Finding non-commutative crepant resolutions for the affine cone of the Grassmannian

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Abstract

We find noncommutative crepant resolutions for the affine cone of the Grassmannian. To do this we use the framework of Špenko and Van den Bergh. By strengthening their results in this specific family of examples we can show both Cohen-Macaulayness and finite global dimension. We actually get two different NCCR's and by relating these NCCR's to categorical resolutions that Kuznetsov gives we show that they are derived equivalent.

Contents

1 Introduction **2**

1.1 History and Background . . . 3

1.2 Our results . . . 4

2 Borel-Weil-Bott and local cohomology **4**

2.1 Borel-Weil-Bott . . . 5

2.2 Local Cohomology . . . 6

3 Algebraic NCCR **7**

3.1 Setup . . . 8

3.2 The Spectral Sequence . . . 9

3.3 Calculations . . . 10

3.4 Finite global dimension . . . 16

4 Geometric NCCR **19**

5 These NCCR's are equivalent **21**

1 Introduction

The idea of trying to resolve a singularity by replacing a space X which is not smooth with another space $\tilde{X}$, birational to X, and smooth has been very useful in the study of schemes. This is a well known and very commonly used tool in algebraic geometry. As resolutions of singularities are not unique so we often ask for extra conditions, one of those conditions is crepancy (the canonical bundle pulls back to the canonical bundle).
This type of resolution is geometric, recently the idea of taking a different type of resolution has appeared, using non-commutative rings or categories instead. This is no longer a geometric idea, however it still has connections to geometry and even to classical resolutions of singularities in certain cases.

Definition 1.1. [[ŠV15, Def. 1.1.1]] Let S be a scheme, such that S is a normal noetherian domain. A non-commutative resolution of S is an algebra over S which has finite global dimension and is of the form $\Lambda = \mathrm{End}_S(M)$, for M a non-zero, finitely generated, reflexive S-module.
It is called crepant if S is Gorenstein and Λ is a maximal Cohen-Macaulay S-module.
We abbreviate with NC(C)R.

Definition 1.2. [[Kuz06, Def. 3.2]] A *categorical resolution* of S is a (regular) triangulated category T and maps

$$\pi_* : T \to D^b(S) \qquad\qquad \pi^* : D^b(S) \to T$$

such that π^* is the left adjoint to π_* on $D^{perf}(S)$ and the natural map $id_{D^{perf}(S)} \to \pi_*\pi^*$ is an isomorphism.
It is weakly crepant if π^* is also the right adjoint to π_* on $D^{perf}(S)$.

A NC(C)R gives a categorical resolution and in some cases, the reverse is also true.

1.1 History and Background

For motivation and discussion see the survey paper by Leuschke [Leu11]. In section K the definition of a NCCR is given, it is different to the one above, but later it is shown to be equivalent to the above definition, this paper also contains many other results and thoughts on NC(C)R's. In there is also a brief discussion of categorical desingularizations (resolutions) and how they relate to NCCR's.

The first mentions of non-commutative resolutions were by Bondal and Orlov [BO02] and Van den Bergh [Van04], not as a main idea, but as something that could be useful to show other results. Van den Bergh only defines an NCCR in the appendix of [Van04], it had been used as an intermediary step in proving two derived categories were equivalent. Looking at derived categories and equivalences was also the reason that Bondal and Orlov thought of it.
Later, Špenko and Van den Bergh, [ŠV15], gave a framework for proving the existence of NC(C)R's for quotients by a reductive group, G. They prove the existence of NC(C)R's under specific conditions.
Kuznetsov, [Kuz06], defined a kind of categorical resolution and proved an existence result, again under specific conditions. Later Kuznetsov with Lunts, [KL12], proved the existence of a categorical resolution for the derived category of any separated scheme with characteristic 0.

Špenko and Van den Bergh build potential NC(C)R's by building M from some finite collection of irreducible representations of G, and considering the module of covariants associated to those representations, motivated by the case of finite groups, where taking all irreducible representations works.
They then build resolutions of these modules of covariants, under specific conditions they show that these resolutions prove finite global dimension of Λ.
Earlier, Van den Bergh [Van91], proved results about the Cohen-Macaulayness of modules of covariants, they also show when those results give Cohen-Macaulayness of Λ.

Kuznetsov starts with a geometric resolution, $\tilde{S}$ of S, whose exceptional divisor has a Lefschetz decomposition. The decomposition is used to find a subcategory of $D^b(\tilde{S})$ that gives the categorical resolution.

For more details about both constructions see Sections 3 and 4.

1.2 Our results

We consider an explicit family of examples, let V be a k dimension vector space over $\mathbb{C}$ and let $X = \mathrm{Hom}(\mathbb{C}^n, V)$ where $n > k+1$. Consider the standard action of $G = SL(V)$ on X. (Acting by left matrix multiplication), then X/G is the affine cone of the Grassmannian, $Gr(n, V)$. It has one singular point, the origin, and has a natural geometric resolution $\pi : \mathcal{O}(-1)_{Gr(n,V)} \to X/G$. We want to find a NCCR of X/G using the framework of [ŠV15]. We do this in Section 3, assuming that n and k are coprime. If n, k are not coprime then many of the results do not go through.
We can not just apply the main result of [ŠV15] as our example is not quasi-symmetric, so we will need to strengthen results about Cohen-Macaulayness and finite global dimension. First of all we deal with Cohen-Macaulayness, going back to [Van91] we analyse the spectral sequence provided there and show which modules of covariants are Cohen-Macaulay. Second, we explain how a result of Fonarev [Fon13] can be easily adapted to show finite global dimension.
This gives us an NCCR

$$\mathrm{End}\left(\bigoplus (\mathbb{S}^\alpha V^* \otimes \mathrm{Sym}^\bullet X^*)^G\right)$$

where $\mathbb{S}^\alpha V^*$ is a Schur power and the sum is over all Young Diagrams that fit inside a triangle of length $n-k$ and height k.
An isomorphic singularity is $\mathrm{Hom}(S, \mathbb{C}^n)/SL(S)$ and we get a very similar NCCR, however the algebra we get is not isomorphic to one above. They should however be derived equivalent and the rest of the paper is dedicated to proving this, see Section 5 and in particular, Theorem 5.2.

We do not do this directly, we go via categorical resolutions as X/G has a geometric resolution, $\pi : \mathcal{O}(-1)_{Gr(n,V)} \to X/G$, whose exceptional divisor has a Lefschetz decomposition, [Fon13]. The first block of the decomposition is generated by vector bundles indexed by the same collection of Young diagrams as the NCCR. (In fact this is the motivation for considering the collection we do for the NCCR.) In Section 4 we explain what the end result is and how we get a few different categorical resolutions which are all easily shown to be equivalent. This section has nothing original apart from a few short proofs (which are all straightforward and probably already known).

Finally in Section 5 we show that the NCCR from Section 3 (the above one) is equivalent to a geometric one obtained from the categorical resolutions from Section 4. The isomorphic singularity also has an NCCR equivalent to one coming from a categorical resolution and as all the categorical resolutions are equivalent we get the wanted result, Theorem 5.2.

We finally note that even though Kuznetsov's categorical resolutions and Van den Bergh's NCCR are different (but related) ideas, the proof for finite global dimension of the NCCR is effectively the same as the proof that gives us generation in the Lefschetz decomposition. However the way of proving crepancy is rather different.
It would not be surprising if this turned out to be true in other situations as well.

2 Borel-Weil-Bott and local cohomology

We will be using and calculating cohomology repeatedly, so we will give a brief overview of the Borel-Weil-Bott theorem which is used for calculations and then of local cohomology which is mainly used as a tool to find Cohen-Macaulay sheaves.

2.1 Borel-Weil-Bott

Theorem 2.1 (Borel-Weil-Bott). *Let G be a reductive group, let $B \subset G$ be a Borel subgroup. Let α be a weight of B, it corresponds to a line bundle $\mathcal{O}(\alpha)$ on G/B. Let ρ be half the sum of the positive roots of G.*
There exists a unique element, σ, of the Weyl group of G that takes $\alpha+\rho$ to a dominant weight, let $\alpha^+ = \sigma(\alpha+\rho)-\rho$. Then we have two possible situations

- $H^i(G/B, \mathcal{O}(\alpha)) = 0$ for all i when α^+ is not dominant.

- $H^i(G/B, \mathcal{O}(\alpha)) = V(\alpha^+)$ for $i = l(\alpha^+)$ and zero for all other i when α^+ is dominant. Here $V(\alpha^+)$ denotes the irreducible representation of G with highest weight α^+ and $l(\alpha^+) = l(\sigma)$ is the smallest number k such that σ is the product of k simple reflections, called the length of σ.

Remark 2.2. G/B is a flag variety, in the special case of $G = GL(V)$ we have G/B is the complete flag on V.

We also need a result about calculating cohomology on the Grassmannian, which follows from Borel-Weil-Bott.

Let V be a vector space of dimension n.
Consider $Gr(k,V) = Gr(V, n-k)$, the Grassmannian of k-planes in V, or identically the same as $(n-k)$-quotients of V.
We have two dual tautological short exact sequences on $Gr(k,V)$.

$$0 \to S \to V \otimes \mathcal{O}_{Gr(k,V)} \to Q \to 0$$

$$0 \to Q^* \to V^* \otimes \mathcal{O}_{Gr(k,V)} \to S^* \to 0$$

Where S is the tautological sub-bundle and Q is a tautological quotient bundle.

Let α be a Young Diagram, i.e. $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ such that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$. These ordered collections of positive integers also correspond to highest weights of $SL(V)$ (as long as $\alpha_n = 0$) and it turns out that if we let $\mathbb{S}^\alpha$ denote the Schur functor associated to α, then $\mathbb{S}^\alpha V$ gives the irreducible representation of $SL(V)$ with highest weight α.
In a similar way every Young Diagram also corresponds to a highest weight of $GL(V)$ and again $\mathbb{S}^\alpha V$ gives the irreducible representation of $GL(V)$ with highest weight α. This enables us to extend $\mathbb{S}^\alpha V$ to any highest weight α of $GL(V)$.

Proposition 2.3. *Let $\beta \in \mathbb{Z}^k, \gamma \in \mathbb{Z}^{n-k}$ be two non-increasing sequences of integers and let $\alpha = (\beta, \gamma) \in \mathbb{Z}^n$.*
Then we have

$$H^\bullet(Gr(k,V), \mathbb{S}^\beta S^* \otimes \mathbb{S}^\gamma Q^*) \cong \mathbb{S}^{\alpha'} V^*[-l(\alpha')]$$

Where α' is the unique dominant weight in the twisted weyl group orbit of α, and $l(\alpha')$ is the length. (If such an element doesn't exist, the cohomology is zero.)

Remark 2.4. This proposition is the result we will mainly use. It follows from applying Borel-Weil-Bott twice. First we apply it to the two flag bundles, $F(S^*)$ and $F(Q^*)$ over $Gr(k, V)$, then by using the Künneth formula we find out that $\mathcal{O}(\alpha)$ on $F(S^*) \times F(Q^*) = F(V^*)$ pushes down to $\mathbb{S}^\beta S^* \otimes \mathbb{S}^\gamma Q^*$. Second we apply it to $G = GL(V^*)$ with $G/B = F(V^*)$ and the line bundle $\mathcal{O}(\alpha)$.

As a special case, if α is dominant then we just have H^0 being $\mathbb{S}^\alpha V^*$ and there is no higher cohomology.
Also note, by considering the vector space V^* instead, any result we get for S^*, Q^* also holds for S, Q up to appropriate swapping of k and $n-k$.

We use the Littlewood-Richardson rule to calculate what Young diagrams can appear in the tensor product of two diagrams.

2.2 Local Cohomology

Let X be a scheme and let $Y \subset X$ be a closed subscheme, we can then define $\Gamma_Y(\mathcal{F})$ to be the sections of $\mathcal{F}$ with support in Y, deriving we get $H^i_Y(X, \mathcal{F})$, which are called the *local cohomology groups.*
There is also another definition.
Let $U = X \backslash Y$ and let $j : U \to X$ be the inclusion, then $\mathcal{H}^\bullet_Y(X, \mathcal{F}) = Cone(\mathcal{F} \to j_* j^* \mathcal{F})$ where the map comes from adjunction. Take global sections to get $H^i_Y(X, \mathcal{F})$.

There are two sequences that hold for local cohomology that will be useful to us. First we have the following long exact sequence.

$$\begin{aligned} 0 \to & H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to \\ & H^1_Y(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}|_U) \to \cdots \end{aligned}$$

For the second sequence, let $Z \subset Y \subset X$ be a sequence of closed subspaces, we then have an exact triangle.

$$\mathcal{H}^\bullet_Z(X, \mathcal{F}) \to \mathcal{H}^\bullet_Y(X, \mathcal{F}) \to \mathcal{H}^\bullet_{Y \backslash Z}(X \backslash Z, \mathcal{F}|_{X \backslash Z}) \tag{1}$$

We will use local cohomology in the following way. Let X be a scheme and $\mathcal{M}$ be a bundle on X, then given a family Z of closed subschemes over X that lives inside $\mathcal{M}$ we have $\mathcal{H}^i_Z(\mathcal{M}, -)$, we will use this to denote the pushdown to X.
In the special case of $\mathcal{N} \subset \mathcal{M}$, where $\mathcal{N}$ is a sub-bundle of $\mathcal{M}$ we know exactly what the local cohomology of $\mathcal{O}_\mathcal{M}$ is.

Lemma 2.5. *Let X be a smooth scheme, let $\mathcal{M}$ be a bundle over X and $\mathcal{N}$ be a sub-bundle of $\mathcal{M}$. Then*

$$\mathcal{H}^{\mathrm{codim}\,\mathcal{N}}_\mathcal{N}(\mathcal{M}, \mathcal{O}_\mathcal{M}) = \mathrm{Sym}^\bullet\left(\mathcal{N}^\vee\right) \otimes \mathrm{Sym}^\bullet\left(\mathcal{M}/\mathcal{N}\right) \otimes \det\left(\mathcal{M}/\mathcal{N}\right)$$

and all other cohomology is zero.

Proof. See [Van93]. □

The exact triangle also holds in this case.

Finally there is a third way of thinking about local cohomology, it is the same as relative De Rham homology, which is defined as follows, given a diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr] & \tilde{X} \arrow[d] \\ & X \end{tikzcd}$$

where the horizontal map is a closed immersion and the vertical one is smooth, define $H_i^{DR}(Y/X) = H_Y^{2n-i}(\tilde{X}, \Omega^\bullet_{\tilde{X}/X})$ where n is the relative dimension of $\tilde{X}/X$.
It is functorial for proper maps, so if we have $Y' \to Y$ proper, we get a map $H_i^{DR}(Y'/X) \to H_i^{DR}(Y/X)$ and we have

$$H_{-i}^{DR}(Y/X) = H_Y^i(X, \mathcal{O}_X)$$

for $Y \subset X$ closed.
For more details on relative De Rham homology see [Van91].

3 Algebraic NCCR

Let $X = \mathrm{Hom}(\mathbb{C}^n, V)$, let $G = SL(V)$ acting on X. We want to find a NCCR for X/G. There are not any general methods for finding NCCR's of an arbitrary singular scheme, however for specific types of singularities there are some approaches. Our singularity is a quotient by a reductive group, for this type of singularity, Špenko and Van den Bergh [ŠV15] provide a general method for finding NCCR's.
Based on the finite group case, for H a reductive group acting on Y, they consider the algebra

$$\Lambda = (\mathrm{End}(U) \otimes k[Y])^H$$

where $U = \bigoplus_{\chi \in \mathcal{L}} V(\chi)$ with $V(\chi)$ being the irreducible representation with highest weight χ and $\mathcal{L}$ is some finite collection of highest weights.

We want this algebra to have finite global dimension and be Cohen-Macaulay. It splits into components $(V(\mu) \otimes k[Y])^H$ where $V(\mu)$ appears in the decomposition of $V(\chi)^* \otimes V(\chi')$ for $\chi, \chi' \in \mathcal{L}$. It is sufficient to prove that each individual component is Cohen-Macaulay. Špenko and Van den Bergh just reference a result from [Van91] to show Cohen-Macaulayness. We will strengthen this result.
Most of the paper [ŠV15] is dedicated to showing finite global dimension, they first show that is sufficient to prove that each H-equivariant Λ-module of the form $P_\chi = \mathrm{Hom}\,(U \otimes k[Y], V(\chi) \otimes k[Y])$ has finite projective dimension. Then they build a resolution of P_χ as follows.
Pick a one parameter subgroup λ of $T \subset B \subset H$ (T a maximal torus inside a Borel subgroup B). Let $Z_\lambda \subset Y$ be the subspace of points that flow to zero as λ tends to zero.
Let $\widetilde{Z_\lambda} = Z_\lambda \times_B H$, it is a resolution of HZ_λ and a sub-bundle of $\widetilde{Y}$ where both are bundles over H/B. Taking the Koszul resolution of $\widetilde{Z_\lambda}$ gives an exact complex, we can also tensor it with $\widetilde{V(\chi)}$ and then applying $\mathrm{Hom}(U \otimes k[Y], -)$ gives us a complex. If χ and λ are suitably related then the cohomology of this complex is exact and is a resolution of P_χ. They also describe all the modules P_μ that appear in this resolution.

The idea to show finite global dimension is as follows. Take any module P_χ, if $\chi \in \mathcal{L}$, then P_χ is projective. If $\chi \notin \mathcal{L}$ then we use the resolution constructed above, the aim is to prove that by iterating these resolutions a finite number of times we can find a projective resolution of P_χ. If we can do this for all χ, we have finite global dimension. We will explain briefly how Fonarev, [Fon13] shows this after we have shown Cohen-Macaulayness.

3.1 Setup

For us, we let $\mathcal{UP}_{n,k}$ be the collection of all Young Diagrams α such that $\alpha_i \leq (n-k)\frac{k-i}{k}$. These are all diagrams that fit inside a triangle of length $n-k$ and height k, see Figure 1 for an example.

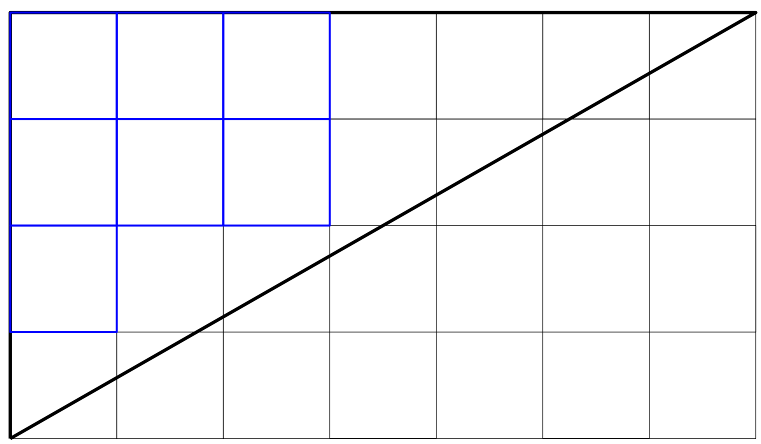

Figure 1: An example of a Young Diagram in $\mathcal{UP}_{11,4}$

We want to show that the algebra

$$\Lambda = \mathrm{End}\left(\bigoplus_{\alpha \in \mathcal{UP}_{n,k}} (\mathbb{S}^\alpha V^* \otimes \mathrm{Sym}^\bullet X^*)^G\right)$$

gives a NCCR of $X/G = \mathrm{Hom}\left(\mathbb{C}^n, V\right)/SL(V)$.
We will do this by using the framework of Špenko and Van den Bergh.
First we show that Λ is Cohen-Macaulay, to do that we need to show that each representation that appears in Λ is Cohen-Macaulay.
We use the techniques from [Van91]. The same result is proved using different methods in [RWW14].

Second we will show that Λ has finite global dimension.

Let R_α^G be the component of $\mathcal{O}_X$ with weight α (split $\mathcal{O}_X$ into irreducible representations of G). We have R_α^G is Cohen-Macaulay iff $(\mathbb{S}^\alpha V^* \otimes \mathrm{Sym}^\bullet X^*)^G$ is Cohen-Macaulay, so we will show R_α^G is Cohen-Macaulay, to do this we will use the following result proved in [Van89]. (The above if and only if is also shown in this paper).

Lemma 3.1. *Let $X^u \subset X$ be the null cone, let $h = \dim \mathcal{O}_X^G$, then R_α^G is Cohen-Macaulay iff there is no representation with weight α in $H^i_{X^u}(X, \mathcal{O}_X)$, where $i = 0, 1, \ldots, h-1$.*

As X^u is singular, it is hard to calculate $H^i_{X^u}(X, \mathcal{O}_X)$ directly so we will use a spectral sequence from [Van91] that converges to $H^i_{X^u}(X, \mathcal{O}_X)$.

Fix a Borel subgroup $B \subset G$, all parabolic subgroups will contain B.
Let $w_1 = (1, 0, \ldots, 0), w_2 = (1, 1, 0 \ldots, 0), \ldots, w_{k-1} = (1, \ldots, 1, 0)$. These are the fundamental weights of $SL(V)$, i.e. any dominant weight can be written as $\sum a_i w_i$ where $a_i \geq 0$.
(Note: our weights live in $\mathbb{R}^k$ up to shifts of $(a, a, \ldots, a)$, so we shift all our weights to have a zero in the final position.)

3.2 The Spectral Sequence

We want to calculate $H^i_{X^u}(X, \mathcal{O}_X)$, to do this we will find a spectral sequence where the terms are local cohomology terms $H^*_Y(\tilde{X}, -)$ where Y is a sub-bundle of $\tilde{X}$ and $\tilde{X}$ is $X \times G/P$, for some P a parabolic subgroup. Y will be closely related to a resolution of a stratification of X^u.

First we will do the first two cases by hand, then give the general construction in [Van91] applied in our situation. Let $\dim V = 2$, then $X^u = \{rank \leq 1 \ \ maps\}$ and $0 \subset X^u \subset X$, so we get a triangle

$$H^\bullet_0(X, \mathcal{O}_X) \to H^\bullet_{X^u}(X, \mathcal{O}_X) \to H^\bullet_{X^u \backslash 0}(X \backslash 0, \mathcal{O}_{X \backslash 0})$$

We think of this as replacing $H^\bullet_{X^u}(X, \mathcal{O}_X)$ with $H^\bullet_0(X, \mathcal{O}_X)$, which we understand, and $H^\bullet_{X^u \backslash 0}(X \backslash 0, \mathcal{O}_{X \backslash 0})$. To deal with this term we use the relation between local cohomology and relative De Rham homology. We have the following diagram

$$\begin{tikzcd}
X^u \backslash 0 \arrow[r, "\sim"] \arrow[d, hook] \arrow[drr] & Y \backslash Gr(1, V) \arrow[r, hook] & X \times Gr(1, V) \arrow[d] \\
X \arrow[rr] & & X
\end{tikzcd}$$

where $Y = \mathrm{Hom}(\mathbb{C}^n, L)_{Gr(1,V)}$, rank $L = 1$. Y is a resolution of X^u and also a sub-bundle of X thought of as a trivial bundle over $Gr(1, V)$. This diagram shows that we have

$$H^\bullet_{X^u \backslash 0}(X, \mathcal{O}_X) \cong H^{DR}_\bullet(X^u \backslash 0 / X) \cong H^\bullet_{Y \backslash Gr(1,V)}(X \times Gr(1, V), \Omega^\bullet_{X \times Gr(1,V)/X})$$

Then using the exact triangle (1), we can replace $H^\bullet_{Y \backslash Gr(1,V)}(X \times Gr(1, V), \Omega^\bullet_{X \times Gr(1,V)/X})$ with $H^\bullet_Y(X \times Gr(1, V), \Omega^\bullet_{X \times Gr(1,V)/X})$ and $H^\bullet_{0 \times Gr(1,V)}(X \times Gr(1, V), \Omega^\bullet_{X \times Gr(1,V)/X})$, we can calculate both of them using Lemma 2.5, so we are done. (Up to working out maps and gradings).

The next case is $\dim V = 3$, here we have $0 \subset X^1 \subset X^2 = X^u$, where X^i are the maps of rank $\leq i$.
If we consider $0 \subset X^2 \subset X$ as above we get the following. Let $Y = \mathrm{Hom}(\mathbb{C}^n, L)_{Gr(2,V)}$ where rank $L = 2$. Again Y is a resolution of X^u, however this time $Z^u \backslash 0 \ncong Y \backslash Gr(2, V)$. Over the rank 2 maps the map $Y \to X^u$ is an isomorphism, however a rank 1 map has $\mathbb{P}^1$ as a fibre over it. What is instead true is that $X^u \backslash X^1 \cong Y \backslash Y^1$ where Y^1 is the preimage of X^1.
This means that we should consider $0 \subset X^1 \subset X$ and $X^1 \subset X^2 \subset X$ and in the same way as above, using (1), we can replace the local cohomology of X^u by local cohomology of 0, Y, $0 \times Gr(1, V)$, $Z = \mathrm{Hom}(\mathbb{C}^n, L')_{Gr(1,V)}$ (where rank $L' = 1$) and a term Y^1. We can also deal with this final term

in a similar way.
We have $U = \mathrm{Hom}(\mathbb{C}^n, M)_{Fl(1,2,3)}$ resolving Y^1 (M is rank 1) and $Y^1 \backslash 0 \times Gr(2, V) \cong U \backslash 0 \times Fl(1, 2, 3)$ so we can replace the local cohomology of Y^1 by that of $0 \times Gr(2, V)$, U and $0 \times Fl(1, 2, 3)$. All of these terms are linear sub-bundles so we can use Lemma 2.5.

They all fit together as in the following diagram

$$\begin{tikzcd}
 & 0 \times Gr(2,3) \arrow[dl] \arrow[dd] & & 0 \times Fl(1,2,3) \arrow[ll] \arrow[dl] \arrow[dd] \\
\mathrm{Hom}(\mathbb{C}^n, L)_{Gr(2,3)} \arrow[dd] & & \mathrm{Hom}(\mathbb{C}^n, M)_{Fl(1,2,3)} \arrow[ll] \arrow[dd] & \\
 & 0 \arrow[dl] & & 0 \times Gr(1,3) \arrow[ll] \arrow[dl] \\
X^u & & \mathrm{Hom}(\mathbb{C}^n, L')_{Gr(1,3)} \arrow[ll] &
\end{tikzcd}$$

In general we have exactly the same idea. Stratify X^u by the X^i, take a specific resolution, Y^i, of each X^i and then consider a stratification of each Y^i given by the preimages of the X^j with $j < i$, resolve those stratifications and repeat this procedure until we only have local cohomology of sub-bundles. The general case is explained in [Van91], here is a brief summary of the result in our situation using some of their language.

Consider the stratification $0 \subset X^1 \subset \cdots \subset X^{k-1} = X^u$ where $\dim V = k$.
Let Y^i be a linear subspace of X such that $GY^i = X^i$. Let $P \supseteq B$ be any parabolic subset preserving Y^i, consider $Y^i \times_P G$, this is a sub-bundle of $X \times G/P$ and maps down to X^i. These give the local cohomology terms in the spectral sequence. We get maps $Y \times_P G \to Y' \times_{P'} G$ whenever $Y \subseteq Y'$ and $P \subseteq P'$.

3.3 Calculations

Now that we know what terms appear in the spectral sequence we next calculate all the possible representations that can appear in these terms, this will also tell us what representations can not appear in $H^\bullet_{X^u}(X, \mathcal{O}_X)$.

The result we will get is the following

Lemma 3.2. *Any representation appearing in $H^i_{X^u}(X, \mathcal{O}_X)$ for $i < h$ has weight $\sum a_i w_i$ where at least one $a_i \geq n - k$ and all the a_i are non-negative.*

This is enough to prove that Λ is a Cohen-Macaulay algebra, see Proposition 3.7 for a proof.

The rest of this section will be devoted to proving this result. A slightly stronger result was proved independently using another method of calculating local cohomology in [RWW14].

Lemma 3.1 tells us that we only need to check what representations appear in the first $h = \dim X - \dim G$ degrees. We have $\dim X = \dim \operatorname{Hom}(\mathbb{C}^n, V) = nk$ and $\dim G = \dim SL(V) = k^2 - 1$ which gives us that $h = nk - k^2 + 1$.

We have $H^\bullet_Y(X', \mathcal{O}_{X'})$ lives in degree codimY when Y is a sub-bundle and in general we have the following diagram

$$\begin{array}{ccc} H^{DR}_{-i}(0 \times F/X) & \xrightarrow{\sim} & H^{2m+i}_{0\times F}(X \times F, \Omega^\bullet_{X\times F/X}) \\ \downarrow & & \downarrow \\ H^{DR}_{-i}(X^u/X) & \xrightarrow{\sim} & H^i_{X^u}(X, \mathcal{O}_X) \end{array}$$

where m is the relative dimension of $X \times F/X = \dim F$. At worst F is the complete flag variety of V which has dimension $\frac{k(k-1)}{2}$ and as $0 \times F$ has codimension nk, we have a map

$$H^{nk}_{0\times F}(X \times F, \Omega^*_{X\times F/X}) \to H^{nk-2m}_{X^u}(X, \mathcal{O}_X)$$

and $m \leq \frac{k(k-1)}{2}$ so at worst this maps into degree $nk - k(k-1) = nk - k^2 + k > h$. We also have

$$\Omega^*_{X\times F/X} = \mathcal{O}_{X\times F/X} \oplus \Omega^1_{X\times F/X}[-1] \oplus \cdots \oplus \Omega^{\dim F}_{X\times F/X}[-\dim F]$$

and $H^i(Y, \mathcal{F}[-j]) = H^{i-j}(Y, \mathcal{F})$, so $H^{nk}_{0\times F}(X \times F, \Omega^j_{X\times F/X}[-j])$ appears in degree $nk+j$ and so gets mapped to $H^{nk-2m+j}_{X^u}(X, \mathcal{O}_X)$.
This shows that we can ignore the local cohomology of any of the terms where $Y = 0 \times G/P$ where P is any parabolic (including $P = G$).

Next we calculate what representations appear in the remaining terms, to do this we will use Borel-Weil-Bott and Lemma 2.5.
Consider the situation $X \times F \xrightarrow{q} F \xrightarrow{\pi} pt$. We have

$$H^\bullet_Y(X \times F, \Omega^\bullet_{X\times F/X}) \cong H^\bullet(F, \mathcal{H}^\bullet_Y(X \times F, \mathcal{O}_{X\times F}) \otimes \Omega^\bullet_F))$$

using $\Omega^\bullet_{X\times F/X} = \mathcal{O}_{X\times F} \otimes q^*\Omega^*_F$, the projection formula and that calculating cohomology is the same pushing forward along $\pi \circ q$ (Recall that we are using $\mathcal{H}^\bullet_Y(X \times F, -)$ to denote the pushdown along q to F). We will first calculate what potential representations we get from the $\mathcal{H}^\bullet_Y(X \times F, \mathcal{O}_{X\times F})$ term, then we will add in Ω^*_F to get the result afterwards.

Lemma 3.3. *Let $Y = \operatorname{Hom}(\mathbb{C}^n, W)_F$ where F is some flag variety and $\dim W = l$. Then $\mathcal{H}^\bullet_Y(X \times F, \mathcal{O}_{X\times F})$ only contains terms that are the pushdown of line bundles $\mathcal{O}(\gamma)$ on the complete flag, where $\gamma = \sum a_i w_i$, with $a_l \geq n, a_i \geq 0$.*

Proof. We can push the bundle $\operatorname{Hom}(\mathbb{C}^n, W)$ down to $Gr(l, V)$ without changing the weights that appear. (Another way of saying this is that the line bundle which pushes down to $\operatorname{Hom}(\mathbb{C}^n, W)$ doesn't depend on the partial flag that it is over). So we can assume that $F = Gr(l, V)$.
By Lemma 2.5 we have

$$\mathcal{H}^\bullet_Y(X \times Gr(l, V), \mathcal{O}_{X\times Gr(l,V)}) = \operatorname{Sym}^\bullet(\operatorname{Hom}(\mathbb{C}^n, W)^*) \otimes \operatorname{Sym}^\bullet(\operatorname{Hom}(\mathbb{C}^n, V/W)) \otimes (\det V/W)^{\otimes n}$$

Consider the first term, let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l \geq 0$, then using

$$\mathrm{Sym}^{\bullet}\left(\bigoplus W^*\right) \cong \bigotimes \mathrm{Sym}^{\bullet}\left(W^*\right)$$

and the Littlewood-Richardson rule on

$$\mathrm{Sym}^{\alpha_1}\left(W^*\right) \otimes \mathrm{Sym}^{\alpha_2}\left(W^*\right) \otimes \cdots \otimes \mathrm{Sym}^{\alpha_l}\left(W^*\right) \otimes \mathrm{Sym}^{0}\left(W^*\right) \otimes \cdots \mathrm{Sym}^{0}\left(W^*\right)$$

we find that $\mathbb{S}^{\alpha} W^*$ appears for any $\alpha = (\alpha_1, \ldots, \alpha_l)$ a Young diagram.

Looking at the final term, we have $\det V/W \cong \det W^* \cong \mathcal{O}(1)$. This comes from the Young diagram $(n, n, \ldots, n)$ as we have n copies of $\det V/W$, on the complete flag this corresponds to nw_l.

Finally considering the middle term and writing $Q = V/W$, we get $\mathbb{S}^{\beta'} Q$ appearing for any β' a Young diagram in exactly the same way as for W. Rewriting in terms of Q^* we get $\mathbb{S}^{\beta} Q^*$ where $\beta = (-\beta'_{n-k}, \ldots, -\beta'_1)$. When we lift to the line bundle $\mathcal{O}(\gamma) = \mathcal{O}(\alpha, \beta)$ on the complete flag, we are allowed to shift by $(1, 1, \ldots, 1)$ as we only care about $SL(V)$ weights, not $GL(V)$ weights, so shift by β'_1 to get that γ must appear in the wanted form. $\square$

Proof of Lemma 3.2. If a representation appears in $H^i_{X^u}(X, \mathcal{O}_X)$, it must appear somewhere in the spectral sequence, i.e. it must appear in

$$H^{\bullet}_Y(X \times F, \Omega^{\bullet}_{X \times F/X}) \cong H^{\bullet}(F, \mathcal{H}^{\bullet}_Y(X \times F, \mathcal{O}_{X \times F}) \otimes \Omega^{\bullet}_F))$$

For some Y, F as above. By the earlier discussion we do not need to consider the terms where $Y = O \times F$ for any F.

In Lemma 3.3 we showed how to deal with $\mathcal{H}^{\bullet}_Y(X \times F, \mathcal{O}_{X \times F})$, so we need to deal with $\Omega^{\bullet}_F$, where $\mathcal{O}_F \oplus \Omega^1_F[-1] \oplus \cdots \oplus \Omega^p_F[-p]$, $\dim F = p$.

In the first case of $F = Gr(l, V)$ we have $\Omega^1_{Gr(l,V)} \cong \mathrm{Hom}(Q, W) \cong Q^* \otimes W$ using the notation of Lemma 3.3 and $\Omega^p_{Gr(l,V)} \cong \det(Q^* \otimes W) \cong \mathcal{O}(-k)$. In the other extreme case, on the complete flag, $F(V)$, we have $\Omega^p_{F(V)} \cong \mathcal{O}(-2\rho)$ where $\rho = (k-1, k-2, \ldots, 1, 0)$. ($\rho$ is half the sum of the positive weights).

In general there is not an easy way of describing $\Omega^{\bullet}_F$, but we can describe the line bundles that push down to the components of $\Omega^{\bullet}_F$. Using this description on G/P we get that $\Omega^1_{G/P}$ has associated line bundles $\mathcal{O}(\gamma)$, where γ corresponds to the collection of roots of G with the roots of P removed. As $B \subset P$ we get some collection of negative roots, and for $P = B$ we get all the negative roots. (The negative roots look like $(0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0)$)
Taking wedge products corresponds to adding the weights together, for example on the complete flag, the canonical bundle corresponds to adding all the negative weights together, this is exactly -2ρ as above.

This means that we need to consider the line bundle corresponding to $nw_l + \sum a_i w_i + \sum_{p \in S^-} p$ where S^- is some collection of negative roots. This is not written in the right form to find what the weight is so we need to rewrite without the $\sum_{p \in S^-} p$ term. We can assume all the a_i are zero as they only make the situation better, so let $\beta = nw_l + \sum_{p \in S^-} p$.

Before we can analyse this further we will prove a bound on the maximum difference that can appear in $\gamma = \sum_{p\in S^-} p$. The claim is that in this situation we have $\gamma_j - \gamma_i \leq k + j - i - 1$, for $j > i$. We will show this by induction.

Fix k, pick $i < j$. If $i = 1, j = k$ then the weight -2ρ gives the worst case. If at least one of i, j is not $1, k$ then let $k' = j - i + 1 < k$. By assumption, only using weights that fit into the window $[i, j]$, we get that $\gamma'_j - \gamma'_i \leq k' + j - i - 1 = 2j - 2i$. We can at most increase γ'_j by $i - 1$ as we have that many choices for a negative weight with the -1 in one of the first $i - 1$ entries and the 1 in the j^{th} entry, similarly we can reduce γ'_i by at most $k - j$. Putting these together we get

$$\gamma_j - \gamma_i \leq \gamma'_j - \gamma'_i + i - 1 + k - j \leq 2j - 2i + i + k - j - 1 = k + j - i - 1$$

The base case of $k = 2$ is trivial.

Now that we have this bound we can complete the proof.

If β is dominant, at most we can have $\beta_l - \beta_{l+1} = n - (k + l + 1 - l - 1) = n - k$. Rewriting in terms of the w_i basis we get the result.

If β is not dominant then we need to apply Borel-Weil-Bott in full generality. Let $\beta' = \sum_{\rho\in S^-} \rho$ and let β_i be the smallest entry with $i \leq l, \beta_j$ the largest entry with $j > l$. After applying the twisted Weyl action from Borel-Weil-Bott we get α (we assume that we get a dominant weight α, if we get nothing it is fine) and we have the following bounds

$$\begin{aligned}\alpha_l &\geq \beta_i + (k - i) - (k - l) = \beta_i + l - i\\ \alpha_{l+1} &\leq \beta_j + (k - j) - (k - (l + 1)) = \beta_j + l - j + 1\end{aligned}$$

as $\beta_i = n + \beta'_i$ and $\beta_j = \beta'_j$ we have

$$\begin{aligned}\alpha_l - \alpha_{l+1} &\geq n + \beta'_i + l - i - (\beta'_j + l - j - 1)\\ &\geq n + (-k - j + i + 1) + l - i - l + j - 1\\ &\geq n - k\end{aligned}$$

Again, rewriting in the w_i basis gives us the result. □

Remark 3.4. It is easy to show that these bounds are attained and that given any representation such that at least one of the coefficients in front of the w_i is at least $n - k$ we can find a term in the spectral sequence where it appears.
In seems like [Van99] enables us to strengthen this and say that these representations also appear in $H^i_{X^u}(X, \mathcal{O}_X)$, this would give us the same result as [RWW14].

Remark 3.5. We can also see exactly where we get a stronger result then the one from [Van91]. In the above work we have shown that the collection of representations that we get does not get any bigger after applying Borel-Weil-Bott in full generality. I.e. everything that appears after applying the twisted Weyl group action can also be found directly. This is not true in [Van91], more precisely, we have strengthened Corollary 6.8 of that paper.

Now that we have found out what weights are Cohen-Macaulay, we can answer the original question of whether our weights give a Cohen-Macaulay algebra.
To do this we use a result about what weights can appear in the decomposition of $\mathbb{S}^\alpha \otimes \mathbb{S}^\beta$, the full decomposition is given by the Littlewood-Richardson rule, we just need the following

Lemma 3.6. *Assume that $\mathbb{S}^\gamma \subset \mathbb{S}^\alpha \otimes \mathbb{S}^\beta$ and that all diagrams have at most k rows. Then we have the following bounds on the entries of γ.*

$$\alpha_i + \beta_k \leq \gamma_i \leq \alpha_1 + \beta_i$$

Proof. We will first prove the upper bound then the lower one.
A straightforward application of the rule says that the boxes from the i^{th} row of β can only appear in rows i to k. This gives us $\gamma_1 \leq \alpha_1 + \beta_1$. As each column has to have strictly increasing boxes, once we have filled up to column α_1 the columns have to look like 1234 etc. This shows that row i can not be longer than $\alpha_1 + \beta_i$.

Using the first observation also tells us that $\gamma_k \geq \alpha_k + \beta k$ and we can generalize this using part of the rule. It tells us that there must be at least β_k boxes from β_{k-1} placed in the rows above the k^{th} row, combining this with the fact that boxes from β_{k-1} can only appear in rows $k-1$ and k tells us that at least β_k of them must appear in row $k-1$. I.e. $\gamma_{k-1} \geq \alpha_{k-1} + \beta_k$.
In exactly the same way one can show that we need to place at least β_k boxes from β_i in the i^{th} row. This gives us the lower bound $\gamma_i \geq \alpha_i + \beta_k$. □

This is enough to show that our algebra is Cohen-Macaulay.

Proposition 3.7. *The algebra*

$$\Lambda = End\left(\bigoplus_{\alpha \in \mathcal{UP}_{n,k}} (\mathbb{S}^\alpha V^* \otimes \mathrm{Sym}^\bullet X^*)^G\right)$$

is Cohen-Macaulay

Proof. It is sufficient to show that every piece of the decomposition of Λ is Cohen-Macaulay. In this situation each piece appears in the decomposition of $\mathbb{S}^{\alpha^*} V^* \otimes \mathbb{S}^\beta V^*$ where $\alpha^* = (\alpha_1 - \alpha_k, \alpha_1 - \alpha_{k-1}, \ldots, \alpha_1 - \alpha_1)$ and $\alpha_i, \beta_i \leq (n-k)\frac{k-i}{k}$.
Applying Lemma 3.6 we get the following bounds on the γ_i.

$$\alpha_1 - \alpha_{k-i+1} \leq \gamma_i \leq \alpha_1 + \beta_i$$

(as $\alpha_k = \beta_k = 0$). This shows that

$$\begin{aligned}
\gamma_i - \gamma_{i+1} &\leq \alpha_1 + \beta_i - (\alpha_1 - \alpha_{k-i}) \\
&\leq \beta_i + \alpha_{k-i} \\
&\leq (n-k)\frac{k-i}{k} + (n-k)\frac{k-(k-i)}{k} = n-k
\end{aligned}$$

As n and k are coprime we have a strict less than and this shows that if we write γ in terms of the w_i each coefficient must be less then $n-k$.
So our algebra Λ is Cohen-Macaulay! □

As an aside, in general a collection of weights will give a Cohen-Macaulay endomorphism algebra if it is "small" enough and that algebra will have finite global dimension if the collection is "big" enough. As we are looking for a NCCR which requires both CM and finite global dimension, our collection should be on the boundary of both conditions. It turns out that we can prove that if we add any weight to $\mathcal{UP}_{n,k}$ then the resulting endomorphism algebra is no longer Cohen-Macaulay.

To show this is true we need first need another simple result about the Littlewood-Richardson rule, it follows by simply building β given α and γ.

Lemma 3.8. *Let α, β, γ be weights with at most l rows, then a $\mathbb{S}^\gamma$ such that $\gamma_1 = \alpha_1$ appears in the decomposition of $\mathbb{S}^\alpha \otimes \mathbb{S}^\beta$ if and only if $\beta \leq (\alpha_1 - \alpha_l, \alpha_1 - \alpha_{l-1}, \dots, \alpha_1 - \alpha_1)$.*

Lemma 3.9. *Let β be any weight that doesn't appear in $\mathcal{UP}_{n,k}$, i.e there exists a β_i that doesn't satisfy $\beta_i \leq (n-k)\frac{k-i}{k}$.*
Then there exists a weight $\alpha \in \mathcal{UP}_{n,k}$ such that at least one representation that appears in the decomposition of $\mathbb{S}^{\alpha^*} \otimes \mathbb{S}^\beta$ that is not Cohen-Macaulay.

Proof. Let $l = \max\{i | \beta_i > (n-k)\frac{k-i}{k}\}$ by the assumptions on β such a l exists.
Let

$$\bar{\alpha} = (\alpha_1, \alpha_2, \cdots \alpha_l, 0, \dots, 0)$$

where α_i is the largest allowed value. Then we have that

$$\bar{\alpha}^* = (\alpha_1, \alpha_1, \dots, \alpha_1, \alpha_1 - \alpha_{k-l}, \alpha_1 - \alpha_{k-l-1}, \dots, \alpha_1 - \alpha_2, 0)$$

We claim that we can find $\mathbb{S}^\gamma \subset \mathbb{S}^{\bar{\alpha}^*} \otimes \mathbb{S}^\beta$ such that $\mathbb{S}^\gamma$ is not Cohen-Macaulay.
It is sufficient to show that there exists a γ with $\gamma_l - \gamma_{l+1} \geq n-k$.

Let $\gamma = (\alpha_1 + \beta_1, \alpha_1 + \beta_2, \dots, \alpha_1 + \beta_l, ?, \dots, ?)$.
In applying the Littlewood Richardson rule (algorithm) we have used up all the entries of β up to and including the l^{th} row, we also can not change the first l entries of γ so we have reduced the problem to applying the Littlewood Richardson rule to the situation

$$(\alpha_1 - \alpha_{k-l}, \alpha_1 - \alpha_{k-l-1}, \dots, \alpha_1 - \alpha_2, 0) \otimes (\beta_{l+1}, \dots, \beta_{k-1}, 0)$$

We want to find a diagram in the decomposition with the first entry equal to $\alpha_1 - \alpha_{k-l}$.
To check that we can do this we need to show that our weights satisfy the conditions of the above lemma.

So we need to show that $\beta_{l+i} \leq (\alpha_1 - \alpha_{k-l}) - (\alpha_1 - \alpha_i) = \alpha_i - \alpha_{k-l}$.

Note that we have $\alpha_i + \alpha_{k-j} \leq \alpha_{i-j}$ for $i \geq j$. (We are assuming that α_i is the maximum possible value for all i.)[1]
Applying this gives us that

$$\alpha_i - \alpha_{k-l} \geq \alpha_{l+i}$$

and by assumption

$$\beta_{l+i} \leq (n-k)\frac{k-(l+i)}{k}$$

so using that α_{l+i} is as large as possible, we get that $\beta_{l+i} \leq \alpha_{l+i}$ and so we have found a γ which looks like $(\dots, \alpha_1 + \beta_l, \alpha_1 - \alpha_{k-l}, \dots)$.
And we have $\beta_l + \alpha_{k-1} \geq n-k$ (use that α_{k-l} is maximal and that $\beta_l > (n-k)\frac{k-l}{k}$). ☐

[1] This follows as $\lfloor a \rfloor + b \leq a + b$ which implies that $\lfloor a \rfloor + \lfloor b \rfloor = \lfloor \lfloor a \rfloor + b \rfloor \leq \lfloor a + b \rfloor$.

3.4 Finite global dimension

To complete the proof that Λ is a NCCR, we need to show that Λ has finite global dimension. To do this we will use the resolution from [ŠV15] described at the start of this section. The same resolution is also found in [Fon13] and [DS14] for the special case of $G = GL(V)$ but they describe it differently (staircase complexes), it turns out that their description is more helpful for us.
We will analyse this resolution and show that by iterating it a finite number of times, any module P_α has a finite resolution by modules P_β for $\beta \in \mathcal{UP}_{n,k}$.

First if P_α is any module such that $\alpha_1 > n-k$, then using the description in [ŠV15] it is easy to see that all MP_β that appear in the resolution satisfy $\beta_1 < \alpha_1$, see Theorem 3.11 for the proof. This tells us that every weight has a finite resolution by weights that fit into a rectangle of size $(n-k) \times k$. Therefore it is sufficient to prove that all weights of size at most $(n-k) \times k$ have finite projective dimension.

To do this we will use the description of the resolution, as found in [Fon13] and [DS14], called a staircase complex.
This is all found in [Fon13], we reproduce it here for completeness, and also simplify the arguement. (Fonarev uses $GL(V)$ representations, our representations are $SL(V)$ ones).

We can represent a Young diagram, α, using a binary sequence of length n containing k 1's and $n-k$ 0's. Representing diagrams in this way gives us a natural action of $\mathbb{Z}/n\mathbb{Z}$ by rotation, under this action exactly one diagram in each orbit will be upper triangular.

There is a geometric way of describing this operation as well.
Take a diagram α with $\alpha_1 \leq n-k$ and glue them together at the corners. (In terms of the binary sequence, repeat the sequence infinitely in both directions.) Then applying i rotations is the same as starting at the bottom left corner of the diagram and taking i steps along the edge of α, then creating a new diagram with the bottom left being at this point.
I.e. take a box of size $(n-k) \times k$ and move it along the edge of the diagram α to give the other diagrams in the orbit.

In Figure 2 we see an example where $n = 11, k = 4, \alpha = (6, 3, 3, 0)$. The green box labelled 1 is the original diagram, the box labelled 2 is the diagram $(4, 3, 0, 0)$ and corresponds to $4 \cdot \alpha$. Finally, the box labelled 3 is the diagram $(4, 4, 1, 0)$ and corresponds to $9 \cdot \alpha$.

Using this description we can also easily show that there exists a unique upper triangular diagram in the orbit. To see this draw a diagonal line with gradient $(n-k)/k$ through each corner of the diagram, one of these lines will be the lowest (and in fact only one, using the fact that n and k are coprime.). Take the diagram that has as corners, the points where this line touches the repeating diagram, it will be upper triangular. See Figure 3 for an example, again with $\alpha = (6, 3, 3, 0)$.

Let $d^{upp}(\alpha)$ be the number of rotations needed to take α to an upper triangular diagram, or equivalently the number of steps needed to move to the corner which gives the lowest line. Note that $d^{upp}(\alpha) = 0$ for all diagrams in $\mathcal{UP}_{n,k}$.
Next we describe the terms that appear in the staircase complex, these come from $SL(V)$ representations, but we won't remove any complete columns when we draw the diagrams.
Let α be a diagram such that $\alpha_1 \leq n-k$, create a new diagram by adding a strip to the edge of α

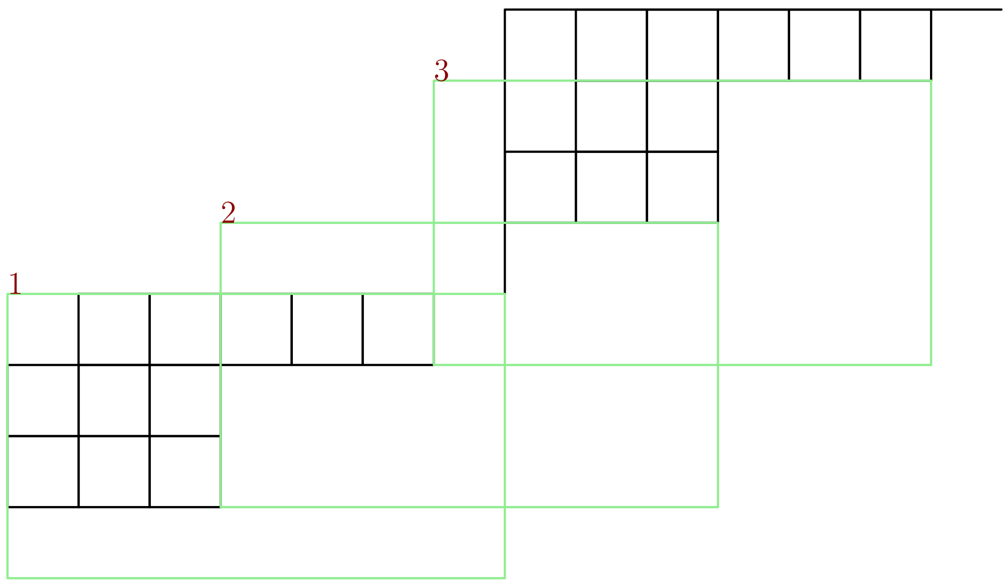

Figure 2: The effect of the $\mathbb{Z}/n\mathbb{Z}$ action

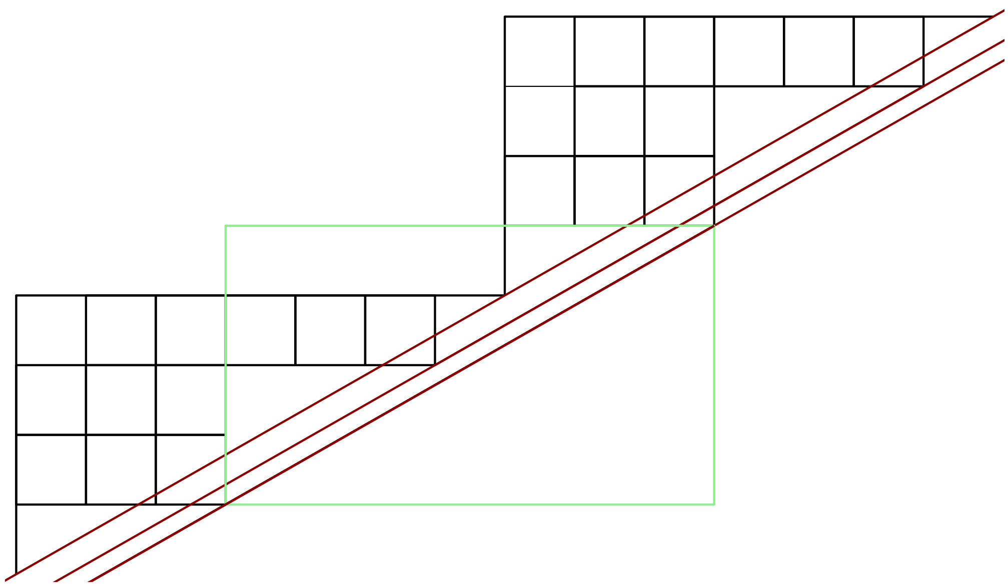

Figure 3: Detecting the upper triangular element in the orbit

as seen in Figure 4 and extend along the first row until we have added n boxes. Then the first term in the complex has diagram corresponding to the new diagram, but where we have only completed the first column, and the i^{th} term has diagram corresponding the new diagram, but this time with only the new terms from the first i columns. (It is also possible to find the multiplicities of the terms, but we don't need that detail.)
Looking at Figure 4, the i^{th} term is the diagram corresponding to the original diagram as well as all boxes labelled i or less.

Lemma 3.10. *Let α be a diagram not in $\mathcal{UP}_{n,k}$ and such that $\alpha_1 \leq n-k$, let P_β be any term that appears in the given resolution of P_α.*

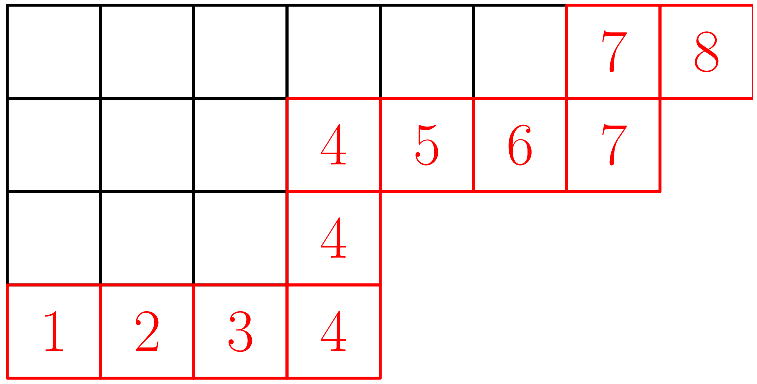

Figure 4: The staircase resolution

Then we have $d^{upp}(\beta) < d^{upp}(\alpha)$

Proof. We will take the definition of d^{upp} as the number of steps to get to the corner that gives the lowest diagonal. It is clear that adding more boxes can only make the lowest diagonal lower so we can assume β is the final term in the resolution.
Also note that removing a column reduces d^{upp} by 1 and β has a full column so at least one can be removed, this shows it is sufficient to show $d^{upp}(\beta') \leq d^{upp}(\alpha)$ where β' is β without excess columns removed.

As β' looks like α shifted diagonally by 1, if the lowest diagonal comes from a corner not on the first row then $d^{upp}(\beta') = d^{upp}(\alpha)$ so it is sufficient to show that the lowest diagonal can't come from the corner on the first row.

At worst $\beta'_1 = n - k + 1$, if this is the case we need to remove the first column so that we fit into a box of size $(n-k) \times k$. Assume this is the case and that the lowest diagonal meets here. This means that the weight $(\beta'_2 - 1, \dots, \beta_k - 1, 0)$ is upper triangular. We have $\beta_i = \alpha_{i-1} + 1$ for $i \geq 2$ which shows that $(\alpha_1, \dots, \alpha_{k-1}, 0)$ is in fact upper triangular. This contradicts the original assumption. □

This result is found in [Fon13, Prop. 5.6] as a note.
We can now show the main result, that Λ is a NCCR for X/G.

Theorem 3.11. *The algebra Λ has finite global dimension and gives a NCCR of X/G.*

Proof. It is sufficient to show that P_α has finite projective dimension for any Young diagram α. By construction all the P_α where $\alpha \in \mathcal{UP}_{n,k}$ are projective, so let α be a weight not in $\mathcal{UP}_{n,k}$, if $\alpha_1 \leq n-k$ then Lemma 3.10 shows that we have a finite resolution of P_α by projectives.
Another way of describing the terms, P_β, in the resolution is that β is one of the weights that arise by applying Borel-Weil-Bott to $\alpha + (0, 0, \dots, 0, i)$ for $i = 1, \dots n$.
We will use this description to deal with diagrams α such that $\alpha_1 > n-k$.

As $i \leq n < \alpha_1 + k$, after applying Borel-Weil-Bott we still have that α_1 is the largest entry, and the final element is at least 1. This shows that any P_β in the resolution satisfies $\beta_1 < \alpha_1$.

Therefore any module P_α with $\alpha_1 > n-k$ has a finite resolution by modules P_β such that $\beta_1 \leq n-k$.

Putting these two results together we get that each P_α has finite projective dimension.

We have already shown that Λ is Cohen-Macaulay in Proposition 3.7 so we get Λ is an NCCR of X/G. $\square$

4 Geometric NCCR

There is another to find a NCCR of X/G.
We have a geometric resolution of X/G given by $\mathcal{O}(-1)_{Gr(n,k)}$ and the exceptional divisor is the zero locus, isomorphic to the Grassmannian.

$$\begin{tikzcd} Gr(n,V) \arrow[r, hook, shift left, "i"] \arrow[d] & \mathcal{O}_{Gr(n,V)}(-1) \arrow[l, shift left, "p"] \arrow[d, "\pi"] \\ 0 \arrow[r, hook] & \mathrm{Hom}(\mathbb{C}^n, V)/SL(V) \end{tikzcd}$$

We also have the standard exact sequence of vector bundles on $Gr(n,V)$.

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{Gr(n,V)} \to Q \to 0$$

Fonarev, [Fon13], tells us that we have a rectangular Lefschetz decomposition of $D^b(Gr(k,n))$ with the first block given by Schur powers of the universal quotient-bundle, Q, on the Grassmannian. These Schur powers are indexed by upper triangular Young diagrams of size $(n-k) \times k$, exactly the ones in $\mathcal{UP}_{n,k}$ as earlier.

Kuznetsov, [Kuz06], gives a way to find a categorical (crepant) resolution in this situation. He works in the slightly more general situation of a geometric resolution with an exceptional divisor that has a Lefschetz decomposition. He uses this Lefschetz decomposition to build a semiorthogonal collection on the resolution. Then the left orthogonal of this collection gives a categorical (crepant) resolution.

Applying this to our situation gives us that the subcategory $\tilde{D} = \{\mathcal{F} | i^*\mathcal{F} \in B_0\}$ is a categorical weakly crepant resolution where B_0 is the subcategory generated by the Schur powers mentioned above. As $p \circ i = id$, we have that $\tilde{D}$ is generated by $p^*\mathbb{S}^\alpha Q$.
If in addition $\tilde{D}$ is generated by a tilting vector bundle, then Kuznetsov proves that the endomorphism algebra of vector bundle gives an NCCR.

Lemma 4.1. *The vector bundle $\bigoplus p^*\mathbb{S}^\alpha Q$ where the sum is over $\alpha \in \mathcal{UP}_{n,k}$ is tilting over $\mathrm{Hom}(\mathbb{C}^n, V)/SL(V)$.*

Proof. As $\mathrm{Hom}(\mathbb{C}^n, V)/SL(V)$ is affine we just need to show that

$$R\mathrm{Hom}(p^*\mathbb{S}^\alpha Q, p^*\mathbb{S}^\beta Q) = \mathrm{Hom}(p^*\mathbb{S}^\alpha Q, p^*\mathbb{S}^\beta Q)$$

for all $\alpha, \beta \in \mathcal{UP}_{n,k}$.

We have

$$R\mathrm{Hom}(p^*\mathbb{S}^\alpha Q, p^*\mathbb{S}^\beta Q) \cong H^\bullet\left(\mathcal{O}(-1), p^*\left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)\right)$$
$$\cong H^\bullet\left(Gr(n,V), (\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q \otimes p_*\mathcal{O}_{\mathcal{O}(-1)}\right)$$

As $p_*\mathcal{O}_{\mathcal{O}(-1)} = \bigoplus_{i\geq 0}\mathcal{O}(i)$ we need to show that $(\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q \otimes \mathcal{O}(i)$ has no higher cohomology for all $i \geq 0$.
To do this consider what terms can appear in the decomposition of $(\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q \cong \mathbb{S}^{\alpha^*} Q \otimes \mathbb{S}^\beta Q$ From Lemma 3.6 we get the following bounds on any $\mathbb{S}^\gamma Q$ appearing

$$-(n-k)(k-1)/k + i \leq \gamma_k \leq i$$

If $\gamma_k \geq 0$ it is a dominant weight and we only have cohomology in degree 0 (using Borel-Weil-Bott). Else we have

$$\gamma_k \geq i - (n-k)(k-1)/k \geq -(n-k)(k-1)/k$$

so when we apply the twisted Weyl action, we first add ρ which has k^th entry $n-k+1$, then we rearrange and as $n-k+1 > (n-k)(k-1)/k$ we get two entries with the same value. Therefore this weight is not dominant and we have no cohomology at all.
In both cases we get the wanted result. □

Corollary 4.2. $\tilde{D} \cong D^b\left(End\left(\bigoplus p^*\mathbb{S}^\alpha Q\right)\right))$

Proof. See [Kuz06, Section 5] □

This process actually gives us four categorical resolutions, as we have two different Lefschetz decompositions, one with terms of the form $\mathbb{S}^\alpha Q$, and one with terms $\mathbb{S}^{\alpha^t} S$, where α^t is the transposed Young diagram. We then get two more by using $\mathbb{S}^\alpha Q^*$ and $\mathbb{S}^{\alpha^t} S^*$. As we have $(\mathbb{S}^\alpha Q)^* = \mathbb{S}^\alpha Q^*$ and dualizing is an anti-auto-equivalence this shows us straightforwardly that we only have two potentially different categorical resolutions.
We can in fact go further and show that the remaining two resolutions are equivalent.

Lemma 4.3. *The two collections* $\{\mathbb{S}^\alpha Q\}_{\alpha\in\mathcal{UP}_{n,k}}$ *and* $\{\mathbb{S}^{\alpha^t} S\}_{\alpha\in\mathcal{UP}_{n,k}}$ *generate the same subcategory* $\tilde{D}$ *in* $D^b(\mathcal{O}_{Gr(n,V)}(-1))$.

Proof. We can take Schur powers of a sequence, as follows.
Given a short exact sequence

$$0 \to A \to B \to C \to 0$$

we get

$$0 \to \mathbb{S}^\alpha A \to \mathbb{S}^\alpha B \to \cdots \to \bigoplus\left(\mathbb{S}^\beta B \otimes \mathbb{S}^{\gamma^t} C\right)^{\oplus m^\alpha_{\beta\gamma}} \to \cdots \to \mathbb{S}^{\alpha^t} C \to 0$$

is exact, where the sum is over $|\gamma| = p, |\lambda| = |\alpha| - p$ and $m^\alpha_{\beta\gamma}$ is the Littlewood-Richardson coefficients. It is non-zero only if γ is a sub-digram of α. Applying this to

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{Gr(n,V)} \to Q \to 0$$

we get that

$$\langle \mathbb{S}^{\alpha^t} S | \alpha \in \mathcal{L}_{n,k} \rangle \subset \langle \mathbb{S}^{\alpha} Q | \alpha \in \mathcal{UP}_{n,k} \rangle$$

One can also expand $\mathbb{S}^{\alpha} C$ by terms involving Schur powers of A and B, this gives us the reverse inclusion.
Therefore our two collections are just different generators of the same subcategory. $\square$

This shows that all four categorical resolutions that arise from the geometric resolution are equivalent, and therefore all four NCCR's are derived equivalent.

5 These NCCR's are equivalent

There is a conjecture that all NCCR's should be derived equivalent, we want to show it in the special case above. We have already shown that the four different NCCR's from the geometric resolution using Kuznetsov's construction are equivalent, we also have at least two NCCR's from the algebraic picture, one in terms of $(\mathbb{S}^{\alpha} V^* \otimes \mathrm{Sym}^{\bullet} X^*)^G$ and another in terms of $\left(\mathbb{S}^{\alpha^t} W \otimes \mathrm{Sym}^{\bullet} X'^*\right)^G$, where $X' = \mathrm{Hom}(W, \mathbb{C}^n), \dim W = n - k$ and these Schur powers are thought of as $SL(V), SL(W)$ representations, but they are indexed by exactly the same Young diagrams, $\alpha \in \mathcal{UP}_{n,k}$.
One can show that even in the first non-trivial case ($n = 5, k = 2$) we get two different endomorphism algebras. Representing the algebras as quivers and writing $X = \mathrm{Hom}(U, V)$ and $X' = \mathrm{Hom}(W, U)$ we get

$$\begin{tikzcd} \cdot \arrow[loop left, "\bigoplus \mathbb{S}^{(a,a)}U"] \arrow[r, bend right, "\bigoplus \mathbb{S}^{(a+1,a)}U"'] & \cdot \arrow[l, bend right, "\bigoplus \mathbb{S}^{(a+1,a)}U"'] \arrow[loop right, "\bigoplus \mathbb{S}^{(a,a)}U \bigoplus \mathbb{S}^{(a+2,a)}U"] \end{tikzcd}$$

for X (we haven't mentioned what the relations are), and

$$\begin{tikzcd} \cdot \arrow[loop left, "\bigoplus \mathbb{S}^{(a,a,a)}U^*"] \arrow[r, bend right, "\bigoplus \mathbb{S}^{(a+1,a+1,a)}U^*"'] & \cdot \arrow[l, bend right, "\bigoplus \mathbb{S}^{(a+1,a,a)}U^*"'] \arrow[loop right, "\bigoplus \mathbb{S}^{(a,a,a)}U^* \bigoplus \mathbb{S}^{(a+2,a+1,a)}U^*"] \end{tikzcd}$$

for X'. Picking a volume form for U gives us an isomorphism between some of the components ($\bigoplus \mathbb{S}^{(a,a,a)} U^* \cong \bigoplus \mathbb{S}^{(a,a)} U$), but not all of them. For example in degree 0 ($a = 0$), we have U maps from the left to the right in the first algebra and $\bigwedge^2 U^*$ maps in the second algebra. These have ranks 5 and 10 respectively.

We will prove that these two are derived equivalent by showing that they are equivalent to those from the geometric resolution.

We have the following diagram

$$
\begin{tikzcd}
\mathcal{O}_{Gr(n,V)}(-1) \arrow[dr] & & {[\mathrm{Hom}(\mathbb{C}^n, V)/SL(V)]} \arrow[dl] \\
& \mathrm{Hom}(\mathbb{C}^n, V)/SL(V) &
\end{tikzcd}
$$

and while there is no map between the geometric resolution and the stack resolution, we can find an open set in each of them which are isomorphic.
On the geometric side we have the complement of the zero section and on the stack side we have the full rank locus, these opens are both isomorphic to the nonsingular part of $\mathrm{Hom}(\mathbb{C}^n, V)/SL(V)$.

We will show that on both sides, we can calculate the endomorphism algebra on the open sets, not just on the whole space, then as they agree on these subsets we are done.

On the stack side this follows for dimension reasons (algebraic Harthog's Lemma, the locus of maps without full rank has codimension n). To show this on the geometric side we will use local cohomology.

Let Y be the total space of the $\mathcal{O}(-1)$ line bundle on $Gr(n,V)$ and let Z be the zero locus of the zero section, $Z \cong Gr(n,V)$, finally let $U = Y \backslash Z$.
The next lemma shows that each component of the endomorphism algebra can be calculated by restricting to the open set U and then taking global sections.

Lemma 5.1. *For $\alpha, \beta \in \mathcal{UP}_{n,k}$ we have*

$$H^0\left(Y, p^*\left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)\right) \cong H^0\left(U, p^*\left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)\Big|_U\right)$$

Proof. We have the following exact sequence for local cohomology

$$0 \to H^0_Z(Y, \mathcal{F}) \to H^0(Y, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to H^1_Z(Y, \mathcal{F}) \to \cdots$$

so it is sufficient for us to prove that $H^0_Z(Y, \mathcal{F})$ and $H^1_Z(Y, \mathcal{F})$ vanish for $\mathcal{F} \cong p^*\left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)$.

We have

$$
\begin{aligned}
H^\bullet_Z(Y, p^*\mathcal{F}) =& H^\bullet_{0\times Gr(n,V)}\left(\mathcal{O}_{Gr(n,V)}(-1), p^*\mathcal{F}\right) \\
=& H^\bullet\left(Gr(n,V), \mathcal{H}^\bullet_{0\times Gr(n,V)}\left(\mathcal{O}_{Gr(n,V)}(-1), \mathcal{O}_{\mathcal{O}_{Gr(n,V)}(-1)}\right) \otimes \mathcal{F}\right)
\end{aligned}
$$

and we also have

$$\mathcal{H}^\bullet_{0\times Gr(n,V)}\left(\mathcal{O}_{Gr(n,V)}(-1), \mathcal{O}_{\mathcal{O}_{Gr(n,V)}(-1)}\right) = \bigoplus_{i\geq 1} \mathcal{O}_{Gr(n,V)}(-i)$$

by Lemma 2.5. Rewriting we get

$$H^\bullet\left(Gr(n,V), \left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)(-i)\right) = R\mathrm{Hom}_{Gr(n,V)}\left(\mathbb{S}^\alpha Q(i), \mathbb{S}^\beta Q\right)$$

For $1 \leq i \leq n$ this vanishes by [Fon13]. (The terms $\mathbb{S}^\beta Q$ give a Lefschetz decomposition with respect to $\mathcal{O}(1)$ for the Grassmannian)

If $i > n$, then we will use Serre Duality to show the result. The canonical bundle is $\mathcal{O}(-k)$ so we have

$$H^i\left(Gr(n,V), \left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)(-i)\right) \cong H^{p-i}\left(Gr(n,V), \left(\left(\mathbb{S}^\beta Q\right)^* \otimes \mathbb{S}^\alpha Q\right)(i-k)\right)^*$$

As we have full symmetry between α and β and $i > n > k$ so $i - k > 0$ we are in the situation of Lemma 4.1. During the proof we showed that such a term only has cohomology in degree 0, pushing this through Serre Duality we get that

$$H^\bullet\left(Gr(n,V), \left((\mathbb{S}^\alpha Q)^* \otimes \mathbb{S}^\beta Q\right)(-i)\right)$$

is non-zero only in the top degree.

Therefore in all situations we have no cohomology in degree 0 and 1, this gives us the wanted result. □

Theorem 5.2. *Let $X = \mathrm{Hom}(\mathbb{C}^n, V)$ and $X' = \mathrm{Hom}(S, \mathbb{C}^n)$.*
Let Λ and Λ' be the NCCR's of $X/SL(V)$ and $X'/SL(S)$ respectively from Theorem 3.11.
Then Λ and Λ' are derived equivalent.

Proof. We have already done all the work;
For dimension reasons we can calculate Λ and Λ' on the open set of maps with full rank, Lemma 5.1 tells us that we can also calculate the geometric NCCR's on this open subset.
On this subset it is clear that the geometric one described in terms of $\mathbb{S}^\alpha V^*$ and the algebraic one described in exactly the same terms are the same, similarly for the other one. In Section 4 we showed that all four NCCR's mentioned in that section are equivalent. Using this equivalence, we get that the two algebraic ones are equivalent as they are both equivalent to the geometric one. □

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