_r(x_0))}{r^2}$ is monotonically almost-decreasing as $r \downarrow 0$ gives that, for $r \le r_0$ (for a small enough r_0), we are dealing with a family of currents $\{T_{x_0,r} \llcorner B_1\}$ that satisfy the hypothesis of Federer-Fleming's compactness theorem (see [8] page 141), thus there exist weak limits (in the sense of currents) as $r \to 0$. Every current T_∞ such that there exists a sequence $r_n \to 0$ for which

$$T_{x_0,r_n} \llcorner B_1 \to T_\infty,$$

turns out to be a cone, a so called **tangent cone** to T at x_0; the density of T_∞ at the origin the same as the density of T at x_0; moreover T_∞ is ω_{x_0}-positive (see [9]).

If there is a unique T_∞ such that $T_{x_0,r_n} \llcorner B_1 \to T_\infty$ independently of the sequence $\{r_n\}$, then T_∞ is the **unique** tangent cone to T at x_0.

The uniqueness of tangent cones is of interest in order to understand *regularity at infinitesimal level* for a current.

Such a uniqueness result is, up to now, only known for some particular classes of integral currents, namely for mass-minimizing integral cycles of dimension 2 ([22]) and for general semi-calibrated integral 2-cycles ([16]).

Passing to normal currents, things are different. As mentioned before the uniqueness of tangent cones to ω-positive normal 2-cycles **fails** in general: this was proven by Kiselman [11]. Further works extended the result to arbitrary dimension and codimension (see [2] and [3], where conditions on the rate of convergence of the mass ratio are given, under which uniqueness holds).

In this work we prove the following result:

Theorem 1. *Given an almost complex $2n$-dimensional manifold $(\mathcal{M}, J, \omega, g)$ as above, let T be a positive-$(1,1)$ normal cycle, i.e. a ω-positive normal 2-cycle. Let x_0 be a point of positive density $\nu(x_0) > 0$ and assume that there is a sequence $x_m \to x_0$ of points $x_m \ne x_0$ all having positive densities $\nu(x_m)$ and such that $\nu(x_m) \to \nu(x_0)$.*

Then the tangent cone at x_0 is unique and it is given by $\nu(x_0)[\![D]\!]$ for a certain J_{x_0}-invariant disk D.

The notation $[\![D]\!]$ stands for the current of integration on D. The key idea of the proof, given in [1], is an analysis implementation of the classical blow up of curves in algebraic or symplectic geometry. The same argument actually yields the following stronger result, a sort of a partial uniqueness for tangent cones:

Theorem 2. *Given an almost complex $2n$-dimensional manifold $(\mathcal{M}, J, \omega, g)$, let T be a ω-positive normal 2-cycle. Let x_0 be a point of positive density $\nu(x_0) > 0$ and assume that there is a sequence of points $\{x_m\}$ such that $x_m \to x_0$, $x_m \ne x_0$ and the x_m have positive densities with $\kappa := \underline{\lim}\, \nu(x_m) > 0$.*

Let $H : S^{2n-1} \to \mathbb{CP}^{n-1}$ denote the standard Hopf projection, where S^{2n-1} is the unit sphere in $T_{x_0}\mathcal{M}$, which is endowed with coordinates such that the complex structure J_{x_0} is the standard one.

Let $\{y_\alpha\}_{\alpha\in A}$ be the set of accumulation points in $\mathbb{CP}^{n-1}$ for the sequence $H\left(\frac{x_m-x_0}{|x_m-x_0|}\right)$. Denote by D_α the flat J_{x_0}-holomorphic disk in $T_{x_0}\mathcal{M}$ centered at 0 and containing the circle $H^{-1}(y_\alpha)$.

Then the set A is finite and any tangent cone T_∞ to T at x_0 is such that it "contains" the sum of the currents of integration on the disks D_α multiplied by κ, in other words $T_\infty - \oplus_{\alpha\in A}\kappa[\![D_\alpha]\!]$ is ω_{x_0}-positive.

In addition to the interest for tangent cones themselves, theorems 1 and 2 can serve as a first step towards a regularity result analogous to the one in [19], this time in the non-integrable setting.

The need for such a regularity result comes from several possible applications in geometry, namely problems where the structure must be perturbated from a complex to almost complex one, in order to ensure some transversality conditions. Related discussions can be found in [6], [17], [20], [21].

Let us see an example, related to the study of pseudo-holomorphic maps into algebraic varieties, as those analyzed in [17]. Indeed, if $u: M^4 \to \mathbb{CP}^1$ is pseudoholomorphic and weakly approximable as in [17], with M^4 a compact closed 4-dimensional almost-complex manifold, denoting by ϖ the symplectic form on $\mathbb{CP}^1$, then the 2-current U defined by $U(\beta) := \int_{M^4} u^*\varpi\wedge\beta$ is a normal positive-$(1,1)$ cycle in M^4. As explained in [17], the singular set of u is of zero $\mathcal{H}^2$-measure and is the set where the density (Lelong number) of U is $\geq \epsilon$, for a positive ϵ depending on (M^4, g) (this is a so-called ε-regularity result, see [18]). Then we would be reduced, in order to understand singularities of u, to the study of points of density $\geq \epsilon$ of a normal $(1,1)$-cycle. The knowledge that such a set is made of pseudoholomorphic subvarieties, together with the fact that it is $\mathcal{H}^2$-null, would imply that the singular set is made of isolated points, the same result achieved in [17] with different techniques.

The strategy might then be applied to other problems, in which ε-regularity results and calibrations play a role. An important example is, in Gauge Theory, the case of anti self-dual instantons in a 6-dimensional almost complex manifold (see section 5 of [21]).

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